DYNAMICAL HIERARCHICAL MODELS FOR ELASTIC SHELLS

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Abstract

In the present paper dynamical three-dimensional model for linearly elastic shells in curvilinear coordinates is considered and a hierarchy of two-dimensional models of the corresponding initial-boundary value problem is constructed. The well-posedness of the two-dimensional problems is investigated in suitable spaces and the accuracy of approximation of the solution to the original problem by the vector functions restored from the solutions of the reduced problems is estimated.

Key words and phrases: Initial boundary value problems for elastic shells, modelling error estimation.

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The two-dimensional models of elastic shells are widely used for constructing various engineering structures and hence investigation of dimensional reduction algorithms and justification of mathematical models of elastic shells is important both from practical and from theoretical point of view. In the paper [1] I. Vekua suggested a new method of constructing the two-dimensional models for prismatic shells in the theory of elasticity. In order to reduce the three-dimensional boundary or initial boundary value problems to two-dimensional ones, I. Vekua expands the components of displacement vector function, stress and strain tensors into orthogonal Fourier-Legendre series with respect to the plate thickness variable x_3 and multiplying the constitutive equations by Legendre polynomials and integrating them with respect to x_3 an infinite system with unknown vector function defined on two-dimensional domain is constructed. Then, considering only the first $N+1$ terms of the expansions and the corresponding $N+1$ equations, a sequence of two-dimensional models of linearly elastic prismatic shells was obtained. Note that I. Vekua considered the three-dimensional problems and reduction algorithms only in the spaces of classical smooth enough functions and did not investigate the relation of the constructed two-dimensional models to the original problem. These issues first were studied in the papers [2, 3]. More precisely, in the static case the two-dimensional boundary value problem obtained by I. Vekua was investigated in Sobolev spaces in [2] and the rate of convergence of the sequence of vector functions of three variables restored from the solutions of the reduced problems to the solution of the original problem in the spaces of classical regular functions was estimated in [3]. Later on, static and dynamical hierarchical models for prismatic shells constructed using Vekua's dimensional reduction method were systematically studied in Sobolev spaces in the papers [4-7].

It should be pointed out that the reduction method considered in [1] for prismatic shells and its generalizations are widely used for constructing lower-dimensional approximations of various problems in the theory of elasticity and mathematical physics [8-20]. Particularly, in [8, 11], under certain geometric assumptions, there were obtained two-dimensional models for thin shallow shells and further, applying Vekua's reduction method, were constructed two-dimensional models for linear nonshallow shells in [12] and for nonlinear nonshallow shells in [16]. Note that in the papers devoted to the construction of two-dimensional models of shells there are mainly considered such diffeomorphisms defining the shell, that their components are polynomials with respect to the thickness variable x_3. In the present paper we consider the dynamical three-dimensional model for linearly elastic shell in curvilinear coordinates defined by an arbitrary twice continuously differentiable diffeomorphism. Applying I. Vekua's idea to variational formulation of the corresponding initial boundary value problem, we construct a hierarchy of two-dimensional models for elastic shell. Moreover, we show that the vector functions of three variables restored from the solutions of reduced problems can be considered as approximations to the solution of the original three-dimensional problem and we estimate the rate of approximation.

Shell is an elastic body occupying a reference configuration of a specific shape, consisting of all points within a given distance from a certain surface. Consequently, when constructing two-dimensional models of elastic shells instead of Cartesian coordinates, it is more convenient to use curvilinear coordinates that follow the geometry of the shell in a most natural way. Thus, first let us consider the three-dimensional model of linearly elastic body in curvilinear coordinates with initial configuration $\overline{\Omega^*} = \boldsymbol{\theta}(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, $\boldsymbol{\theta}$ is a twice continuously differentiable diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega^*}$, such that the vectors $\boldsymbol{g}_i(x) = \partial\boldsymbol{\theta}(x)/\partial x_i$, $i = 1, 2, 3$ are linearly independent at all points $x = (x_i)_{i=1}^3 \in \overline{\Omega}$, where $\overline{\Omega}$ and $\overline{\Omega^*}$ denote the closures of the domains Ω and Ω^*, respectively. The coordinates x_i ($i = \overline{1,3}$) of x defined by $\boldsymbol{\theta}$ are called the curvilinear coordinates x^*. The triplets of vectors $\{\boldsymbol{g}_i(x)\}_{i=1}^3$ and

$\{\boldsymbol{g}^i(x)\}_{i=1}^3$ form the covariant and contravariant bases, respectively, at the point $x^* = \boldsymbol{\theta}(x)$, where $\boldsymbol{g}^j(x) \cdot \boldsymbol{g}_i(x) = \delta_{ij}$, for all $i, j = \overline{1,3}$, δ_{ij} designates Kronecker's symbol and $\boldsymbol{a} \cdot \boldsymbol{b}$ is the Euclidean scalar product of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. The mapping $\boldsymbol{\theta}$ defines the metric tensor of the domain $\overline{\Omega^*}$ with covariant $(g_{ij})_{i,j=1}^3$ and contravariant $(g^{ij})_{i,j=1}^3$ components and the Christoffel symbols Γ_{ij}^p which are given by

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \quad g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \quad \Gamma_{ij}^p = \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j, \quad i, j, p = 1, 2, 3.$$

We study boundary value problems for elastic shell applying variational methods in suitable functional spaces. For each real $s \geq 0$, we denote by $H^s(\Omega)$ and $H^s(\widetilde{\Gamma})$ Sobolev space of order s based on $H^0(\Omega) = L^2(\Omega)$ and $H^0(\widetilde{\Gamma}) = L^2(\widetilde{\Gamma})$, respectively, where $\widetilde{\Gamma}$ is an element of a Lipschitz dissection of the boundary $\Gamma = \partial\Omega$ [22]. The trace operator from $H^1(\Omega)$ into $H^{1/2}(\widetilde{\Gamma})$ we denote by $tr_{\widetilde{\Gamma}}$. $H_0^s(\Omega)$ denotes the closure of the set $\mathfrak{D}(\Omega)$ of infinitely differentiable functions with compact support in Ω in the space $H^s(\Omega)$. The spaces of vector-valued functions we denote by $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^3$, $\mathbf{H}_0^s(\Omega) = [H_0^s(\Omega)]^3$, $\mathbf{H}^s(\widetilde{\Gamma}) = [H^s(\widetilde{\Gamma})]^3$, $s \geq 0$. The generalized derivative with respect to the variable x_i we designate by ∂_i $(i = 1, 2, 3)$. For Banach space X, $C([0, T]; X)$ is the space of continuous vector functions on $[0, T]$ with values in X. $L^m(0, T; X)$, $m \geq 1$, denotes the space of measurable vector functions $h : (0, T) \to X$ such that $\|h\|_X \in L^m(0, T)$ and the generalized derivative of h we denote by $h' = dh/dt$.

Let us consider the linear dynamical three-dimensional model of elastic shell which consists of homogeneous and isotropic material with the Lamé constants λ, μ. Assume that the shell is clamped along the part $\boldsymbol{\theta}(\Gamma_0)$ of Lipschitz dissection of the boundary $\boldsymbol{\theta}(\partial\Omega)$, a surface force with density $\boldsymbol{\sigma}^* = (\sigma^{*i})_{i=1}^3$ is acting on the remaining part $\boldsymbol{\theta}(\Gamma_1)$, $\Gamma_1 = \partial\Omega \backslash \overline{\Gamma_0}$, of the boundary and the shell is subjected to applied body forces with density $\boldsymbol{f}^* = (f^{*i})_{i=1}^3$. The corresponding initial boundary value problem admits the following variational formulation [23], when it is expressed in terms of the curvilinear coordinates: Find a vector function $\boldsymbol{u} \in C([0, T]; \mathbf{V}(\Omega))$, $\boldsymbol{u}' \in C([0, T]; \mathbf{L}^2(\Omega))$, which satisfies the equation

$$\frac{d}{dt} \sum_{i,j=1}^{3} \int_{\Omega} \sqrt{g} g^{ij} u_j' v_i dx + \sum_{i,j,p,q=1}^{3} \int_{\Omega} A^{ijpq} e_{p||q}(\boldsymbol{u}) e_{i||j}(\boldsymbol{v}) \sqrt{g} dx =$$

$$= \sum_{i=1}^{3} \int_{\Omega} f^i \sqrt{g} v_i dx + \sum_{i=1}^{3} \int_{\Gamma_1} \sigma^i \sqrt{g} tr_{\Gamma_1}(v_i) d\Gamma, \qquad \forall \boldsymbol{v} \in \mathbf{V}(\Omega), \qquad (1)$$

in the sense of distributions in $(0, T)$, together with the following initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{\varphi}, \qquad \boldsymbol{u}'(0) = \boldsymbol{\psi}, \qquad (2)$$

where $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^3 \in \mathbf{V}(\Omega) = \{\boldsymbol{v} = (v_i)_{i=1}^3 \in \mathbf{H}^1(\Omega);\ tr_{\Gamma_0}(v_i) \equiv 0, i = \overline{1,3}\}$, $\boldsymbol{\psi} = (\psi_i)_{i=1}^3 \in \mathbf{L}^2(\Omega)$, $A^{ijpq} = \lambda g^{ij} g^{pq} + \mu(g^{ip}g^{jq} + g^{iq}g^{jp})$ are the contravariant components of the three-dimensional elasticity tensor, $e_{p||q}(\boldsymbol{v}) = 1/2(\partial_p v_q + \partial_q v_p) - \sum\limits_{i=1}^{3} \Gamma_{pq}^i v_i$ designate the covariant components of the linearized strain tensor in curvilinear coordinates, g is the determinant of the matrix $(g_{ij})_{i,j=1}^3$, f^i, σ^i are the contravariant components of the applied body and surface force densities, respectively, u_i are the covariant components of the displacement vector field $\sum\limits_{i=1}^{3} u_i \boldsymbol{g}^i$ $(i,j,p,q = \overline{1,3})$. The bilinear form with respect to $\boldsymbol{u}$ and $\boldsymbol{v}$ in the left-hand side of the equation (1) we denote by $A(\boldsymbol{u},\ \boldsymbol{v})$ and let $L(\boldsymbol{v})$ be the linear form in the right-hand side of (1).

For the formulated three-dimensional problem (1), (2) we have the following theorem.

Theorem 1. If the Lamé constants λ, μ are such that $\mu > 0$, $3\lambda + 2\mu > 0$ and $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma'} \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$, $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$, then the problem (1), (2) possesses a unique solution $\boldsymbol{u}$, which satisfies the energy equality

$$(\boldsymbol{G}\boldsymbol{u}'(t), \boldsymbol{u}'(t))_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{u}(t), \boldsymbol{u}(t)) = (G\boldsymbol{\psi}, \boldsymbol{\psi})_{\mathbf{L}^2(\Omega)} + \\ + A(\boldsymbol{\varphi},\ \boldsymbol{\varphi}) + 2\widetilde{L}(\boldsymbol{u})(t), \quad \forall t \in [0,T], \tag{3}$$

where $\boldsymbol{G} = (G_{ij})_{i,j=1}^3$ is 3×3 matrix with (i,j) element $G_{ij} = g^{ij}\sqrt{g}$, $i,j = \overline{1,3}$, and

$$\widetilde{L}(\boldsymbol{u})(t) = \int\limits_0^t (\sqrt{g}\boldsymbol{f}(\tau), \boldsymbol{u}'(\tau))_{\mathbf{L}^2(\Omega)} d\tau + (\sqrt{g}\boldsymbol{\sigma}(t), \boldsymbol{tr}_{\Gamma_1}(\boldsymbol{u}(t)))_{\mathbf{L}^2(\Gamma_1)} - \\ -(\sqrt{g}\boldsymbol{\sigma}(0), \boldsymbol{tr}_{\Gamma_1}(\boldsymbol{u}(0)))_{\mathbf{L}^2(\Gamma_1)} - \int\limits_0^t \left(\sqrt{g}\boldsymbol{\sigma}'(\tau), \boldsymbol{tr}_{\Gamma_1}(\boldsymbol{u}(\tau))\right)_{\mathbf{L}^2(\Gamma_1)} d\tau.$$

Note that the formulated existence and uniqueness theorem for the three-dimensional problem in curvilinear coordinates is a consequence of more general result for an abstract second order evolution problem. More precisely, let V and H be Hilbert spaces, V is dense in H and is continuously imbedded in it. The dual space of V we denote by V^* and H is identified with its dual space with respect to the scalar product, then we have $V \hookrightarrow H \hookrightarrow V^*$, with continuous and dense imbeddings. The duality relation between the spaces V^* and V we denote by $\langle .,. \rangle$.

Assume that a and l are bilinear forms on $H \times H$, b_1 is a bilinear form on $V \times V$ and b_2 is a bilinear form on $(V \times H) \cup (H \times V)$, which satisfy the following conditions for all $u_1, v_1 \in H$, $u_2, v_2 \in V$,

$$\begin{aligned} &a(u_1,v_1) = a(v_1,u_1),\ a(u_1,u_1) \geq \beta \left\| u_1 \right\|_H^2,\ \beta > 0, \\ &|a(u_1,v_1)| \leq c_a \left\| u_1 \right\|_H \left\| v_1 \right\|_H, |l(u_1,v_1)| \leq c_l \left\| u_1 \right\|_H \left\| v_1 \right\|_H, \end{aligned} \tag{4}$$

$$\begin{aligned} &b_1(u_2,v_2) = b_1(v_2,u_2),\ |b_1(u_2,v_2)| \leq c_{b_1} \|u_2\|_V \|v_2\|_V, \\ &b_1(u_2,u_2) \geq \beta_1 \|u_2\|_V^2 - \beta_2 \left\| u_2 \right\|_H^2,\ \beta_1 > 0,\ \ \beta_2 \in \mathbb{R}, \end{aligned} \tag{5}$$

$$|b_2(\tilde{u},\tilde{v})| \leq \begin{cases} c_{b_2} \|\tilde{u}\|_V \|\tilde{v}\|_H, & \forall \tilde{u} \in V, \tilde{v} \in H, \\ c_{b_2} \|\tilde{u}\|_H \|\tilde{v}\|_V, & \forall \tilde{u} \in H, \tilde{v} \in V. \end{cases} \tag{6}$$

Let us consider the following variational problem: Find a vector function $y \in C([0,T];V)$, $y' \in C([0,T];H)$, which satisfies the equation

$$\begin{aligned} &\frac{d}{dt} a(y'(.),v) + b_1(y(.),v) + b_2(y(.),v) + \\ &\qquad + l(y'(.),v) = (F(.),v)_H + \langle \widetilde{F}(.),v \rangle, \quad \forall v \in V, \end{aligned} \tag{7}$$

in the sense of distributions in $(0,T)$, together with the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1, \tag{8}$$

where $y_0 \in V,\ y_1 \in H,\ F \in L^2(0,T;H),\ \widetilde{F}, \widetilde{F}' \in L^2(0,T;V^*)$.

For the posed abstract problem the following theorem is true.

Theorem 2. If the conditions (4)-(6) are valid, then the problem (7), (8) possesses a unique solution y, which satisfies the energy equality

$$\begin{aligned} &a(y'(t),y'(t)) + b_1(y(t),y(t)) + 2\int_0^t b_2(y(\tau),y'(\tau))d\tau \\ &+2\int_0^t l(y'(\tau),y'(\tau))d\tau = a(y_1,y_1) + b_1(y_0,y_0) + 2\int_0^t (F(\tau),y'(\tau))_H d\tau \\ &+2\langle \widetilde{F}(t),y(t)\rangle - 2\langle \widetilde{F}(0),y_0\rangle - 2\int_0^t \langle \widetilde{F}'(\tau),y(\tau)\rangle d\tau, \qquad \forall t \in [0,T]. \end{aligned}$$

The existence result of Theorem 2 can be proved in a standard way applying Faedo-Galerkin's method [24], while the energy equality can be obtained through the usual regularization and limiting procedure.

In the present paper we consider the particular case of the body $\overline{\Omega^*}$, when the Lipschitz domain Ω is of the following form

$$\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3; \quad h^-(x_1,x_2) < x_3 < h^+(x_1,x_2), \quad (x_1,x_2) \in \omega\},$$

where $\omega \subset \mathbb{R}^2$ is a two-dimensional domain with boundary $\partial\omega$, h^+ and h^- are Lipschitz continuous on $\overline{\omega}$, i.e. $h^{\pm} \in Lip(\overline{\omega})$, $h^+(x_1, x_2) > h^-(x_1, x_2)$, for $(x_1, x_2) \in \omega \cup \gamma_1$, $\gamma_1 \subset \partial\omega$, and $h^+(x_1, x_2) = h^-(x_1, x_2)$, for $(x_1, x_2) \in \gamma_2 = \partial\omega \backslash \gamma_1$. For each diffeomorphism $\boldsymbol{\theta}$, the obtained body $\overline{\Omega^*}$ can be considered as an elastic shell with thickness vanishing on a part of the boundary, which coincides with well-known shell-type bodies, when $\boldsymbol{\theta}(x_1, x_2, x_3) = \boldsymbol{\theta}_1(x_1, x_2, x_3) + x_3\boldsymbol{\theta}_2(x_1, x_2, x_3)$, $(x_1, x_2, x_3) \in \overline{\Omega}$. The upper and lower faces of Ω, which are given by the equations $x_3 = h^+(x_1, x_2)$ and $x_3 = h^-(x_1, x_2)$, $(x_1, x_2) \in \omega$, we denote by Γ^+ and Γ^-, respectively, and the lateral face we denote by $\widetilde{\Gamma} = \partial\Omega \backslash (\overline{\Gamma^+ \cup \Gamma^-})$. We assume that the shell is clamped on the part $\boldsymbol{\theta}(\Gamma_0)$ of the lateral face, where $\Gamma_0 = \{(x_1, x_2, x_3) \in \widetilde{\Gamma}; (x_1, x_2) \in \gamma_0 \subset \gamma_1\}$, γ_0 is a Lipschitz curve.

Let us consider the subspaces $\mathbf{V_N}(\Omega)$ and $\mathbf{H_N}(\Omega)$, $\mathbf{N} = (N_1, N_2, N_3)$, of $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, which consist of vector functions the i-th $(1 \le i \le 3)$ components of which are polynomials of degree $N_i \ge 0$ with respect to the variable x_3, i.e. the subspaces $\mathbf{V_N}(\Omega)$ and $\mathbf{H_N}(\Omega)$ are given as follows

$$\mathbf{V_N}(\Omega) = \{\boldsymbol{v}_\mathbf{N} = (v_{\mathbf{N}i})_{i=1}^3 \in \mathbf{H}^1(\Omega);\ v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{v}_{\mathbf{N}i}\, P_{r_i}(z),$$
$$tr_{\Gamma_0}(v_{\mathbf{N}i}) \equiv 0, \quad \overset{r_i}{v}_{\mathbf{N}i} \in L^2(\omega), \quad 0 \le r_i \le N_i,\ i = \overline{1,3}\},$$

$$\mathbf{H_N}(\Omega) = \{\boldsymbol{v}_\mathbf{N} = (v_{\mathbf{N}i})_{i=1}^3 \in \mathbf{L}^2(\Omega);\ v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{v}_{\mathbf{N}i}\, P_{r_i}(z),$$
$$\overset{r_i}{v}_{\mathbf{N}i} \in L^2(\omega),\ 0 \le r_i \le N_i,\ i = \overline{1,3}\},$$

where $z = (x_3 - \bar{h})/h$, $h = (h^+ - h^-)/2$, $\bar{h} = (h^+ + h^-)/2$ and P_r is the Legendre polynomial of order $r \in \mathbb{N} \cup \{0\}$.

In order to construct dynamical two-dimensional models of elastic shell we consider the original three-dimensional problem (1), (2) on the subspaces $\mathbf{V_N}(\Omega)$ and $\mathbf{H_N}(\Omega)$: Find a vector function $\boldsymbol{w}_\mathbf{N} \in C([0,T]; \mathbf{V_N}(\Omega))$, $\boldsymbol{w}'_\mathbf{N} \in C([0,T]; \mathbf{H_N}(\Omega))$, which satisfies the equation

$$\frac{d}{dt}(\boldsymbol{G}\boldsymbol{w}'_\mathbf{N}(.), \boldsymbol{v}_\mathbf{N})_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{w}_\mathbf{N}(.), \boldsymbol{v}_\mathbf{N}) = L(\boldsymbol{v}_\mathbf{N}), \ \ \forall \boldsymbol{v}_\mathbf{N} \in \mathbf{V_N}(\Omega), \tag{9}$$

in the sense of distributions in $(0, T)$, together with the following initial conditions

$$\boldsymbol{w}_\mathbf{N}(0) = \boldsymbol{\varphi}_\mathbf{N}, \qquad \boldsymbol{w}'_\mathbf{N}(0) = \boldsymbol{\psi}_\mathbf{N}, \tag{10}$$

where $\boldsymbol{\varphi}_\mathbf{N} \in \mathbf{V_N}(\Omega)$ and $\boldsymbol{\psi}_\mathbf{N} \in \mathbf{H_N}(\Omega)$.

From the definition of the space $\mathbf{H_N}(\Omega)$, taking into account the orthogonality property of the Legendre polynomials, it follows that $\boldsymbol{v}_\mathbf{N} \in \mathbf{H_N}(\Omega)$

if and only if $h^{-1/2} \overset{r_i}{v}_{\mathbf{N}i} \in L^2(\omega)$, $r_i = \overline{0, N_i}$, $i = \overline{1,3}$. Since $h^{\pm} \in Lip(\overline{\omega})$ and h is positive in ω from Rademacher's theorem [25] we have that h^+, h^- and $1/h$ are differentiable almost everywhere in ω, $\partial_\alpha h^{\pm} \in L^\infty(\omega), \alpha = 1, 2$ and $\partial_\alpha(1/h) \in L^\infty(\widehat{\omega})$, for all subdomains $\widehat{\omega} \subset \omega$, $\overline{\partial\widehat{\omega} \cap \gamma_1} \subset \gamma_1$. For any vector function $\boldsymbol{v}_{\mathbf{N}} = (v_{\mathbf{N}i})_{i=1}^3 \in \mathbf{V}_{\mathbf{N}}(\Omega)$ the following equality is valid in the sense of distributions in ω

$$\partial_\alpha \overset{r_i}{v}_{\mathbf{N}i} = \int\limits_{h^-}^{h^+} (\partial_\alpha v_{\mathbf{N}i}) P_{r_i}(z) dx_3 + \int\limits_{h^-}^{h^+} v_{\mathbf{N}i}\, \partial_\alpha (P_{r_i}(z)) dx_3 + \\ + tr_{\Gamma^+}(v_{\mathbf{N}i}) \partial_\alpha h^+ - tr_{\Gamma^-}(v_{\mathbf{N}i}) \partial_\alpha h^- (-1)^{r_i}, \tag{11}$$

where $r_i = \overline{0, N_i}$, $i = \overline{1,3}$. The right-hand side of the equality (11) belongs to the space $L^2(\widehat{\omega})$, $\overline{\widehat{\omega}} \subset \omega$, and hence $\overset{r_i}{v}_{\mathbf{N}i} \in H^1(\widehat{\omega})$, i.e. $\overset{r_i}{v}_{\mathbf{N}i} \in H^1_{loc}(\omega)$, $r_i = \overline{0, N_i}$, $i = \overline{1,3}$. Applying the expressions for derivatives of the Legendre polynomials

$$\begin{aligned} P_r'(t) &= \sum_{s=0}^{r-1} \left(s + \frac{1}{2}\right) (1 - (-1)^{r+s}) P_s(t), \\ tP_r'(t) &= rP_r(t) + \sum_{s=0}^{r-1} \left(s + \frac{1}{2}\right) (1 + (-1)^{r+s}) P_s(t), \end{aligned} \qquad r \in \mathbb{N}, \tag{12}$$

we obtain that for any $\boldsymbol{v}_{\mathbf{N}} = (v_{\mathbf{N}i})_{i=1}^3 \in \mathbf{V}_{\mathbf{N}}(\Omega)$,

$$\partial_\alpha v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h} \left(r_i + \frac{1}{2}\right) P_{r_i}(z) \left(\partial_\alpha \overset{r_i}{v}_{\mathbf{N}i} - (r_i+1) \frac{\partial_\alpha h}{h} \overset{r_i}{v}_{\mathbf{N}i} - \right. \\ \left. - \sum_{s_i=r_i+1}^{N_i} \frac{\overset{s_i}{v}_{\mathbf{N}i}}{h} \left(s_i + \frac{1}{2}\right) (\partial_\alpha h^+ - (-1)^{r_i+s_i} \partial_\alpha h^-) \right), \tag{13}$$

$$\partial_3 v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h} \left(r_i + \frac{1}{2}\right) P_{r_i}(z) \sum_{s_i=r_i}^{N_i} \frac{1}{h} \left(s_i + \frac{1}{2}\right) (1 - (-1)^{r_i+s_i})\, \overset{s_i}{v}_{\mathbf{N}i}, \tag{14}$$

and using the orthogonality property of the Legendre polynomials we infer the explicit expression for the norm $\|.\|_*$ of $\vec{v}_{\mathbf{N}}$ in the space $[H^1_{loc}(\omega)]^{N_{1,2,3}}$, $N_{1,2,3} = N_1 + N_2 + N_3 + 3$, with components $\overset{r_i}{v}_{\mathbf{N}i}$, $r_i = \overline{0, N_i}$, $i = \overline{1,3}$, such that $\|\vec{v}_{\mathbf{N}}\|_* = \|\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)}$,

$$\|\vec{v}_{\mathbf{N}}\|_*^2 = \sum_{q=1}^{3}\Bigg(\sum_{r_q=0}^{N_q}\left(r_q+\frac{1}{2}\right)\left\|h^{-1/2}\overset{s_q}{v}_{\mathbf{N}q}\right\|^2_{L^2(\omega)} +$$
$$+\sum_{s_q=0}^{N_q}\left(s_q+\frac{1}{2}\right)\left\|\sum_{r_q=s_q}^{N_q}\left(r_q+\frac{1}{2}\right)(1-(-1)^{r_q+s_q})h^{-3/2}\overset{r_q}{v}_{\mathbf{N}q}\right\|^2_{L^2(\omega)} +$$
$$+\sum_{\alpha=1}^{2}\sum_{s_q=0}^{N_q}\left(s_q+\frac{1}{2}\right)\Big\|h^{-1/2}\partial_\alpha \overset{s_q}{v}_{\mathbf{N}q} -(s_q+1)h^{-3/2}\partial_\alpha h\,\overset{s_q}{v}_{\mathbf{N}q} -$$
$$-\sum_{r_q=s_q+1}^{N_q}\left(r_q+\frac{1}{2}\right)(\partial_\alpha h^+ -(-1)^{r_q+s_q}\partial_\alpha h^-)h^{-3/2}\overset{r_q}{v}_{\mathbf{N}q}\Big\|^2_{L^2(\omega)}\Bigg), \tag{15}$$

where we assume that the sum with lower limit greater than the upper one equals to zero. From (15) we step by step deduce that $h^{-3/2}\overset{r_i}{v}_{\mathbf{N}i}\in L^2(\omega)$ and $h^{-1/2}\partial_\alpha \overset{r_i}{v}_{\mathbf{N}i}\in L^2(\omega)$, $\alpha=1,2$, for $r_i=N_i$, N_i-1, N_i-2, ..., 1, $i=1,2,3$. Moreover, from (11) it follows that $\overset{0}{v}_{\mathbf{N}i}\in H^1(\omega)$ and thus $\overset{r_i}{v}_{\mathbf{N}i}\in H^1(\omega)$, for all $0\le r_i\le N_i$, $i=\overline{1,3}$.

So, the problem (9), (10) is equivalent to the following initial boundary value problem on two-dimensional space domain: Find the unknown vector function $\vec{w}_{\mathbf{N}}\in C([0,T];\vec{V}_{\mathbf{N}}(\omega))$, $\vec{w}'_{\mathbf{N}}\in C([0,T];\vec{H}_{\mathbf{N}}(\omega))$, which satisfies the equation

$$\frac{d}{dt}G_{\mathbf{N}}(\vec{w}'_{\mathbf{N}}(.),\vec{v}_{\mathbf{N}}) + A_{\mathbf{N}}(\vec{w}_{\mathbf{N}}(.),\vec{v}_{\mathbf{N}}) = L_{\mathbf{N}}(\vec{v}_{\mathbf{N}}), \qquad \forall\vec{v}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega), \tag{16}$$

in the sense of distributions in $(0,T)$, together with the initial conditions

$$\vec{w}_{\mathbf{N}}(0)=\vec{\varphi}_{\mathbf{N}}, \qquad \vec{w}'_{\mathbf{N}}(0)=\vec{\psi}_{\mathbf{N}}, \tag{17}$$

where the spaces $\vec{V}_{\mathbf{N}}(\omega)=\{\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})\in[H^1(\omega)]^{N_{1,2,3}};\ \|\vec{v}_{\mathbf{N}}\|_*<\infty,$ $tr_{\gamma_0}(\vec{v}_{\mathbf{N}})\equiv 0\}$ and $\vec{H}_{\mathbf{N}}(\omega)=\{\vec{v}_{\mathbf{N}}\in[L^2(\omega)]^{N_{1,2,3}};\ \|\vec{v}_{\mathbf{N}}\|^2_{\vec{H}_{\mathbf{N}}(\omega)}=\sum\limits_{i=1}^{3}\sum\limits_{r_i=0}^{N_i}$ $\|h^{-1/2}\overset{r_i}{v}_{\mathbf{N}i}\|^2_{L^2(\omega)}<\infty\}$ correspond to the spaces $\mathbf{V}_{\mathbf{N}}(\Omega)$ and $\mathbf{H}_{\mathbf{N}}(\Omega)$, respectively, the bilinear forms $G_{\mathbf{N}}(\vec{w}'_{\mathbf{N}}(.),\vec{v}_{\mathbf{N}})$ and $A_{\mathbf{N}}(\vec{w}_{\mathbf{N}}(.),\vec{v}_{\mathbf{N}})$ are the forms $(\boldsymbol{G}\boldsymbol{w}'_{\mathbf{N}},\boldsymbol{v}_{\mathbf{N}})_{\mathbf{L}^2(\Omega)}$ and $A(\boldsymbol{w}_{\mathbf{N}},\boldsymbol{v}_{\mathbf{N}})$ considered with respect to the components $\overset{r_i}{w}_{\mathbf{N}i}$ and $\overset{r_i}{v}_{\mathbf{N}i}$ of $\vec{w}_{\mathbf{N}}$ and $\vec{v}_{\mathbf{N}}$. The initial data $\vec{\varphi}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)$ and $\vec{\psi}_{\mathbf{N}}\in\vec{H}_{\mathbf{N}}(\omega)$ correspond to the initial data $\boldsymbol{\varphi}_{\mathbf{N}}$ and $\boldsymbol{\psi}_{\mathbf{N}}$, respectively, and the form $L_{\mathbf{N}}(.)$ can be expressed as follows

$$L_{\mathbf{N}}(\vec{v}_{\mathbf{N}}) = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\int\limits_{\omega}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{f^i_g}\overset{r_i}{v}_{\mathbf{N}i}\,d\omega + \sum_{i=1}^{3}\int\limits_{\omega}\left(\kappa_+\,\sigma_{g,i}|_{\Gamma^+}\right.$$
$$\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{v}_{\mathbf{N}i} + \kappa_-\,\sigma_{g,i}|_{\Gamma^-}\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)(-1)^{r_i}\,\left.\overset{r_i}{v}_{\mathbf{N}i}\right)d\omega +$$
$$+\sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\int\limits_{\widetilde{\gamma_0}}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{\sigma}_{g,i}\,tr_{\widetilde{\gamma_0}}(\overset{r_i}{v}_{\mathbf{N}i})d\gamma,$$

where $\widetilde{\gamma_0} = \gamma_1\backslash\overline{\gamma_0}$, $\kappa_\pm = (1+\sum\limits_{\alpha=1}^{2}(\partial_\alpha h^\pm)^2)^{1/2}$, $\boldsymbol{\sigma}_g = \boldsymbol{\sigma}\sqrt{g}$, $\vec{\sigma}_{g,\mathbf{N}} = (\overset{r_i}{\sigma}_{g,i})$,

$$\overset{r_i}{f^i_g} = \int\limits_{h^-}^{h^+} f^i\sqrt{g}P_{r_i}(z)dx_3,\quad \overset{r_i}{\sigma}_{g,i} = \int\limits_{h^-}^{h^+}\sigma_i\sqrt{g}P_{r_i}(z)dx_3,\quad r_i=\overline{0,N_i},\quad i=1,2,3.$$

Note that in particular case, when $\boldsymbol{\theta}(x) = \boldsymbol{\theta}_1(x_1,x_2) + x_3\boldsymbol{\theta}_2(x_1,x_2)$, it can be written the explicit expressions for the bilinear forms $G_{\mathbf{N}}(.,.)$ and $A_{\mathbf{N}}(.,.)$, which are obtained in the papers [12, 16]. For the two-dimensional problem (16), (17) we have the following theorem.

Theorem 3. If the Lame constants $\mu > 0$, $3\lambda + 2\mu > 0$ and $\vec{\varphi}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, $\vec{\psi}_{\mathbf{N}} \in \vec{H}_{\mathbf{N}}(\omega)$, $h^{-1/2}\,\overset{r_i}{f^i_g}\in L^2(0,T;L^2(\omega))$, $\kappa_\pm^{3/4}\boldsymbol{\sigma}_g|_{\Gamma^\pm}$, $\kappa_\pm^{3/4}(\boldsymbol{\sigma}_g|_{\Gamma^\pm})' \in L^2(0,T;\mathbf{L}^{4/3}(\omega))$, $h^{-1/4}\,\overset{r_i}{\sigma}_{g,i}$, $h^{-1/4}(\overset{r_i}{\sigma}_{g,i})' \in L^2(0,T;L^{4/3}(\widetilde{\gamma_0}))$, $r_i = \overline{0,N_i}$, $i=\overline{1,3}$, then the problem (16), (17) possesses a unique solution.

Proof. From the definition of the spaces $\vec{V}_{\mathbf{N}}(\omega)$ and $\vec{H}_{\mathbf{N}}(\omega)$ it follows, that $\vec{V}_{\mathbf{N}}(\omega)$ and $\vec{H}_{\mathbf{N}}(\omega)$ are Hilbert spaces, $\vec{V}_{\mathbf{N}}(\omega)$ is a subspace of $\vec{H}_{\mathbf{N}}(\omega)$ and since $[\mathfrak{D}(\omega)]^{N_{1,2,3}} \subset \vec{V}_{\mathbf{N}}(\omega)$ is dense in $\vec{H}_{\mathbf{N}}(\omega)$ we deduce that $\vec{V}_{\mathbf{N}}(\omega)$ is dense in $\vec{H}_{\mathbf{N}}(\omega)$. Moreover, applying imbedding and trace theorems for Sobolev spaces for any $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ we have that $\boldsymbol{tr}_\Gamma(\boldsymbol{v}) \in \mathbf{L}^4(\Gamma)$ and $\|\boldsymbol{tr}_\Gamma(\boldsymbol{v})\|_{\mathbf{L}^4(\Gamma)} \le c\,\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}$. Therefore, for each $\vec{v}_{\mathbf{N}} = (\overset{r_i}{v}_{\mathbf{N}i}) \in \vec{V}_{\mathbf{N}}(\omega)$ we have

$$\int\limits_{\omega}\kappa_\pm\left(\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{v}_{\mathbf{N}i}\,(\pm1)^{r_i}\right)^4 d\omega =$$
$$=\int\limits_{\Gamma^\pm}(tr_{\Gamma^\pm}(v_{\mathbf{N}i}))^4 d\Gamma^\pm \le c_1\|\boldsymbol{v}_{\mathbf{N}}\|^4_{\mathbf{H}^1(\Omega)} = c_1\|\vec{v}_{\mathbf{N}}\|^4_*,$$

$$\int\limits_{\widetilde{\gamma_0}} h^{-3}(tr_{\widetilde{\gamma_0}}(\overset{r_i}{v}_{\mathbf{N}i}))^4 d\widetilde{\gamma_0} \leq \int\limits_{\widetilde{\gamma_0}} h^{-3}\left(\int\limits_{h^-}^{h^+}(tr_{\widetilde{\Gamma_0}}(v_{\mathbf{N}i}))^2 dx_3 \frac{2h}{2r_i+1}\right)^2 d\widetilde{\gamma_0} \leq$$
$$\leq \frac{8}{(2r_i+1)^2}\int\limits_{\widetilde{\Gamma_0}}\left(tr_{\widetilde{\Gamma_0}}(v_{\mathbf{N}i})\right)^4 d\Gamma \leq c_2\|\boldsymbol{v}_{\mathbf{N}}\|^4_{\mathbf{H}^1(\Omega)} = c_2\|\vec{v}_{\mathbf{N}}\|^4_*,$$

and hence $\kappa_{\pm}^{1/4}\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{v}_{\mathbf{N}i}\ (\pm 1)^{r_i} \in L^4(\omega) = (L^{4/3}(\omega))^*$, $h^{-3/4}$ $tr_{\widetilde{\gamma_0}}(\overset{r_i}{v}_{\mathbf{N}i}) \in L^4(\widetilde{\gamma_0}) = (L^{4/3}(\widetilde{\gamma_0}))^*$, $r_i = \overline{0,N_i}$, $i = \overline{1,3}$. Consequently, we can use Theorem 2 with $V = \vec{V}_{\mathbf{N}}(\omega)$, $H = \vec{H}_{\mathbf{N}}(\omega)$ and suitable F, $\widetilde{F}$ defined by $\overset{r_i}{f^i_g}$, $\boldsymbol{\sigma}_g|_{\Gamma^\pm}$, $\overset{r_i}{\sigma}_{g,i}$ ($r_i = \overline{0,N_i}$, $i = \overline{1,3}$), and hence the problem (16), (17) has a unique solution. □

So, applying variational approach, we have constructed the hierarchy of dynamical two-dimensional models for elastic shell in curvilinear coordinates and have investigated the existence and uniqueness of solution of the corresponding initial boundary value problems in Sobolev spaces. In order to complete the justification of the obtained models it's necessary to study the relation of the two-dimensional problems (16), (17) and the original three-dimensional one. In the following theorem we formulate the approximation result, but before let us introduce the following anisotropic Sobolev spaces

$$\mathbf{H}^{1,1,s}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega);\ \partial_3^{k-1}\boldsymbol{v} \in \mathbf{H}^1(\Omega),\ \text{for } k = 2,3,...,s\},\quad s \in \mathbb{N}.$$

Note that $\mathbf{H}^{1,1,s}(\Omega)$ is a Hilbert space for each $s \in \mathbb{N}$. For simplicity of notes the norms in the spaces $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$ we denote by $\|.\|$ and $|.|$, respectively.

Theorem 4. If the Lamé constants satisfy the following conditions $\mu > 0$, $3\lambda + 2\mu > 0$, $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma'} \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$, $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$ and the vector functions of three variables $\boldsymbol{\varphi}_{\mathbf{N}}$ and $\boldsymbol{\psi}_{\mathbf{N}}$

$$\begin{aligned}\boldsymbol{\varphi}_{\mathbf{N}} &= (\varphi_{\mathbf{N}i})_{i=1}^3, \quad \varphi_{\mathbf{N}i} = \sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{\varphi}_{\mathbf{N}i}\, P_{r_i}(z),\\ \boldsymbol{\psi}_{\mathbf{N}} &= (\psi_{\mathbf{N}i})_{i=1}^3, \quad \psi_{\mathbf{N}i} = \sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{\psi}_{\mathbf{N}i}\, P_{r_i}(z),\end{aligned} \qquad i = 1,2,3,$$

corresponding to $\vec{\varphi}_{\mathbf{N}} = (\overset{r_i}{\varphi}_{\mathbf{N}i})$, $\vec{\psi}_{\mathbf{N}} = (\overset{r_i}{\psi}_{\mathbf{N}i})$ tend to $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, as $N = \min\limits_{1\leq i\leq 3}\{N_i\} \to \infty$, in the spaces $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, then the vector function

$$\boldsymbol{w}_{\mathbf{N}}(t) = (w_{\mathbf{N}i}(t))_{i=1}^{3}, \quad w_{\mathbf{N}i}(t) = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{w}_{\mathbf{N}i}(t) P_{r_i}(z), \quad i = \overline{1,3},$$

restored from the solution $\vec{w}_{\mathbf{N}}(t) = (\overset{r_i}{w}_{\mathbf{N}i}(t))$ of the two-dimensional problem (16), (17) tends to the solution $\boldsymbol{u}(t)$ of the three-dimensional problem (1), (2) in the space $\mathbf{V}(\Omega)$,

$$\begin{array}{ll} \boldsymbol{w}_{\mathbf{N}}(t) \to \boldsymbol{u}(t) & \text{in } \mathbf{V}(\Omega), \\ \boldsymbol{w}'_{\mathbf{N}}(t) \to \boldsymbol{u}'(t) & \text{in } \mathbf{L}^2(\Omega), \end{array} \quad \text{as} \quad N \to \infty, \quad \forall t \in [0,T].$$

Moreover, if $\boldsymbol{u}$ satisfies additional conditions $\boldsymbol{u} \in L^2(0,T;\mathbf{H}^{1,1,s_0}(\Omega))$, $\boldsymbol{u}' \in L^2(0,T;\mathbf{H}^{1,1,s_1}(\Omega))$, $\boldsymbol{u}'' \in L^2(0,T;\mathbf{H}^{1,1,s_2}(\Omega))$, $s_0 \geq s_1 \geq s_2 \geq 1$, $s_1 \geq 2$, then for appropriate initial data $\vec{\varphi}_{\mathbf{N}}$, $\vec{\psi}_{\mathbf{N}}$ the following estimate is valid

$$|\boldsymbol{u}'(t) - \boldsymbol{w}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{u}(t) - \boldsymbol{w}_{\mathbf{N}}(t)\|^2 \leq \frac{1}{N^{2s}} o(T,\mathbf{N}), \quad \forall t \in [0,T],$$

where $s = \min\{s_2, s_1 - 1\}$ and $o(T,\mathbf{N}) \to 0$ as $N \to \infty$.

Proof. First we obtain the estimate of modelling error given in the theorem. Since the solution $\boldsymbol{u}$ of the three-dimensional problem satisfies the equation (1) for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$ and $\mathbf{V}_{\mathbf{N}}(\Omega)$ is a subspace of $\mathbf{V}(\Omega)$, hence $\boldsymbol{u}$ satisfies it for all $\boldsymbol{v}_{\mathbf{N}} \in \mathbf{V}_{\mathbf{N}}(\Omega)$, i.e.

$$\frac{d}{dt}\left(\boldsymbol{G}\boldsymbol{u}', \boldsymbol{v}_{\mathbf{N}}\right)_{\mathbf{L}^2(\Omega)} + A\left(\boldsymbol{u}, \boldsymbol{v}_{\mathbf{N}}\right) = L(\boldsymbol{v}_{\mathbf{N}}), \qquad \forall \boldsymbol{v}_{\mathbf{N}} \in \mathbf{V}_{\mathbf{N}}(\Omega),$$

in the sense of distributions in $(0,T)$. Moreover, $\boldsymbol{w}_{\mathbf{N}}$ satisfies the equation (9) and from the latter equation we have

$$\frac{d}{dt}\left(\boldsymbol{G}(\boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}})', \boldsymbol{v}_{\mathbf{N}}\right)_{\mathbf{L}^2(\Omega)} + A\left(\boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}\right) = 0, \quad \forall \boldsymbol{v}_{\mathbf{N}} \in \mathbf{V}_{\mathbf{N}}(\Omega).$$

Suppose that the additional regularity conditions of the theorem are fulfilled, i.e. $d^p\boldsymbol{u}/dt^p \in L^2(0,T;\mathbf{H}^{1,1,s_p}(\Omega))$, $s_p \in \mathbb{N}$, $p = \overline{0,2}$, $s_0 \geq s_1 \geq s_2 \geq 1$, $s_1 \geq 2$. From the regularity theorems we obtain $\boldsymbol{u} \in C([0,T];\mathbf{H}^{1,1,s_1}(\Omega))$, $\boldsymbol{u}' \in C([0,T];\mathbf{H}^{1,1,s_2}(\Omega))$ [24] and hence $\boldsymbol{\varphi} \in \mathbf{H}^{1,1,s_1}(\Omega)$, $\boldsymbol{\psi} \in \mathbf{H}^{1,1,s_2}(\Omega)$. Let us consider the Fourier-Legendre expansion of the functions u_i with respect to the variable x_3. We denote by $u_{\mathbf{N}i}$ the following approximation of u_i, which is a polynomial of degree N_i with respect to the variable x_3

$$u_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{u_i} P_{r_i}(z) + \sum_{r_i=N_i}^{N_i+1} \frac{1}{2}\, \partial_3 \overset{r_i}{u_i}\, P_{r_i-1}(z),$$

where for any $v \in L^2(\Omega)$ we denote $\overset{r}{v}= \int\limits_{h^-}^{h^+} vP_r(z)dx_3$, $r \in \mathbb{N} \cup \{0\}$. The remainder term we denote by $\boldsymbol{\varepsilon}_{\mathbf{N}} = (\varepsilon_{\mathbf{N}i})_{i=1}^3$ and hence $u_i = u_{\mathbf{N}i} + \varepsilon_{\mathbf{N}i}$, $i = 1, 2, 3$. Let the components of the initial conditions $\vec{\varphi}_{\mathbf{N}} = (\overset{r_i}{\varphi}_{\mathbf{N}i})$ and $\vec{\psi}_{\mathbf{N}} = (\overset{r_i}{\psi}_{\mathbf{N}i})$ of the reduced problem (16), (17) be given as follows: for $i = 1, 2, 3$,

$$\overset{r_i}{\varphi}_{\mathbf{N}i}= \int\limits_{h^-}^{h^+} \varphi_i P_{r_i}(z)dx_3, \quad \overset{r_i}{\psi}_{\mathbf{N}i}= \int\limits_{h^-}^{h^+} \psi_i P_{r_i}(z)dx_3, \qquad 0 \le r_i \le N_i - 2,$$

$$\left.\begin{aligned} \overset{r_i}{\varphi}_{\mathbf{N}i}&= \int\limits_{h^-}^{h^+} \varphi_i P_{r_i}(z)dx_3 + \frac{h}{2r_i+1}\int\limits_{h^-}^{h^+} \partial_3\varphi_i P_{r_i+1}(z)dx_3,\\ \overset{r_i}{\psi}_{\mathbf{N}i}&= \int\limits_{h^-}^{h^+} \psi_i P_{r_i}(z)dx_3 + \frac{h}{2r_i+1}\int\limits_{h^-}^{h^+} \partial_3\psi_i P_{r_i+1}(z)dx_3, \end{aligned}\right. \qquad r_i = N_i - 1, N_i.$$

The conditions of the theorem imply that $\boldsymbol{u}_{\mathbf{N}} = (u_{\mathbf{N}i}) \in C([0,T]; \mathbf{V}_{\mathbf{N}}(\Omega))$, $\boldsymbol{u}'_{\mathbf{N}} \in C([0,T];\, \mathbf{H}_{\mathbf{N}}(\Omega))$. Indeed, since $\boldsymbol{u} \in C([0,T];\, \mathbf{H}^{1,1,s_1}(\Omega))$ we have $\partial_3\boldsymbol{u} \in C([0,T]; \mathbf{H}^{1,1,s_1-1}(\Omega))$ and from the condition $\boldsymbol{tr}(\boldsymbol{u}) \equiv \mathbf{0}$ on Γ_0 it follows that $tr_{\Gamma_0}(u_{\mathbf{N}i}) \equiv 0$ $(i = \overline{1,3})$. Applying the following formulas for the Legendre polynomials and their derivatives

$$\begin{aligned} P_r(t) &= \frac{1}{2r+1}(P'_{r+1}(t) - P'_{r-1}(t)), && r \ge 1,\\ tP'_r(t) &= P'_{r+1}(t) - (r+1)P_r(t), && r \ge 0, \end{aligned}$$

we obtain for almost all $(x_1, x_2) \in \omega$,

$$\overset{r}{u}_i(x_1,x_2) = \frac{h}{2r+1}\left(\overset{r-1}{\partial_3 u_i}(x_1,x_2) - \overset{r+1}{\partial_3 u_i}(x_1,x_2)\right), \qquad r \ge 1, \tag{18}$$

$$\partial_\alpha(\overset{(r)}{u_i}) = \overset{(r)}{\partial_\alpha u_i} + \frac{\partial_\alpha h}{h} r \overset{(r)}{u_i} + \partial_\alpha \bar{h}\, \overset{(r)}{\partial_3 u_i} + \frac{\partial_\alpha h}{h}\left(r+\frac{1}{2}\right)\overset{r+1}{\partial_3 u}, \; r \ge 0, \tag{19}$$

where $\alpha = 1, 2$, $i = \overline{1,3}$, $\overset{(r)}{v}= \frac{1}{h}\left(r+\frac{1}{2}\right)\overset{r}{v}$, $r \ge 0$, for all $v \in L^2(\Omega)$. Using the formulas (18), (19) and the expressions (12) for derivatives of the

Legendre polynomials, we obtain

$$\partial_3(u_{\mathbf{N}i}) = \sum_{r_i=0}^{N_i-1} \frac{1}{h}\left(r_i+\frac{1}{2}\right) \overset{r_i}{\partial_3 u_i}\, P_{r_i}(z),$$

$$\partial_\alpha(u_{\mathbf{N}i}) = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i+\frac{1}{2}\right) \overset{r_i}{\partial_\alpha u_i}\, P_{r_i}(z) + \frac{\partial_\alpha \bar{h}}{h}\left(N_i+\frac{1}{2}\right) \overset{N_i}{\partial_3 u_i}\, P_{N_i}(z) +$$
$$+ \sum_{r_i=N_i}^{N_i+1} \frac{\partial_\alpha h}{h} \overset{r_i}{\partial_3 u_i}\left(r_i+\frac{1}{2}\right) P_{r_i-1}(z) + \sum_{r_i=N_i}^{N_i+1} \frac{1}{2}\left(\partial_\alpha \overset{r_i}{\partial_3 u_i} + \right.$$
$$\left. + \partial_\alpha \bar{h}\, \partial_3 \overset{r_i}{\partial_3 u_i} + \partial_\alpha h\, \partial_3 \overset{r_i+1}{\partial_3 u_i}\right) P_{r_i-1}(z),$$

for $i=\overline{1,3}$, $\alpha=1,2$. Hence from the conditions $u_i, \partial_3 u_i \in C([0,T];H^1(\Omega))$ and $u_i' \in C([0,T];H^1(\Omega))$, it follows that $u_{\mathbf{N}i} \in C([0,T];\, H^1(\Omega))$, $u'_{\mathbf{N}i} \in C([0,T];\, L^2(\Omega))$, $i=\overline{1,3}$. Since $\boldsymbol{\varphi} \in \mathbf{H}^{1,1,s_1}(\Omega)$, $\boldsymbol{\psi} \in \mathbf{H}^{1,1,s_2}(\Omega)$ we similarly obtain that $\boldsymbol{\varphi}_{\mathbf{N}} = (\varphi_{\mathbf{N}i}) \in \mathbf{V}_{\mathbf{N}}(\Omega)$ and $\boldsymbol{\psi}_{\mathbf{N}} = (\psi_{\mathbf{N}i}) \in \mathbf{H}_{\mathbf{N}}(\Omega)$.

Consequently, the vector function $\boldsymbol{\Delta}_{\mathbf{N}} = \boldsymbol{u}_{\mathbf{N}} - \boldsymbol{w}_{\mathbf{N}} \in C([0,T];\mathbf{V}_{\mathbf{N}}(\Omega))$, $\boldsymbol{\Delta}'_{\mathbf{N}} \in C([0,T];\mathbf{H}_{\mathbf{N}}(\Omega))$, is a solution of the following problem

$$\begin{gathered}\frac{d}{dt}\left(\boldsymbol{G}\boldsymbol{\Delta}'_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}\right)_{\mathbf{L}^2(\Omega)} + A\left(\boldsymbol{\Delta}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}\right) = -\left((\boldsymbol{G}\boldsymbol{\varepsilon}''_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}})_{\mathbf{L}^2(\Omega)} + \right. \\ \left. + A(\boldsymbol{\varepsilon}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}})\right), \qquad \forall \boldsymbol{v}_{\mathbf{N}} \in \mathbf{V}_{\mathbf{N}}(\Omega), \\ \boldsymbol{\Delta}_{\mathbf{N}}(0) = \boldsymbol{u}_{\mathbf{N}}(0) - \boldsymbol{\varphi}_{\mathbf{N}} = \mathbf{0}, \quad \boldsymbol{\Delta}'_{\mathbf{N}}(0) = \boldsymbol{u}'_{\mathbf{N}}(0) - \boldsymbol{\psi}_{\mathbf{N}} = \mathbf{0}.\end{gathered} \tag{20}$$

Applying Theorem 2 to the problem (20), we obtain the corresponding energy equality

$$\left(\boldsymbol{G}\boldsymbol{\Delta}'_{\mathbf{N}}(t), \boldsymbol{\Delta}'_{\mathbf{N}}(t)\right)_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{\Delta}_{\mathbf{N}}(t), \boldsymbol{\Delta}_{\mathbf{N}}(t)) = -2A\left(\boldsymbol{\varepsilon}_{\mathbf{N}}(t), \boldsymbol{\Delta}_{\mathbf{N}}(t)\right) -$$
$$-2\int_0^t \left(\boldsymbol{G}\boldsymbol{\varepsilon}''_{\mathbf{N}}(\tau), \boldsymbol{\Delta}'_{\mathbf{N}}(\tau)\right)_{\mathbf{L}^2(\Omega)} d\tau + 2\int_0^t A\left(\boldsymbol{\varepsilon}'_{\mathbf{N}}(\tau), \boldsymbol{\Delta}_{\mathbf{N}}(\tau)\right) d\tau, \quad \forall t \in [0,T].$$

Since the bilinear form A is coercive on $\mathbf{V}(\Omega)$, the matrix $\boldsymbol{G}$ is symmetric and uniformly positive-definite, and $\mathbf{V}(\Omega)$ is continuously imbedded in $\mathbf{L}^2(\Omega)$, from the latter equality we have for all $t \in [0,T]$,

$$|\boldsymbol{\Delta}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{\Delta}_{\mathbf{N}}(t)\|^2 \le c_1 \left(\int_0^t \left(|\boldsymbol{\Delta}'_{\mathbf{N}}(\tau)|^2 + \|\boldsymbol{\Delta}_{\mathbf{N}}(\tau)\|^2\right) d\tau + \right.$$
$$\left. + \int_0^t \left|\boldsymbol{\varepsilon}''_{\mathbf{N}}(\tau)\right|^2 d\tau + \|\boldsymbol{\varepsilon}_{\mathbf{N}}(t)\|^2 + \int_0^t \|\boldsymbol{\varepsilon}'_{\mathbf{N}}(\tau)\|^2 d\tau\right), \quad c_1 = const > 0,$$

and by Gronwall's lemma we obtain that

$$|\boldsymbol{\Delta}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{\Delta}_{\mathbf{N}}(t)\|^2 \le c_2 \left(\int_0^t \left|\boldsymbol{\varepsilon}''_{\mathbf{N}}(\tau)\right|^2 d\tau + \right.$$

$$\left. + \|\boldsymbol{\varepsilon}_{\mathbf{N}}(t)\|^2 + \int_0^t \|\boldsymbol{\varepsilon}'_{\mathbf{N}}(\tau)\|^2 d\tau \right), \quad \forall t \in [0,T]. \quad (21)$$

Applying the orthogonality of the Legendre polynomials, the expressions for the functions $u_{\mathbf{N}i}$, $\partial_3(u_{\mathbf{N}i})$, $\partial_\alpha(u_{\mathbf{N}i})$ $(\alpha = 1,2)$ and the Parseval relation, we obtain

$$\|\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} = \sum_{r_i=N_i+1}^{\infty} \int_\omega \frac{1}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{u_i})^2 d\omega + \sum_{r_i=N_i-1}^{N_i} \int_\omega \frac{h}{2(2r_i+1)} (\overset{r_i+1}{\partial_3 u_i})^2 d\omega,$$

$$\|\partial_3(\varepsilon_{\mathbf{N}i})\|^2_{L^2(\Omega)} = \sum_{r_i=N_i}^{\infty} \int_\omega \frac{1}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{\partial_3 u_i})^2 d\omega,$$

$$\|\partial_\alpha(\varepsilon_{\mathbf{N}i})\|^2_{L^2(\Omega)} \le 3 \left(\sum_{r_i=N_i+1}^{\infty} \int_\omega \frac{1}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{\partial_\alpha u_i})^2 d\omega + \right.$$

$$+ \sum_{r_i=N_i-1}^{N_i} \int_\omega \frac{h}{2(2r_i+1)} \left(\partial_\alpha \overset{r_i+1}{\partial_3 u_i} + \partial_\alpha \bar{h} \overset{r_i+1}{\partial_3 \partial_3 u_i} + \partial_\alpha h \overset{r_i+2}{\partial_3 \partial_3 u_i} \right)^2 d\omega +$$

$$\left. + \int_\omega \frac{(\partial_\alpha \bar{h})^2}{h}\left(N_i + \frac{1}{2}\right)(\overset{N_i}{\partial_3 u_i})^2 d\omega + \sum_{r_i=N_i}^{N_i+1} \int_\omega \frac{(\partial_\alpha h)^2}{h} \frac{(2r_i+1)^2}{2(2r_i-1)} (\overset{r_i}{\partial_3 u_i})^2 d\omega \right),$$

for $\alpha = 1,2$, $i = \overline{1,3}$. Using (18) we get that for any vector function $\boldsymbol{v} = (v_i) \in \mathbf{H}^{1,1,n}(\Omega)$, $n \in \mathbb{N}$,

$$\int_\omega \left(\int_{h^-}^{h^+} \partial_3^{\alpha_1} \partial_k^{\alpha_2} v_i P_r(z) dx_3 \right)^2 d\omega \le$$

$$\le \frac{c_3}{r^{2\widehat{n}}} \sum_{l=r-\widehat{n}}^{r+\widehat{n}} \int_\omega h^{2\widehat{n}} \left(\int_{h^-}^{h^+} \partial_3^{n-\alpha_2} \partial_k^{\alpha_2} v_i P_l(z) dx_3 \right)^2 d\omega,$$

where $r \ge \widehat{n} = n - \alpha_1 - \alpha_2$, $\alpha_1, \alpha_2 = 0,1$, $k,i = 1,2,3$, and $c_6 = const > 0$ is independent of h^+, h^- and r.

From the latter estimate it follows that

$$\int_0^T \|\varepsilon_{\mathbf{N}^i}{}''(\tau)\|^2_{L^2(\Omega)} d\tau \le \frac{1}{N_i^{2s_2}} o(T, N_i),$$
$$\int_0^T \|\varepsilon_{\mathbf{N}^i}{}'(\tau)\|^2_{H^1(\Omega)} d\tau \le \frac{1}{N_i^{2(s_1-1)}} o(T, N_i),$$

where $o(T, N_i) \to 0$ as $N_i \to \infty$, $i = 1, 2, 3$.

Since $\boldsymbol{u} \in C([0,T]; \mathbf{H}^{1,1,s_1}(\Omega))$, $\boldsymbol{u}' \in C([0,T]; \mathbf{H}^{1,1,s_2}(\Omega))$ we have

$$\left.\begin{aligned} &\|\varepsilon_{\mathbf{N}^i}(t)\|^2_{H^1(\Omega)} \le \frac{1}{N_i^{2(s_1-1)}} o(T, N_i), \\ &\|\varepsilon_{\mathbf{N}^i}{}'(t)\|^2_{L^2(\Omega)} \le \frac{1}{N_i^{2s_2}} o(T, N_i), \end{aligned}\right. \qquad \forall t \in [0,T], \tag{22}$$

where $o(T, N_i) \to 0$ as $N_i \to \infty$, $i = 1, 2, 3$.

So, the inequality (21) implies that

$$|\boldsymbol{u}'(t) - \boldsymbol{w}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{u}(t) - \boldsymbol{w}_{\mathbf{N}}(t)\|^2 \le 2 \left(|\boldsymbol{\varepsilon}'_{\mathbf{N}}(t)|^2 + |\boldsymbol{\Delta}'_{\mathbf{N}}(t)|^2 + \right.$$
$$\left. + \|\boldsymbol{\varepsilon}_{\mathbf{N}}(t)\|^2 + \|\boldsymbol{\Delta}_{\mathbf{N}}(t)\|^2\right) \le \frac{1}{N^{2s}} o(T, \mathbf{N}), \qquad \forall t \in [0,T],$$

where $o(T, \mathbf{N}) \to 0$ as $N = \min\limits_{1\le k\le 3}\{N_k\} \to \infty$, and $s = \min\{s_2, s_1 - 1\}$.

Now let us prove the convergence result formulated in the theorem. The conditions of the theorem ensure the validity of the conditions of Theorem 3. Hence for each $\mathbf{N} \in [\mathbb{N} \cup \{0\}]^3$ the two-dimensional problem (16), (17) possesses a unique solution $\vec{w}_{\mathbf{N}}(t)$ and it satisfies the corresponding energy equality, which can be rewritten as follows:

$$\left(\boldsymbol{G}\boldsymbol{w}'_{\mathbf{N}}(t), \boldsymbol{w}'_{\mathbf{N}}(t)\right)_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{w}_{\mathbf{N}}(t), \boldsymbol{w}_{\mathbf{N}}(t)) =$$
$$= (\boldsymbol{G}\boldsymbol{\psi}_{\mathbf{N}}, \boldsymbol{\psi}_{\mathbf{N}})_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{\varphi}_{\mathbf{N}}, \boldsymbol{\varphi}_{\mathbf{N}}) + 2\widetilde{L}(\boldsymbol{w}_{\mathbf{N}})(t), \quad \forall t \in [0,T]. \tag{23}$$

As $\boldsymbol{\varphi}_{\mathbf{N}} \to \boldsymbol{\varphi}$ in $\mathbf{V}(\Omega)$ and $\boldsymbol{\psi}_{\mathbf{N}} \to \boldsymbol{\psi}$ in $\mathbf{L}^2(\Omega)$, from (23), applying Gronwall's lemma, we obtain that for all $t \in [0,T]$,

$$|\boldsymbol{w}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{w}_{\mathbf{N}}(t)\|^2 \le c_4 \left(|\boldsymbol{\psi}|^2 + \|\boldsymbol{\varphi}\|^2 + \int_0^T |\boldsymbol{f}(\tau)|^2 d\tau + \right.$$
$$\left. +2 \max_{0\le t\le T} \|\boldsymbol{\sigma}(t)\|^2_{[L^{4/3}(\Gamma_1)]^3} + \int_0^T \left\|\boldsymbol{\sigma}'(\tau)\right\|^2_{[L^{4/3}(\Gamma_1)]^3} d\tau \right). \tag{24}$$

Hence the sequence $\{\boldsymbol{w}_{\mathbf{N}}\}$ belongs to the bounded set of $L^\infty(0,T;\mathbf{V}(\Omega)) \cap L^2(0,T;\mathbf{V}(\Omega))$, while $\{\boldsymbol{w}'_{\mathbf{N}}(t)\}$ belongs to the bounded set of the space $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2(\Omega))$. Consequently, there exists a subsequence $\{\boldsymbol{w}_{\boldsymbol{\nu}}\}$, $\boldsymbol{\nu}=(\nu_k)$, of $\{\boldsymbol{w}_{\mathbf{N}}\}$ such that

$$\begin{array}{ll} \boldsymbol{w}_{\boldsymbol{\nu}} \to \widetilde{\boldsymbol{u}} & \begin{array}{l}\text{weakly in } L^2(0,T;\mathbf{V}(\Omega)),\\ \text{weakly-}* \text{ in } L^\infty(0,T;\mathbf{V}(\Omega)),\end{array} \\ \boldsymbol{w}'_{\boldsymbol{\nu}} \to \widetilde{\boldsymbol{u}}' & \begin{array}{l}\text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)),\\ \text{weakly-}* \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\end{array} \end{array} \quad \text{as} \min_{1\le k\le 3}\{\nu_k\}\to\infty. \tag{25}$$

Let us prove that $\widetilde{\boldsymbol{u}}$ is a solution of the problem (1), (2). Note that any vector function $\boldsymbol{v} \in \mathbf{V}(\Omega)$ can be extended on domain $\widehat{\Omega}$, $\Omega \subset \widehat{\Omega}$, $\partial\widehat{\Omega} \cap \partial\Omega = \Gamma_0$ and the mentioned extension belongs to the space $\mathbf{H}^1_0(\widehat{\Omega})$ [22]. Since $[\mathfrak{D}(\widehat{\Omega})]^3$ is dense in $\mathbf{H}^1_0(\widehat{\Omega})$, we deduce that the set $[C^\infty(\overline{\Omega})]^3 \cap \mathbf{V}(\Omega)$ is dense in $\mathbf{V}(\Omega)$. Applying the estimates (22), we obtain that for any vector function $\boldsymbol{v} \in [C^\infty(\overline{\Omega})]^3 \cap \mathbf{V}(\Omega)$ there exists a sequence of vector functions from $\mathbf{V}_{\mathbf{N}}(\Omega)$, which converges to $\boldsymbol{v}$. Consequently, the union $\bigcup\limits_{\mathbf{N}\ge\mathbf{0}} \mathbf{V}_{\mathbf{N}}(\Omega)$ of $\mathbf{V}_{\mathbf{N}}(\Omega)$ for all $\mathbf{N} \in [\mathbb{N}]^3$ is dense in $[C^\infty(\overline{\Omega})]^3 \cap \mathbf{V}(\Omega)$ and hence in $\mathbf{V}(\Omega)$. So, for any $\boldsymbol{v}=(v_i) \in \mathbf{V}(\Omega)$ there exists $\boldsymbol{v}_{\mathbf{N}}=(v_{\mathbf{N}i}) \in \mathbf{V}_{\mathbf{N}}(\Omega)$ such that $\boldsymbol{v}_{\mathbf{N}} \to \boldsymbol{v}$ in $\mathbf{V}(\Omega)$ as $N\to\infty$. Then for any scalar function $\zeta \in \mathfrak{D}(0,T)$ and $\boldsymbol{\zeta}=\zeta\boldsymbol{v}$, $\boldsymbol{\zeta}_{\mathbf{N}}=\zeta\boldsymbol{v}_{\mathbf{N}}$ we have

$$\begin{array}{ll} \boldsymbol{\zeta}_{\mathbf{N}} \to \boldsymbol{\zeta} & \text{in } L^2(0,T;\mathbf{V}(\Omega)),\\ \boldsymbol{\zeta}'_{\mathbf{N}} \to \boldsymbol{\zeta}' & \text{in } L^2(0,T;\mathbf{V}(\Omega)),\end{array} \quad \text{as} \min_{1\le k\le 3}\{N_k\}\to\infty. \tag{26}$$

Using the sequence $\{\boldsymbol{\zeta}_{\mathbf{N}}\}$, from (9), (25) and (26) we obtain that $\widetilde{\boldsymbol{u}}$ satisfies the equation (1) and $\widetilde{\boldsymbol{u}} \in L^\infty(0,T;\mathbf{V}(\Omega))$, $\widetilde{\boldsymbol{u}}' \in L^\infty(0,T;\mathbf{L}^2(\Omega))$.

Therefore, $\widetilde{\boldsymbol{u}} \in C([0,T];\mathbf{L}^2(\Omega)) \cap L^\infty(0,T;\mathbf{V}(\Omega))$, $\widetilde{\boldsymbol{u}}' \in C([0,T];\mathbf{V}^*(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ ($\mathbf{V}^*(\Omega)$ denotes the dual space of $\mathbf{V}(\Omega)$). Consequently, $\widetilde{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{u}}'$ are scalarly continuous from $[0,T]$ to the spaces $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$ [24], respectively. Note that $\widetilde{\boldsymbol{u}}$ satisfies the energy equality (3), from which we obtain the continuity of $\widetilde{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{u}}'$, $\widetilde{\boldsymbol{u}} \in C([0,T];\mathbf{V}(\Omega))$, $\widetilde{\boldsymbol{u}}' \in C([0,T];\mathbf{L}^2(\Omega))$ and, as $\boldsymbol{\varphi}_{\mathbf{N}} \to \boldsymbol{\varphi}$ in $\mathbf{V}(\Omega)$, $\boldsymbol{\psi}_{\mathbf{N}} \to \boldsymbol{\psi}$ in $\mathbf{L}^2(\Omega)$, the vector function $\widetilde{\boldsymbol{u}}$ satisfies the initial conditions (2). Since the problem (1), (2) has a unique solution we have $\widetilde{\boldsymbol{u}}=\boldsymbol{u}$ and the sequence $\{\boldsymbol{w}_{\mathbf{N}}\}$ possesses the properties (25).

Let us prove that $\{\boldsymbol{w}_{\mathbf{N}}\}$ possesses the convergence properties stated in the theorem. From the energy equalities (3) and (23) for $\boldsymbol{u}(t)$ and $\boldsymbol{w}_{\mathbf{N}}(t)$ we obtain the following equality for their difference $\boldsymbol{d}_{\mathbf{N}}(t)=\boldsymbol{u}(t)-\boldsymbol{w}_{\mathbf{N}}(t)$:

$$\big(\boldsymbol{G}\boldsymbol{d}'_{\mathbf{N}}(t),\boldsymbol{d}'_{\mathbf{N}}(t)\big)_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{d}_{\mathbf{N}}(t),\boldsymbol{d}_{\mathbf{N}}(t)) = \big(\boldsymbol{G}\boldsymbol{d}'_{\mathbf{N}}(0),\boldsymbol{d}'_{\mathbf{N}}(0)\big)_{\mathbf{L}^2(\Omega)} +$$

$$+A(\boldsymbol{d}_{\mathbf{N}}(0), \boldsymbol{d}_{\mathbf{N}}(0)) + 2\widetilde{L}(\boldsymbol{d}_{\mathbf{N}})(t) + 2J_{\mathbf{N}}(t), \quad \forall t \in [0, T],$$

where

$$J_{\mathbf{N}}(t) = (\boldsymbol{G}\boldsymbol{\psi}, \boldsymbol{\psi}_{\mathbf{N}})_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{N}}) - A(\boldsymbol{u}(t), \boldsymbol{w}_{\mathbf{N}}(t)) - \\ - \big(\boldsymbol{G}\boldsymbol{u}'(t), \boldsymbol{w}'_{\mathbf{N}}(t)\big)_{\mathbf{L}^2(\Omega)} + 2\widetilde{L}(\boldsymbol{w}_{\mathbf{N}})(t).$$

From (24) it follows that for each $t \in (0, T]$ there exists a subsequence $\{\boldsymbol{w}_{\bar{\boldsymbol{\nu}}}(t)\}$ such that

$$\begin{array}{ll} \boldsymbol{w}_{\bar{\boldsymbol{\nu}}}(t) \to \boldsymbol{\chi}_1 & \text{weakly in } \mathbf{V}(\Omega), \\ \boldsymbol{w}'_{\bar{\boldsymbol{\nu}}}(t) \to \boldsymbol{\chi}_2 & \text{weakly in } \mathbf{L}^2(\Omega), \end{array} \quad \text{as } \min_{1 \le k \le 3}\{\bar{\nu}_k\} \to \infty. \tag{27}$$

Let us take $\bar{\zeta} \in C^1([0, T])$, $\bar{\zeta}(0) = 0$, $\bar{\zeta}(t) \neq 0$ and consider the vector functions $\bar{\boldsymbol{\zeta}} = \bar{\zeta}\mathbf{v}$, $\bar{\boldsymbol{\zeta}}_{\mathbf{N}} = \bar{\zeta}\mathbf{v}_{\mathbf{N}}$, which also possess the properties (26). Using the formula for integration by parts we have

$$\int_0^t (\boldsymbol{w}'_{\bar{\boldsymbol{\nu}}}(\tau), \bar{\boldsymbol{\zeta}}_{\bar{\boldsymbol{\nu}}}(\tau))_{\mathbf{L}^2(\Omega)} d\tau = (\boldsymbol{w}_{\bar{\boldsymbol{\nu}}}(t), \bar{\boldsymbol{\zeta}}_{\bar{\boldsymbol{\nu}}}(t))_{\mathbf{L}^2(\Omega)} - \int_0^t (\boldsymbol{w}_{\bar{\boldsymbol{\nu}}}(\tau), \bar{\boldsymbol{\zeta}}'_{\bar{\boldsymbol{\nu}}}(\tau))_{\mathbf{L}^2(\Omega)} d\tau.$$

By (25)-(27), proceeding to the limit in the latter equality as $\min_{1 \le k \le 3}\{\bar{\nu}_k\} \to \infty$ we obtain

$$\int_0^t (\boldsymbol{u}'(\tau), \bar{\boldsymbol{\zeta}}(\tau))_{\mathbf{L}^2(\Omega)} d\tau = (\boldsymbol{\chi}_1, \bar{\boldsymbol{\zeta}}(t))_{\mathbf{L}^2(\Omega)} - \int_0^t (\boldsymbol{u}(\tau), \bar{\boldsymbol{\zeta}}'(\tau))_{\mathbf{L}^2(\Omega)} d\tau.$$

Since

$$\int_0^t (\boldsymbol{u}'(\tau), \bar{\boldsymbol{\zeta}}(\tau))_{\mathbf{L}^2(\Omega)} d\tau = (\boldsymbol{u}(t), \bar{\boldsymbol{\zeta}}(t))_{\mathbf{L}^2(\Omega)} - \int_0^t (\boldsymbol{u}(\tau), \bar{\boldsymbol{\zeta}}'(\tau))_{\mathbf{L}^2(\Omega)} d\tau,$$

we have that $(\boldsymbol{u}(t), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} = (\boldsymbol{\chi}_1, \boldsymbol{v})_{\mathbf{L}^2(\Omega)}$, for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, and the density of $\mathbf{V}(\Omega)$ in $\mathbf{L}^2(\Omega)$ implies $\boldsymbol{\chi}_1 = \boldsymbol{u}(t)$. Hence, as $\boldsymbol{\chi}_1$ is unique and equals to $\boldsymbol{u}(t)$ the sequence $\{\boldsymbol{w}_{\mathbf{N}}(t)\}$ converges to $\boldsymbol{u}(t)$ weakly in $\mathbf{V}(\Omega)$ as $\min_{1 \le i \le 3}\{N_i\} \to \infty$, for all $t \in (0, T]$.

In addition, as $\boldsymbol{w}_{\mathbf{N}}$ satisfies the equation (9), using (25), (27) together with the properties of the sequence $\{\bar{\boldsymbol{\zeta}}_{\bar{\boldsymbol{\nu}}}\}$ we obtain

$$(\boldsymbol{G}\boldsymbol{u}'(t), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} = (\boldsymbol{G}\boldsymbol{\chi}_2, \boldsymbol{v})_{\mathbf{L}^2(\Omega)}, \qquad \forall \boldsymbol{v} \in \mathbf{V}(\Omega).$$

From the latter it follows that $\boldsymbol{\chi}_2 = \boldsymbol{u}'(t)$ having density of $\mathbf{V}(\Omega)$ in $\mathbf{L}^2(\Omega)$ and positive-definiteness of the matrix $\boldsymbol{G}$. So the sequences $\{\boldsymbol{w}_{\mathbf{N}}(t)\}$ and $\{\boldsymbol{w}'_{\mathbf{N}}(t)\}$ possess the following properties

$$\begin{array}{ll} \boldsymbol{w}_{\mathbf{N}}(t) \to \boldsymbol{u}(t) & \text{weakly in } \mathbf{V}(\Omega), \\ \boldsymbol{w}'_{\mathbf{N}}(t) \to \boldsymbol{u}'(t) & \text{weakly in } \mathbf{L}^2(\Omega), \end{array} \qquad \text{as } \min_{1\le k\le 3}\{N_k\} \to \infty. \tag{28}$$

Therefore, proceeding to the limit in $J_{\mathbf{N}}(t)$ as $\min\limits_{1\le k\le 3}\{N_k\} \to \infty$ and using the energy equality (3), we obtain

$$\begin{aligned} J_{\mathbf{N}}(t) \to (\boldsymbol{G}\boldsymbol{\psi}, \boldsymbol{\psi})_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + 2\widetilde{L}(\boldsymbol{u})(t) - \\ - \big(\boldsymbol{G}\boldsymbol{u}'(t), \boldsymbol{u}'(t)\big)_{\mathbf{L}^2(\Omega)} - A(\boldsymbol{u}(t), \boldsymbol{u}(t)) = 0, \quad \forall t \in [0, T]. \end{aligned} \tag{29}$$

From the equality for $\boldsymbol{d}_{\mathbf{N}}$, as $\boldsymbol{G}$ is positive-definite and A is coercive on $\mathbf{V}(\Omega)$, it follows that

$$\begin{aligned} |\boldsymbol{d}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{d}_{\mathbf{N}}(t)\|^2 \le c_5 \int_0^t \big(|\boldsymbol{d}'_{\mathbf{N}}(\tau)|^2 + \|\boldsymbol{d}_{\mathbf{N}}(\tau)\|^2\big)\, d\tau + c_5 \,\big| 2J_{\mathbf{N}}(t) + \\ + \big(\boldsymbol{G}\boldsymbol{d}'_{\mathbf{N}}(0), \boldsymbol{d}'_{\mathbf{N}}(0)\big) + A(\boldsymbol{d}_{\mathbf{N}}(0), \boldsymbol{d}_{\mathbf{N}}(0)) + 2\widetilde{L}(\boldsymbol{d}_{\mathbf{N}})(t)\big|, \end{aligned} \tag{30}$$

Due to the conditions of the theorem $\boldsymbol{d}_{\mathbf{N}}(0) \to \mathbf{0}$ strongly in $\mathbf{V}(\Omega)$ and $\boldsymbol{d}'_{\mathbf{N}}(0) \to \mathbf{0}$ strongly in $\mathbf{L}^2(\Omega)$ as $\min\limits_{1\le i\le 3}\{N_i\} \to \infty$. Hence, (25), (28), (29) imply that as $N \to \infty$,

$$\big(\boldsymbol{G}\boldsymbol{d}'_{\mathbf{N}}(0), \boldsymbol{d}'_{\mathbf{N}}(0)\big)_{\mathbf{L}^2(\Omega)} + A(\boldsymbol{d}_{\mathbf{N}}(0), \boldsymbol{d}_{\mathbf{N}}(0)) + 2J_{\mathbf{N}}(t) + 2\widetilde{L}(\boldsymbol{d}_{\mathbf{N}})(t) \to 0.$$

Consequently, applying Gronwall's lemma to (30), we obtain

$$|\boldsymbol{d}'_{\mathbf{N}}(t)|^2 + \|\boldsymbol{d}_{\mathbf{N}}(t)\|^2 \to 0 \quad \text{as } \min_{1\le i\le 3}\{N_i\} \to \infty,\ \forall t \in [0, T].$$

□

References

1. Vekua I.N. *On a method of computing prismatic shells.* Proc. A. Razmadze Inst. Math. Georgian Acad. Sci. **21** (1955), 191-259 (in Russian).

2. Gordeziani D.G. *On the solvability of some boundary value problems for a variant of the theory of thin shells.* Dokl. Akad. Nauk SSSR **215** (1974), 6, 1289-1292 (in Russian).

3. Gordeziani D.G. *To the exactness of one variant of the theory of thin shells.* Dokl. Akad. Nauk SSSR **216** (1974), 4, 751-754 (in Russian).

4. Avalishvili M. *On a dimensional reduction method in the theory of elasticity.* Rep. of Enlarged Sess. of the Sem. of I. Vekua Inst. Appl. Math. **14** (1999), 3, 16-19.

5. Gordeziani D., Avalishvili M. *Investigation of hierarchic model of prismatic shell.* Bull. Georgian Acad. Sci. **165** (2002), 3, 485-488.

6. Gordeziani D., Avalishvili M. *Investigation of dynamical two-dimensional model of prismatic shell.* Bull. Georgian Acad. Sci. **166** (2002), 1, 16-19.

7. Avalishvili M., Gordeziani D. *Investigation of two-dimensional models of elastic prismatic shell.* Georgian Math. J. **10** (2003), 1, 17-36.

8. Vekua I.N. *Theory of thin shallow shells of variable thickness.* Proc. A. Razmadze Inst. Math. Georgian Acad. Sci. **30** (1965), 3-103 (in Russian).

9. Guliaev V., Baganov V., Lizunov P. *Nonclassical theory of shells.* Vischa Shkola, Lviv, 1978 (in Russian).

10. Vogelius M., Babuška I. *On a dimensional reduction method.* Math. of Comput. **37** (1981), 155, 31-68.

11. Vekua I.N. *Shell theory: general methods of construction.* Pitman (Advanced Publ. Program), Boston, 1985.

12. Meunargia T. *On two-dimensional equations of the linear theory of non-shallow shells.* Proc. of I. Vekua Inst. Appl. Math. **38** (1990), 5-43 (in Russian).

13. Babuška I., Li L. *Hierarchic modeling of plates.* Comput. and Structures **40** (1991), 419-430.

14. Babuška I., Szabò B.A., Actis R.L. *Hierarchic models for laminated composites.* Internat. J. Numer. Methods Engrg. **33** (1992), 3, 503-535.

15. Schwab C., Wright S. *Boundary layer approximation in hierarchical beam and plate models.* Journal of Elasticity **38** (1995), 1-40.

16. Meunargia T. *On nonlinear and nonshallow shells.* Bull. TICMI **2** (1998), 46-49.

17. Vashakmadze T.S. *The theory of anisotropic elastic plates.* Kluwer Academic Publ., Dordrecht, 1999.

18. Jaiani G.V. *On a mathematical model of bars with variable rectangular cross-sections.* ZAMM **81** (2001), 3, 147-173.

19. Avalishvili M. *Investigation of a mathematical model of elastic bar with variable cross-sections.* Bull. Georgian Acad. Sci. **166** (2002), 1, 37-40.

20. Gordeziani D., Avalishvili G. *On the investigation of static hierarchic model for elastic rods.* Appl. Math. Inf. Mech. **8** (2003), 1, 34-46.

21. Dauge M., Faou E., Yosibash Z. *Plates and shells: Asymptotic expansions and hierarchical models.* Encyclopedia of Computational Mechanics, vol. I, Chap. 8 (2004), 199-236.

22. McLean W. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

23. Ciarlet P.G. *Mathematical elasticity, vol. III, Theory of shells.* North-Holland Publ. Co., Amsterdam, 1999.

24. Lions J.-L., Magenes E. *Problèmes aux limites non homogènes et applications, vol. 1.* Travaux et Recherchers Mathématiques, No. 17, Dunod, Paris, 1968.

25. Whitney H. *Geometric integration theory.* Princeton University Press, Princeton, 1967.