THIRD ORDER OF ACCURACY SEQUENTIAL TYPE DECOMPOSITION SCHEMES FOR TWO AND MULTIDIMENSIONAL EVOLUTION PROBLEMS

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Abstract

In the present work sequential type decomposition schemes of the third order of accuracy for the solution of two and multidimensional evolution problems are constructed and investigated. For the considered scheme the explicit *a priori* estimations are obtained.

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1. Introduction

One of the most effective methods to solve multi-dimensional evolution problems is the decomposition method. Decomposition schemes with first and second order accuracy were constructed in the sixties of the XX century (see [7], [11] and references therein). Q. Sheng has proved that in the real number field there do not exist automatically stable decomposition schemes with an accuracy order higher than two (see [12]). Decomposition schemes are called automatically stable if a sum of the absolute values of its split coefficients (coefficients of exponentials' products) equals to one, and the real parts of exponential powers are positive. In the work [1] there is constructed decomposition schemes with the higher order accuracy, but their corresponding decomposition formulas are not automatically stable. In the works [2]-[5] introducing the complex parameter, we have constructed automatically stable sequential-parallel type decomposition schemes with third order accuracy for two- and multi-dimensional evolution problems and with fourth order accuracy for two-dimensional evolution problem (evolution problem with the operator A is called m-dimensional, if it can be represented as a sum of m summands $A = A_1 + ... + A_m$). The new idea is an introduction of a complex parameter, which allows us to break the order 2 barrier.

In the present work symmetrized sequential type decomposition schemes of the third order of accuracy for the solution of two and multidimensional evolution problems are offered. For the considered scheme the explicit *a priori* estimations are obtained.

2. Statement of the Problem and Decomposition Scheme for Two Dimensional Case

Let us consider the Cauchy abstract problem for an evolution equation in the Banach space X:

$$\frac{du(t)}{dt} + Au(t) = 0, \quad t > 0, \quad u(0) = \varphi, \tag{2.1}$$

where A is a linear closed operator with a definition domain $D(A)$, which is everywhere dense in X, φ is a given element from $D(A)$.

Suppose that the operator $(-A)$ generates a strongly continuous semigroup $\{exp(-tA)\}_{t\geq 0}$. Then the solution of problem (2.1) is given by the following formula [8,9]:

$$u(t) = U(t, A)\varphi, \tag{2.2}$$

where $U(t, A) = exp(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where A_i $(i = 1, 2)$ are closed operators, densely defined in X.

Let us introduce a grid set:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Together with problem (2.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\alpha}{2}A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(2)}(t)}{dt} + \alpha A_2 v_k^{(2)}(t) &= 0, \quad v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k), \\
\frac{dv_k^{(3)}(t)}{dt} + \frac{1}{2}A_1 v_k^{(3)}(t) &= 0, \quad v_k^{(3)}(t_{k-1}) = v_k^{(2)}(t_k), \\
\frac{dv_k^{(4)}(t)}{dt} + \overline{\alpha} A_2 v_k^{(4)}(t) &= 0, \quad v_k^{(4)}(t_{k-1}) = v_k^{(3)}(t_k), \\
\frac{dv_k^{(5)}(t)}{dt} + \frac{\overline{\alpha}}{2}A_1 v_k^{(5)}(t) &= 0, \quad v_k^{(5)}(t_{k-1}) = v_k^{(4)}(t_k),
\end{aligned}$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Suppose that the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j)$, $j = 1, 2$ generate strongly continuous semigroups.

$u_k(t)$, $k = 1, 2, ..$, is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = v_k^{(5)}(t). \tag{2.3}$$

We declare function $u_k(t)$ as an approximated solution of problem (2.1) on each interval $[t_{k-1}, t_k]$.

3. Estimate of Error of the Approximated Solution

We need the natural powers (A^s, $s = 2, 3, 4$) of the operator $A = A_1 + A_2$. They are usually defined as follows:

$$\begin{aligned}
A^2 &= \left(A_1^2 + A_2^2\right) + (A_1 A_2 + A_2 A_1), \\
A^3 &= \left(A_1^3 + A_2^3\right) + \left(A_1^2 A_2 + ... + A_2^2 A_1\right) + (A_1 A_2 A_1 + A_2 A_1 A_2), \\
A^4 &= \left(A_1^4 + A_2^4\right) + \left(A_1^3 A_2 + ... + A_2^3 A_1\right) + \left(A_1^2 A_2 A_1 + ... + A_2^2 A_1 A_2\right) \\
&\quad + (A_1 A_2 A_1 A_2 + A_2 A_1 A_2 A_1),
\end{aligned}$$

It is obvious that the definition domain $D(A^s)$ of the operator A^s represents an intersection of definition domains of its addends.

Let us introduce the following notations:

$$\begin{aligned}
\|\varphi\|_A &= \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D(A); \\
\|\varphi\|_{A^2} &= \left\|A_1^2\varphi\right\| + \left\|A_2^2\varphi\right\| + \|A_1 A_2 \varphi\| \\
&+ \|A_2 A_1 \varphi\|, \quad \varphi \in D\left(A^2\right),
\end{aligned}$$

where $\|\cdot\|$ is a norm in X. $\|\varphi\|_{A^s}$, $(s = 3, 4)$ is defined analogously.

Theorem 3.1. *Let the following conditions be fulfilled:*

(a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ $\left(i = \sqrt{-1}\right)$;

(b) Operators $(-\gamma A_j)$, $\gamma = 1, \alpha, \overline{\alpha}$ $(j = 1, 2)$ *and* $(-A)$ *generate strongly continuous semigroups, for which the following estimates are true:*

$$\begin{aligned}
\|U(t, \gamma A_j)\| &\leq e^{\omega t}, \\
\|U(t, A)\| &\leq M e^{\omega t}, \quad M = const > 0;
\end{aligned}$$

(c) $U(s, A)\varphi \in D\left(A^4\right)$ *for each fixed* $s \geq 0$.

Then the following estimate holds:

$$\|u(t_k) - u_k(t_k)\| \leq c e^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^4},$$

where c and ω_0 are positive constants.

Proof. According to the following formula (see [8], p. 603):

$$A\int_r^t U(s,A)\,ds = U(r,A) - U(t,A), \quad 0 \le r \le t,$$

we can obtain the expansion:

$$U(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t,A), \tag{3.1}$$

where

$$R_k(t,A) = (-A)^k \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} U(s,A) ds ds_{k-1} \dots ds_1. \tag{3.2}$$

From formula (2.2) we obtain:

$$u_k(t_k) = V^k(\tau)\,\varphi, \tag{3.3}$$

where

$$V(\tau) = U\left(\tau, \frac{\alpha}{2}A_1\right) U(\tau, \alpha A_2)\, U\left(\tau, \frac{1}{2}A_1\right) U(\tau, \overline{\alpha} A_2)\, U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right).$$

Remark 3.1. *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem 3.1. In this case, for the solving operator, the following estimate holds:*

$$\left\| V^k(\tau) \right\| \le e^{\omega_1 t_k}, \tag{3.4}$$

where ω_1 is a positive constant.

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1,2)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (3.1), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $\left(\tau^2 A_i A_j\right)$ and $\left(-\tau^3 A_i A_j A_k\right)$ $(i,j,k = 1,2)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$ and $[T(\tau)]_{i,j,k}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (3.1) from left to right in such a way that each residual term appears of the fourth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau\sum_{i=1}^{2}[V(\tau)]_i A_i + \tau^2\sum_{i,j=1}^{2}[V(\tau)]_{i,j} A_iA_j \\ &\quad -\tau^3\sum_{i,j,k=1}^{2}[V(\tau)]_{i,j,k} A_iA_jA_k + R_4^{(2)}(\tau). \end{aligned} \tag{3.5}$$

According to the first inequality of the condition *(b)* of the **Theorem 3.1**, for $R_4^{(2)}(\tau)$, the following estimate holds:

$$\left\|R_4^{(2)}(\tau)\varphi\right\| \le ce^{\omega_0\tau}\tau^4\|\varphi\|_{A^4}, \quad \varphi\in D\left(A^4\right), \tag{3.6}$$

where c and ω_0 are positive constants.

Let us calculate the coefficients $[V(\tau)]_i$ corresponding to the first order members in formula (3.5). It is obvious that the members, corresponding to these coefficients, are obtained from the decomposition of only those factors (semigroups) of the operator $V(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identical operators) will participate.

On the whole, we have two cases: $i=1$ and $i=2$. Let us consider the case $i=1$. We obviously have:

$$[V(\tau)]_1 = [U(\tau, A_1)]_1 = 1. \tag{3.7}$$

Analogously for $i=2$ we have:

$$[V(\tau)]_2 = [U(\tau, A_2)]_2 = 1. \tag{3.8}$$

By combining formulas (3.7) and (3.8), we will obtain:

$$[V(\tau)]_i = 1, \quad i=1,2. \tag{3.9}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i,j=1,2)$ corresponding to the second order members included in formula (3.5). On the whole we have two cases: $(i,j)=(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$. Let us consider the case $(i,j)=(1,1)$. We obviously have:

$$[V(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}. \tag{3.10}$$

Analogously for $(i,j)=(2,2)$ we have:

$$[V(\tau)]_{2,2} = [U(\tau, A_2)]_{2,2} = \frac{1}{2}. \tag{3.11}$$

Let us consider the case $(i, j) = (1, 2)$, we obviously have:

$$\begin{aligned} [V(\tau)]_{1,2} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 [U(\tau, \alpha A_2)]_2 \\ &+ \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 [U(\tau, \overline{\alpha} A_2)]_2 \\ &+ \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_1 [U(\tau, \overline{\alpha} A_2)]_2 \\ &= \frac{\alpha}{2}\alpha + \frac{\alpha}{2}\overline{\alpha} + \overline{\alpha}\frac{1}{2} = \frac{\alpha(\alpha+\overline{\alpha}) + \overline{\alpha}}{2} = \frac{1}{2}. \end{aligned} \tag{3.12}$$

For $(i, j) = (2, 1)$ we have:

$$\begin{aligned} [V(\tau)]_{2,1} &= [U(\tau, \alpha A_2)]_2 \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_1 \\ &+ [U(\tau, \alpha A_2)]_2 \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right)\right]_1 \\ &+ [U(\tau, \overline{\alpha} A_2)]_2 \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right)\right]_1 \\ &= \alpha\frac{1}{2} + \alpha\frac{\overline{\alpha}}{2} + \overline{\alpha}\frac{\overline{\alpha}}{2} = \frac{\alpha + \overline{\alpha}(\alpha+\overline{\alpha})}{2} = \frac{1}{2}. \end{aligned} \tag{3.13}$$

Here we used the identity $\alpha + \overline{\alpha} = 1$.

By combining formulas (3.10) - (3.13), we will obtain:

$$[V(\tau)]_{i,j} = \frac{1}{2}, \quad i, j = 1, 2. \tag{3.14}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i, j, k = 1, 2)$ corresponding to the third order members in formula (3.5). On the whole we have eight cases: $(i, j, k) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 1)$, $(2, 1, 2)$, $(2, 2, 1)$, $(2, 2, 2)$. Let us consider the case $(i, j, k) = (1, 1, 1)$. We obviously have:

$$[V(\tau)]_{1,1,1} = [U(\tau, A_1)]_{1,1,1} = \frac{1}{6}. \tag{3.15}$$

Analogously for $(i, j) = (2, 2, 2)$ we have:

$$[V(\tau)]_{2,2,2} = [U(\tau, A_2)]_{2,2,2} = \frac{1}{6}. \tag{3.16}$$

Thus Let us calculate the case $(i, j, k) = (1, 1, 2)$. We have:

$$\begin{aligned} [V(\tau)]_{1,1,2} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1} [U(\tau, \alpha A_2)]_2 \\ &+ \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1} [U(\tau, \overline{\alpha} A_2)]_2 \end{aligned}$$

$$
\begin{aligned}
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2\\
=\;&\frac{\alpha^2}{8}\alpha+\frac{\alpha^2}{8}\overline{\alpha}+\frac{\alpha}{2}\frac{1}{2}\overline{\alpha}+\frac{1}{8}\overline{\alpha}\\
=\;&\frac{\alpha^2\left(\alpha+\overline{\alpha}\right)+2\alpha\overline{\alpha}+\overline{\alpha}}{8}=\frac{\alpha^2+\alpha\overline{\alpha}+\alpha\overline{\alpha}+\overline{\alpha}}{8}\\
=\;&\frac{\alpha\left(\alpha+\overline{\alpha}\right)+\alpha\overline{\alpha}+\overline{\alpha}}{8}=\frac{\left(\alpha+\overline{\alpha}\right)+\alpha\overline{\alpha}}{8}=\frac{1}{6}. \qquad (3.17)
\end{aligned}
$$

For $(i,j,k)=(2,2,1)$ we have:

$$
\begin{aligned}
[V(\tau)]_{2,2,1}=\;&\left[U\left(\tau,\alpha A_2\right)\right]_{2,2}\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\\
&+\left[U\left(\tau,\alpha A_2\right)\right]_{2,2}\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_1\\
&+\left[U\left(\tau,\alpha A_2\right)\right]_2\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_1\\
&+\left[U\left(\tau,\overline{\alpha}A_1\right)\right]_{2,2}\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1\\
=\;&\frac{\alpha^2}{2}\frac{1}{2}+\frac{\alpha^2}{2}\frac{\overline{\alpha}}{2}+\alpha\overline{\alpha}\frac{\overline{\alpha}}{2}+\frac{\overline{\alpha}^2}{2}\frac{\overline{\alpha}}{2}\\
=\;&\frac{\alpha^2+\overline{\alpha}\left(\alpha^2+\overline{\alpha}^2\right)+2\alpha\overline{\alpha}^2}{4}\\
=\;&\frac{\alpha\left(1-\overline{\alpha}\right)+\frac{1}{3}\overline{\alpha}+\frac{2}{3}\overline{\alpha}}{4}=\frac{\alpha-\frac{1}{3}+\overline{\alpha}}{4}=\frac{1}{6}. \qquad (3.18)
\end{aligned}
$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus let us calculate the case $(i,j,k)=(1,2,2)$. We have:

$$
\begin{aligned}
[V(\tau)]_{1,2,2}=\;&\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\alpha A_2\right)\right]_{2,2}\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_{2,2}\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\alpha A_2\right)\right]_2\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_{2,2}\\
=\;&\frac{\alpha}{2}\frac{\alpha^2}{2}+\frac{\alpha}{2}\frac{\overline{\alpha}^2}{2}+\frac{\alpha}{2}\alpha\overline{\alpha}+\frac{1}{2}\frac{\overline{\alpha}^2}{2}\\
=\;&\frac{\alpha\left(\alpha^2+\overline{\alpha}^2\right)+2\alpha^2\overline{\alpha}+\overline{\alpha}^2}{4}
\end{aligned}
$$

$$= \frac{\frac{1}{3}\alpha + \frac{2}{3}\alpha + \overline{\alpha}\,(1-\alpha)}{4} = \frac{(\alpha+\overline{\alpha}) - \alpha\overline{\alpha}}{4} = \frac{1}{6}. \quad (3.19)$$

For $(i,j,k) = (2,1,1)$ we have:

$$\begin{aligned}
[V(\tau)]_{2,1,1} &= [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1} \\
&\quad + [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_{1,1} \\
&\quad + [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_1 \\
&\quad + [U(\tau,\overline{\alpha} A_1)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_{1,1} \\
&= \alpha\frac{1}{8} + \alpha\frac{\overline{\alpha}^2}{8} + \alpha\frac{1}{2}\frac{\overline{\alpha}}{2} + \overline{\alpha}\frac{\overline{\alpha}^2}{8} \\
&= \frac{\alpha + \overline{\alpha}^2\,(\alpha+\overline{\alpha}) + 2\alpha\overline{\alpha}}{8} \\
&= \frac{\alpha + \overline{\alpha}\,(\alpha+\overline{\alpha}) + \alpha\overline{\alpha}}{4} = \frac{\alpha+\overline{\alpha}-\frac{1}{3}}{4} = \frac{1}{6}. \qquad (3.20)
\end{aligned}$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus let us calculate the case $(i,j,k) = (1,2,1)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,2,1} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha} A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha} A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1 \\
&= \frac{\alpha}{2}\alpha\frac{1}{2} + \frac{\alpha}{2}\alpha\frac{\overline{\alpha}}{2} + \frac{\alpha}{2}\overline{\alpha}\frac{\overline{\alpha}}{2} + \frac{1}{2}\overline{\alpha}\frac{\overline{\alpha}}{2} \\
&= \frac{\left(\alpha^2+\overline{\alpha}^2\right) + \alpha\overline{\alpha}\,(\alpha+\overline{\alpha})}{4} = \frac{1}{6}. \qquad (3.21)
\end{aligned}$$

For $(i,j,k) = (2,1,2)$ we have:

$$\begin{aligned}
[V(\tau)]_{2,1,2} &= [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha} A_2)]_2 \\
&= \alpha\frac{1}{2}\overline{\alpha} = \frac{1}{6}. \qquad (3.22)
\end{aligned}$$

Here we used the identities $\alpha+\overline{\alpha}=1$, $\alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

By combining formulas (3.15) - (3.22), we will obtain:

$$[V(\tau)]_{i,j,k}=\frac{1}{6}, \quad i,j,k=1,2. \tag{3.23}$$

From equality (3.5), taking into account formulas (3.9), (3.14) and (3.23), we will obtain:

$$\begin{aligned} V(\tau) &= I-\tau\sum_{i=1}^{2}A_i+\frac{1}{2}\tau^2\sum_{i,j=1}^{2}A_iA_j-\frac{1}{6}\tau^3\sum_{i,j,k=1}^{2}A_iA_jA_k+R_4^{(2)}(\tau) \\ &= I-\tau\sum_{i=1}^{2}A_i+\frac{1}{2}\tau^2\left(\sum_{i=1}^{2}A_i\right)^2-\frac{1}{6}\tau^3\left(\sum_{i=1}^{2}A_i\right)^3+R_4^{(2)}(\tau) \\ &= I-\tau A+\frac{1}{2}\tau^2A^2-\frac{1}{6}\tau^3A^3+R_4^{(2)}(\tau). \end{aligned} \tag{3.24}$$

According to formula (3.1) we have:

$$U(\tau,A)=I-\tau A+\frac{1}{2}\tau^2A^2-\frac{1}{6}\tau^3A^3+R_4(\tau,A). \tag{3.25}$$

According to condition *(b)* of the second inequality of the **Theorem 3.1**, for $R_4^{(2)}(\tau)$, the following estimate holds:

$$\|R_4(\tau,A)\varphi\|\le ce^{\omega\tau}\tau^4\|\varphi\|_{A^4}. \tag{3.26}$$

According to equalities (3.24) and (3.25):

$$U(\tau,A)-V(\tau)=R_4(\tau,A)-R_4^{(2)}(\tau).$$

From here, taking into account (3.2) and (3.26), we will obtain the following estimate:

$$\|[U(\tau,A)-V(\tau)]\varphi\|\le ce^{\omega_2\tau}\tau^4\|\varphi\|_{A^4}. \tag{3.27}$$

From equalities (2.2) and (3.3), taking into account inequalities (3.4) and (3.27), we will obtain:

$$\begin{aligned} \|u(t_k)-u_k(t_k)\| &= \left\|\left[U(t_k,A)-V^k(\tau)\right]\varphi\right\|=\left\|\left[U^k(\tau,A)-V^k(\tau)\right]\varphi\right\| \\ &= \left\|\left[\sum_{i=1}^{k}V^{k-i}(\tau)\left[U(\tau,A)-V(\tau)\right]U((i-1)\tau,A)\right]\varphi\right\| \\ &\le \sum_{i=1}^{k}\|V(\tau)\|^{k-i}\|[U(\tau,A)-V(\tau)]U((i-1)\tau,A)\varphi\| \end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{k} e^{\omega_1(k-i)\tau} c e^{\omega_2\tau}\tau^4 \left\| U\left((i-1)\tau, A\right)\varphi\right\|_{A^4} \\
&\le c e^{\omega_0 t_k}\tau^4 \sum_{i=1}^{k} \left\| U\left((i-1)\tau, A\right)\varphi\right\|_{A^4} \\
&\le k c e^{\omega_0 t_k}\tau^3 \sup_{s\in[o,t_k]} \left\| U\left(s, A\right)\varphi\right\|_{A^4} \\
&\le c e^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[o,t_k]} \left\| U\left(s, A\right)\varphi\right\|_{A^4} \qquad \blacksquare
\end{aligned}$$

4. Decomposition Scheme for Multidimensional Case

Let us consider multidimensional case of the problem (2.1). Let $A = A_1 + ... + A_m$ $(m \geq 2)$, where $A_i \quad (i = 1, .., m)$ are closed operators, densely defined in X. Together with problem (2.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\alpha}{2}A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\alpha}{2}A_i v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dv_k^{(m)}(t)}{dt} + \alpha A_m v_k^{(m)}(t) &= 0, \quad v_k^{(m)}(t_{k-1}) = v_k^{(m-1)}(t_k), \\
\frac{dv_k^{(2m-i)}(t)}{dt} + \frac{\alpha}{2}A_2 v_k^{(2m-i)}(t) &= 0, \quad v_k^{(2m-i)}(t_{k-1}) = v_k^{(2m-i-1)}(t_k), \\
i &= m-1, ..., 2, \\
\frac{dv_k^{(2m-1)}(t)}{dt} + \frac{1}{2}A_1 v_k^{(2m-1)}(t) &= 0, \quad v_k^{(2m-1)}(t_{k-1}) = v_k^{(2m-2)}(t_k), \\
\frac{dv_k^{(2m-2+i)}(t)}{dt} + \frac{\overline{\alpha}}{2}A_i v_k^{(2m-2+i)}(t) &= 0, \quad v_k^{(2m-2+i)}(t_{k-1}) = v_k^{(2m-3+i)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dv_k^{(3m-2)}(t)}{dt} + \overline{\alpha}A_m v_k^{(3m-2)}(t) &= 0, \quad v_k^{(3m-2)}(t_{k-1}) = v_k^{(3m-3)}(t_k), \\
\frac{dv_k^{(4m-2-i)}(t)}{dt} + \frac{\alpha}{2}A_i v_k^{(4m-2-i)}(t) &= 0, \quad v_k^{(4m-2-i)}(t_{k-1}) = v_k^{(4m-3-i)}(t_k), \\
i &= m-1, ..., 1,
\end{aligned}$$

where α is a same complex number as for two dimensional case. Suppose that the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j)$, $j = 1, ..., m$ generate strongly continuous semigroups.

$u_k(t)$, $k = 1, 2, ..,$ is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = v_k^{(4m-3)}(t). \tag{4.1}$$

We declare function $u_k(t)$ as an approximated solution of problem (2.1) on each interval $[t_{k-1}, t_k]$.

5. Estimate of Error of the Approximated Solution for Multidimensional Case

Natural powers (A^s, $s = 2, 3, 4$) of the operator $A = A_1 + ... + A_m$ are defined analogously as for two dimensional case.

Theorem 5.1. *Let the conditions of Theorem3.1 be satisfied, where $j = 1, ..., m$. Then estimation of Theorem3.1 is true, where $u_k(t_k)$ is defined by formula (4.1).*

Proof. From formula (4.1) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{5.1}$$

where

$$\begin{aligned} V(\tau) \;=\;& U\left(\tau, \frac{\alpha}{2}A_1\right)...U\left(\tau, \frac{\alpha}{2}A_{m-1}\right)U(\tau, \alpha A_m) \\ &\times U\left(\tau, \frac{\alpha}{2}A_{m-1}\right)...U\left(\tau, \frac{\alpha}{2}A_2\right)U\left(\tau, \frac{1}{2}A_1\right) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)...U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right)U(\tau, \overline{\alpha}A_m) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right)...U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right). \end{aligned}$$

Let us define $[T(\tau)]_i$, $[T(\tau)]_{i,j}$ and $[T(\tau)]_{i,j,k}$ $(i, j, k = 1, ..., m)$ analogously as for two dimensional case.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (3.1) from left to right in such a way that each residual term appears of the fourth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) \;=\;& I - \tau\sum_{i=1}^{m}[V(\tau)]_i A_i + \tau^2\sum_{i,j=1}^{m}[V(\tau)]_{i,j}A_iA_j \\ &-\tau^3\sum_{i,j,k=1}^{m}[V(\tau)]_{i,j,k}A_iA_jA_k + R_4^{(m)}(\tau). \end{aligned} \tag{5.2}$$

According to the first inequality of the condition *(b)* of the **Theorem 5.1**, for $R_4^{(m)}(\tau)$, the following estimate holds:

$$\left\| R_4^{(m)}(\tau)\varphi \right\| \leq ce^{\omega_0\tau}\tau^4 \|\varphi\|_{A^4}, \quad \varphi \in D\left(A^4\right), \tag{5.3}$$

where c and ω_0 are positive constants.

Let us compute coefficients $[V(\tau)]_i$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V(\tau)$ which are generated by operators $(-\gamma A_i)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V(\tau)]_i = 1, \qquad i = 1, ..., m.$$

Let us compute coefficients $[V(\tau)]_{i,j}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$ which are generated by operators $(-\gamma A_i)$and $(-\gamma A_j)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V(\tau)]_{i,j} = \left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U(\tau, \alpha A_{i_2})U\left(\tau, \frac{1}{2}A_{i_1}\right)U(\tau, \overline{\alpha}A_{i_2})U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j},$$

where (i_1, i_2) is a pair of i and j indices, arranged in an increasing order. According to the **Theorem 3.1** we have**:**

$$\left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U(\tau, \alpha A_{i_2})U\left(\tau, \frac{1}{2}A_{i_1}\right)U(\tau, \overline{\alpha}A_{i_2})U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j} = \frac{1}{2}.$$

So we have

$$[V(\tau)]_{i,j} = \frac{1}{2}, \qquad i, j = 1, 2, ..., m.$$

Let us compute coefficients $[V(\tau)]_{i,j,k}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V(\tau)$, which are generated by operators $(-\gamma A_i), (-\gamma A_j)$ and $(-\gamma A_k)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$\begin{aligned}[V(\tau)]_{i,j,k} &= U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)U(\tau, \alpha A_{i_3})U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)\\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_2}\right)U(\tau, \overline{\alpha}A_{i_3})U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_2}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right),\end{aligned}$$

where (i_1, i_2, i_3) is a triple of i, j and k indices, arranged in an increasing order.

First let us consider the case when $i = j = k$, we have:

$$[V(\tau)]_{i,j,k} = [U(\tau, A_i)]_{i,i,i} = \frac{1}{6}$$

Now let us consider the case when only two of i, j, k indices are different. In this case we have:

$$[V(\tau)]_{i,j,k} = \left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right) U(\tau, \alpha A_{i_2}) U\left(\tau, \frac{1}{2}A_{i_1}\right) U(\tau, \overline{\alpha} A_{i_2}) U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j,k},$$

where (i_1, i_2) is pair of different indices of i, j and k triple, arranged in an increasing order. According to the **Theorem 3.1** we have**:**

$$\left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right) U(\tau, \alpha A_{i_2}) U\left(\tau, \frac{1}{2}A_{i_1}\right) U(\tau, \overline{\alpha} A_{i_2}) U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j,k} = \frac{1}{6}.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \qquad i, j, k = 1, 2, ..., m.$$

Now let us consider the case when i, j, k indices are different. We have six variants. Let us consider each one separately:

Case 1. If $i < j < k$, then

$$\begin{aligned}
[V(\tau)]_{i,j,k} &= \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U(\tau, \alpha A_k)\right]_{i,j,k} \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U(\tau, \overline{\alpha} A_k)\right]_{i,j,k} \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U(\tau, \overline{\alpha} A_k)\right]_{i,j,k} \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) U(\tau, \overline{\alpha} A_k)\right]_{i,j,k} \\
&\quad + \left[U\left(\tau, \frac{1}{2}A_i\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) U(\tau, \overline{\alpha} A_k)\right]_{i,j,k} \\
&= \frac{\alpha}{2}\frac{\alpha}{2}(\alpha + \overline{\alpha}) + \frac{\alpha}{2}\left(\frac{\alpha}{2} + \frac{\overline{\alpha}}{2}\right)\overline{\alpha} + \frac{1}{2}\frac{\overline{\alpha}}{2}\overline{\alpha} \\
&= \frac{\alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2}{4} = \frac{1}{6}.
\end{aligned}$$

Case 2. If $i < k < j$, then

$$\begin{aligned}
[V(\tau)]_{i,j,k} &= \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U(\tau, \alpha A_j) U\left(\tau, \frac{\alpha}{2}A_k\right)\right]_{i,j,k} \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U(\tau, \alpha A_j) U\left(\tau, \frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}
\end{aligned}$$

$$\begin{aligned}
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\alpha A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\overline{\alpha} A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\frac{1}{2}A_i\right)U\left(\tau,\overline{\alpha} A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
=\ &\frac{\alpha}{2}\alpha\left(\frac{\alpha}{2}+\frac{\overline{\alpha}}{2}\right)+\frac{\alpha}{2}\left(\alpha+\overline{\alpha}\right)\frac{\overline{\alpha}}{2}+\frac{1}{2}\overline{\alpha}\frac{\overline{\alpha}}{2}\\
=\ &\frac{\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2}{4}=\frac{1}{6}.
\end{aligned}$$

Case 3. If $j<i<k$, then

$$\begin{aligned}
[V(\tau)]_{i,j,k} &= \left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\frac{1}{2}A_j\right)U\left(\tau,\overline{\alpha}A_k\right)\right]_{i,j,k}\\
&= \frac{\alpha}{2}\frac{1}{2}\overline{\alpha}=\frac{1}{6}.
\end{aligned}$$

Case 4. If $j<k<i$, then

$$\begin{aligned}
[V(\tau)]_{i,j,k} &= \left[U\left(\tau,\alpha A_i\right)U\left(\tau,\frac{1}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\alpha A_i\right)U\left(\tau,\frac{1}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&= \alpha\frac{1}{2}\frac{\overline{\alpha}}{2}+\alpha\frac{1}{2}\frac{\overline{\alpha}}{2}=\frac{1}{6}.
\end{aligned}$$

Case 5. If $k<i<j$, then

$$\begin{aligned}
[V(\tau)]_{i,j,k} &= \left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\alpha A_j\right)U\left(\tau,\frac{1}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\alpha A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\overline{\alpha} A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\overline{\alpha} A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_i\right)U\left(\tau,\overline{\alpha} A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k}\\
&= \frac{\alpha}{2}\alpha\frac{1}{2}+\frac{\alpha}{2}\left(\alpha+\overline{\alpha}\right)\frac{\overline{\alpha}}{2}+\left(\frac{\alpha}{2}+\frac{\overline{\alpha}}{2}\right)\overline{\alpha}\frac{\overline{\alpha}}{2}
\end{aligned}$$

$$= \frac{\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2}{4}=\frac{1}{6}.$$

Case 6. If $k<j<i$, then

$$\begin{aligned}
[V(\tau)]_{i,j,k} &= \left[U(\tau,\alpha A_i)\,U\left(\tau,\frac{\alpha}{2}A_j\right)U\left(\tau,\frac{1}{2}A_k\right)\right]_{i,j,k} \\
&+\left[U(\tau,\alpha A_i)\,U\left(\tau,\frac{\alpha}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k} \\
&+\left[U(\tau,\alpha A_i)\,U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k} \\
&+\left[U(\tau,\alpha A_i)\,U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k} \\
&+\left[U(\tau,\overline{\alpha} A_i)\,U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\right]_{i,j,k} \\
&= \alpha\frac{\alpha}{2}\frac{1}{2}+\alpha\left(\frac{\alpha}{2}+\frac{\overline{\alpha}}{2}\right)\frac{\overline{\alpha}}{2}+(\alpha+\overline{\alpha})\frac{\overline{\alpha}}{2}\frac{\overline{\alpha}}{2} \\
&= \frac{\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2}{4}=\frac{1}{6}.
\end{aligned}$$

Finally, for any triple (i,j,k) we have:

$$[V(\tau)]_{i,j,k}=\frac{1}{6}.$$

Inserting in (5.2) the obtained coefficients, we will get:

$$\begin{aligned}
V(\tau) &= I-\tau\sum_{i=1}^{m}A_i+\frac{1}{2}\tau^2\sum_{i,j=1}^{m}A_iA_j-\frac{1}{6}\tau^3\sum_{i,j,k=1}^{m}A_iA_jA_k+R_4^{(m)}(\tau) \\
&= I-\tau\sum_{i=1}^{m}A_i+\frac{1}{2}\tau^2\left(\sum_{i=1}^{m}A_i\right)^2-\frac{1}{6}\tau^3\left(\sum_{i=1}^{m}A_i\right)^3+R_4^{(m)}(\tau) \\
&= I-\tau A+\frac{1}{2}\tau^2A-\frac{1}{6}\tau^3A^3+R_4^{(m)}(\tau).
\end{aligned} \tag{5.4}$$

According to the second inequality of the condition b)) of the **Theorem 5.1** the following estimation is true for $R_4^{(m)}(\tau)$:

$$\left\|R_4^{(m)}(\tau)\,\varphi\right\|\le ce^{\omega\tau}\tau^4\,\|\varphi\|_{A^4}. \tag{5.5}$$

According to (5.4) and (3.25) we have:

$$U(\tau,A)-V(\tau)=R_4(\tau,A)-R_4^{(m)}(\tau).$$

Hence using (3.2) and (5.5) we can get the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \leq c e^{\omega_2 \tau} \tau^4 \|\varphi\|_{A^4}. \tag{5.6}$$

From equalities (2.2) and (5.1), taking into account inequalities (3.4) and (5.6), we will obtain sought estimation ■

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