SIMPLE TECHNIQUE FOR EVALUATING RESIDUAL TERM OF FINITE VOLUME SCHEMES

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Abstract

We develop here special technique for evaluating residual in finite volume schemes for nonlinear scalar conservation laws. Traditionally for evaluating of similar terms BV or weak BV type estimates are needed, or some special requirement on regularity of mesh refinement procedure is needed in order to get the residual convergent to zero. The technique we introduce here is called simple because it uses just uniform L^∞ estimate on approximate solutions constructed by means of kinetic finite volume schemes. Coupling this technique with abstract convergence theorem introduced by Botchorishvili, Perthame, Vasseur [3] we prove convergence of the explicit kinetic finite volume schemes.

Key words and phrases: hyperbolic conservation laws, finite volume schemes, residual term, weak* convergence.

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Introduction

Consider the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial A_i(u)}{\partial x_i} = 0, \quad t \geq 0,\ x \in \mathrm{I\!R}^{\mathrm{N}}, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_0(x) \in L^\infty(\mathrm{I\!R}^{\mathrm{N}}), \tag{1.2}$$

with smooth functions $A_i(.)$, $\;A_i \in C^1(\mathrm{I\!R})$, $1 \leq i \leq N$. The equation (1.1) is endowed with the full family of entropy inequalities

$$\frac{\partial S(u)}{\partial t} + \sum_{i=1}^{N} \frac{\partial \eta_i(u)}{\partial x_i} \leq 0, \tag{1.3}$$

for all convex entropy functions $S(\cdot)$ and corresponding entropy fluxes $\eta_i(\cdot)$,$1 \le i \le N$, that are defined in accordance with the relation

$$\eta_i\prime(u) = S'(u)a_i(u), \qquad a_i(u) = A_i'(u), \;\; 1 \le i \le N, \tag{1.4}$$

see Kruzkov [10], Lax [12] for more details.

For numerical solution of the problem (1.1),(1.2) several different approaches exist, see e.g. [9],[13]. Finite volume schemes have proven to be most suitable for this purpose. Standard monotone explicit finite volume scheme on arbitrary meshes for the equation (1.1) writes:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{|C_j|}\sum_k \sum_l |\Gamma_{jk}^l| A(u_j^n, u_k^n, \overrightarrow{n}_{jk}^l) = 0, \tag{1.5}$$

where Δt is discretization step in time, $A(u_j^n, u_k^n, \overrightarrow{n}_{jk})$ is a monotone numerical flux function [15],[1] satisfying usual requirements on consistency:

$A(u, u, \overrightarrow{n}) =< A(u), \overrightarrow{n} >$, $< \cdot, \cdot >$ is a scalar product in $\mathbb{R}^{\mathrm{N}}$,
$A(u, v, \overrightarrow{n})$ is Lifschitz continuous with respect to u, v,

and on monotonicity:

$$A(u, v, \overrightarrow{n}) \text{ is nondecreasing in } u \text{ and nonincreasing in } v;$$

u_j^n is approximate solution at time t_n in nodal point $\overrightarrow{x_j}$ of a finite volume mesh, $\overrightarrow{x_j} \in \mathbb{R}^{\mathrm{N}}$, $\mathrm{j} = 0, 1, ..$; C_j are cells associated with node $\overrightarrow{x_j}$ and Γ_{jk} is the interface between cells C_j and C_k, $\Gamma_{jk} = C_j \cap C_k$, $\Gamma_{jk} = \cup_l \Gamma_{jk}^l$, $\overrightarrow{n}_{jk}^l$ is the unit normal of Γ_{jk}^l directed into C_k. Notice that cell interface Γ_{jk} can be composed by several subinterfaces Γ_{jk}^l. Notice also that here and onward superscript l refers to subinterface number. We supply finite volume scheme (1.5) with the following initial condition:

$$u_j^0 = \frac{1}{|C_j|}\int_{C_j} u_0(x)dx. \tag{1.6}$$

Because of the nonlinearity of the problem under consideration and because of the low regularity of its solution one of the main difficulties associated with the investigation of convergence of finite volume schemes is finding of suitable compactness framework. From this standpoint we can characterize several different approaches as follows:

- approaches using BV estimates in case of (1.2) with initial value function of bounded variation, see e.g.[11],[15].

- approaches using compensated compactness and DiPerna's uniqueness theorem[8] for measure valued solutions, see e.g. [7],[16].
- approaches based on weak BV type estimates, see e.g. [6].
- approaches based on kinetic formulation of scalar conservation laws[14] and the uniqueness theorem for kinetic solutions[3],[2],[4].

For more information about the first three approaches see also review papers [1], [17] and references therein. Here we concentrate on the fourth approach in the above list. In [3] for numerical approximations to scalar conservation laws abstract convergence theorem was introduced and convergence of kinetic schemes was proven in one space dimension. Notice that this abstract convergence theorem provides general framework for any space dimensions and contains just necessary set of suppositions for the proof of convergence. However when using similar technique in several space dimensions on arbitrary finite volume meshes problem arises with estimation of some residual term. In order to get such residual term vanishing some additional supposition on a mesh refinement process was introduced in [2] and convergence of kinetic schemes was proven. In the present paper we introduce special technique that is simple and enables proving of convergence for arbitrary finite volume meshes just using abstract convergence theorem from [3].

The rest of the paper is organized as follows. In the section 2 we define the residual term and estimate it using different approaches in one space dimension. In the section 3 we develop simple technique and apply it for evaluating mesh size dependent behavior of a residual of kinetic schemes in one space dimension. In the section 4 we couple simple technique with the abstract convergence theorem [3] and prove the convergence of kinetic finite volume schemes on arbitrary unstructured meshes in several space dimensions. Finally, conclusions are given in the last section.

2. Residual term in one space dimension

2.1. Numerical scheme

Monotone finite volume scheme (1.5) and its initial condition in one space dimension writes:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{\mathbb{A}(\mathrm{u}^{\mathrm{n}}_{\mathrm{j+1}},\mathrm{u}^{\mathrm{n}}_{\mathrm{j}})-\mathbb{A}(\mathrm{u}^{\mathrm{n}}_{\mathrm{j}},\mathrm{u}^{\mathrm{n}}_{\mathrm{j-1}})}{h_j}=0, \tag{2.1}$$

$$u_j^0 = \frac{1}{h_j}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, \tag{2.2}$$

where $h_j = x_{j+1/2} - x_{j-1/2} = 0.5(h_{j+1/2} + h_{j-1/2})$, $x_{j+1/2} = 0.5(x_{j+1} + x_j)$, $h_{j+1/2} = x_{j+1} - x_j$, $\mathbb{A}(.,.)$ is consistent and monotone numerical flux function. Notice that the scheme (2.1),(2.2) is well investigated in scientific literature and just for the convenience of further exposition we recall some of its properties. We set:

$$\begin{aligned} u_h(t,x) = u_j^n,\ (t,x) \in (t_n, t_{n+1}) \times C_j, \\ t_{n+1} = t_n + \Delta t,\quad C_j = (x_{j-1/2}, x_{j+1/2}). \end{aligned} \tag{2.3}$$

Proposition 1. Suppose $u_0(x) \in L^\infty(\mathbb{R}) \bigcap \mathrm{L}^1(\mathbb{R}) \bigcap \mathrm{BV}(\mathbb{R})$ and the following CFL-condition

$$\frac{\Delta t}{h_j}\Big(\max_{|u|,|v|\le\|u_0\|_\infty} |\mathbb{A}'_{\mathrm{u}}(\mathrm{u},\mathrm{v})| + \max_{|\mathrm{u}|,|\mathrm{v}|\le\|\mathrm{u}_0\|_\infty} |\mathbb{A}'_{\mathrm{v}}(\mathrm{u},\mathrm{v})|\Big) \le 1 \tag{2.4}$$

is valid. Then the following estimates hold true:

$$\|u_h(t,.)\|_{L^\infty} \le \|u_0\|_{L^\infty}, \|u_h(t,.)\|_{L^1} \le \|u_0\|_{L^1}, var_x[u_h(t,x)] \le var_x[u_0], \tag{2.5}$$

$$\frac{S(u_j^{n+1}) - S(u_j^n)}{\Delta t} + \frac{\boldsymbol{\eta}(u_{j+1}^n, u_j^n) - \boldsymbol{\eta}(u_j^n, u_{j-1}^n)}{h_j} \le 0, \tag{2.6}$$

where $\boldsymbol{\eta}$ is numerical entropy flux consistent with the entropy function S and numerical flux function $\mathbb{A}$.

2.2. Controlling residual term using BV estimate

For proving the convergence of finite volume schemes several approaches exist. One way is to show that the family of approximate solutions is compact and its any subsequence satisfies (1.1),(1.3) in the weak sense. Clearly, the equation and the entropy condition are not satisfied exactly by approximate solution. In particular, written in the weak form for u_h they contain some extra term called residual. For the simplicity of exposition consider weak form of the finite volume scheme (2.1) that writes:

$$\int\int (u_h g_t + A(u_h) g_x) dx dt + \Psi_h = 0, \tag{2.7}$$

where u_h is defined by (2.3), g is sufficiently smooth compactly supported function, Ψ_h is the so called residual term,

$$\begin{aligned} -\Psi_h &= \Delta t \sum_n \sum_j g_j^n h_j \Big(\frac{\mathbb{A}(\mathrm{u}_{\mathrm{j}+1}^{\mathrm{n}}, \mathrm{u}_{\mathrm{j}}^{\mathrm{n}}) - \mathbb{A}(\mathrm{u}_{\mathrm{j}}^{\mathrm{n}}, \mathrm{u}_{\mathrm{j}-1}^{\mathrm{n}})}{h_j} - \frac{A(u_{j+1}^n) - A(u_{j-1}^n)}{2h_j}\Big) \\ &+ O(\Delta t + h) = \psi_h + O(\Delta t + h). \end{aligned} \tag{2.8}$$

The residual term ψ_h equivalently writes:

$$\psi_h = \Delta t \sum_n \sum_j (g_j^n - g_{j+1}^n)[\mathbb{A}(\mathrm{u}_{\mathrm{j+1}}^{\mathrm{n}}, \mathrm{u}_{\mathrm{j}}^{\mathrm{n}}) - \frac{1}{2}(\mathrm{A}(\mathrm{u}_{\mathrm{j+1}}^{\mathrm{n}}) + \mathrm{A}(\mathrm{u}_{\mathrm{j}}^{\mathrm{n}}))], \qquad (2.9)$$

$$|\psi_h| \leq \frac{h}{2}\|g_x\|(\max_{|u|,|v|\leq\|u_0\|_\infty} |\mathbb{A}'_{\mathrm{u}}(\mathrm{u},\mathrm{v})| + \max_{|u|,|v|\leq\|u_0\|_\infty} |\mathbb{A}'_{\mathrm{v}}(\mathrm{u},\mathrm{v})|)\mathrm{var}[\mathrm{u}_0](\max_{\mathrm{g_x(t,x)}\neq 0}\{\mathrm{t}\} + \Delta\mathrm{t}). \qquad (2.10)$$

2.3. Estimating residual term using regular mesh refinement

Suppose no uniform BV or weak BV type estimates are available for approximate solutions. Then application of the technique considered in the previous subsection is impossible. Though in case of uniform mesh and the numerical flux function defined according to formula

$$\mathbb{A}(\mathrm{u},\mathrm{v}) = \frac{1}{2}(\mathrm{A}(\mathrm{u}) + \mathrm{A}(\mathrm{v})) - \frac{1}{2}(\tilde{\mathrm{A}}(\mathrm{u}) - \tilde{\mathrm{A}}(\mathrm{v})), \quad \tilde{\mathrm{A}}'(\mathrm{u}) \geq |\mathrm{A}'(\mathrm{u})|, \qquad (2.11)$$

the problem is easily resolved by means of using integration by parts at a discrete level. In particular, putting (2.11) in (2.12) yields:

$$\begin{aligned}\psi_h &= \frac{\Delta t}{2}\sum_n\sum_j (g_{j+1}^n - g_j^n)(\tilde{A}(u_{j+1}^n) - \tilde{A}(u_j^n)) \\ &= -\frac{\Delta t}{2}\sum_n\sum_j (g_{j+1}^n - 2g_j^n + g_{j-1}^n)\tilde{A}(u_j^n), \end{aligned} \qquad (2.12)$$

$$g_{j+1}^n - 2g_j^n + g_{j-1}^n = (x_{j+1} - 2x_j + x_{j-1})g_{xj}^n + 0(h^2),$$

$$\psi_h = -\frac{\Delta t}{2}\sum_n\sum_j \frac{x_{j+1} - 2x_j + x_{j-1}}{x_{j+1/2} - x_{j-1/2}} g_{xj}^n \tilde{A}(u_j^n)h_j + 0(h^2). \qquad (2.13)$$

Notice that if mesh is uniform, i.e. $h_j = h$, then

$$x_{j+1} - 2x_j + x_{j-1} = 0 \qquad (2.14)$$

and therefore ψ_h vanishes together with h. In case of arbitrary meshes (2.14) is not valid, though its left hand side could be used as some measure of mesh regularity. Regular mesh refinement [2],[4] claims that mesh must be refined so that the following inequality be valid

$$|\frac{x_{j+1} - 2x_j + x_{j-1}}{x_{j+1/2} - x_{j-1/2}}| \leq Kh^\gamma, \ 0 < \gamma < 1, \qquad (2.15)$$

where K, γ are constants independent of a sequence of meshes as $h \to 0$. In other words this means that on such meshes second order finite differences vanish faster then the first order finite differences do. Notice that if (2.15) is valid then the residual term ψ_h can be easily controlled.

3. Simple technique

The techniques presented in the previous section for evaluating residual term in finite volume schemes have drawbacks in several space dimensions: in the first case it is impossible to evaluate total variation of the approximate solutions on arbitrary finite volume meshes and in the second case mesh refinement is restricted by similar to (2.15) condition. Simple technique presented below is free of these drawbacks. It's main component is weak* continuity of uniformly bounded sequence of grid functions.

3.1. Weak* continuity of grid functions

Let $\omega(x)$ be smooth nonnegative compactly supported in unit cube function such that

$$\int_{\mathbb{R}^N} \omega(x) = 1, \quad \omega(x) = -\omega(x). \tag{3.1}$$

Then $\omega(x)$ can be used as regularization kernel for L^∞ functions, e.g. as follows:

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon}\omega(\frac{x}{\varepsilon}), \quad \varepsilon > 0,$$

$$v_\varepsilon(x) = \int_{\mathbb{R}^N} \omega_\varepsilon(x-y)v(y)dy, \quad v \in L^\infty(\mathbb{R}^N).$$

Lemma 2.(Regularization lemma.) Let h be characterizing size of finite volume mesh, $u_h(t,x)$ be uniformly bounded sequence of L^∞ functions such that u_h is piecewise constant at each finite volume cell. Suppose $u_h \longrightarrow u$ in L^∞ weak* and $0 < \varepsilon(h),\ \varepsilon(h) \longrightarrow 0$ as $h \longrightarrow 0$. Then the following hold true:

$$(u_{h\varepsilon(h)} - u_h) \longrightarrow 0, \text{ in } L^\infty \text{ weak* as } h \longrightarrow 0, \tag{3.2}$$

$$|u_{h\varepsilon(h)}(t,x_j) - u_h(t,x_k)| \le K_\omega \frac{|x_j - x_k|}{\varepsilon}\|u_h\|_{L^\infty}, \tag{3.3}$$

where

$$u_{h\varepsilon}(t,x) = \int_{\mathbb{R}^N} \omega_\varepsilon(x-y)u(t,y)dy,$$

$$|\overrightarrow{x}_j - \overrightarrow{x}_k| = (\sum_{i=1}^{N}(x_{ji} - x_{ki})^2)^{1/2}, \qquad K_\omega = N^{\frac{1}{2}} \max_{x,i} |\frac{\partial}{\partial x_i}\omega(x)|.$$

Proof: Let $g(t,x) \in L^1(\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N})$. Then we can write:

$$
\begin{aligned}
& \int_{\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}} u_{h\varepsilon}(t,x) g(t,x) dx dt \\
= & \int_{\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}} \int_{\mathrm{IR}^\mathrm{N}} u_{h\varepsilon}(t,x) \omega_\varepsilon(x-y) g(t,x) dy dx dt \\
= & \int_{\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}} u_h(t,x) g_\varepsilon(t,x) dx dt \\
= & \int_{\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}} u_h(t,x) g(t,x) dx dt \\
+ & \int_{\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}} u_h(t,x)(g_\varepsilon(t,x) - g(t,x)) dx dt. \qquad (3.4)
\end{aligned}
$$

Notice that for any continuous and compactly supported function g we have

$$
\lim_{h\to 0} \int_{\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}} u_h(t,x)(g_\varepsilon(t,x) - g(t,x)) dx dt = 0.
$$

Therefore (3.4) yields that (3.2) is proven.

$$
\begin{aligned}
|u_{h\varepsilon}(t,\overrightarrow{x}_j) - u_{h\varepsilon}(t,\overrightarrow{x}_k)| & |< \nabla u_{h\varepsilon}(t,\overrightarrow{\zeta}), \overrightarrow{x}_j - \overrightarrow{x}_k >| \\
& \leq (\sum_{i=1}^N (\frac{\partial u_{h\varepsilon}}{\partial x_i})^2)^{\frac{1}{2}} (\sum_{i=1}^N (x_j^i - x_k^i)^2)^{\frac{1}{2}} \qquad (3.5)
\end{aligned}
$$

where $\overrightarrow{\zeta} = \theta \overrightarrow{x}_j + (1-\theta)\overrightarrow{x}_k, \quad 0 < \theta < 1$.

$$
\begin{aligned}
(\sum_{i=1}^N (\frac{\partial u_{h\varepsilon}}{\partial x_i})^2)^{\frac{1}{2}} & = (\sum_{i=1}^N (\int_{\mathrm{IR}^\mathrm{N}} \frac{\partial \omega_\varepsilon(x-y)}{\partial x_i} u_h(t,y) dy)^2)^{\frac{1}{2}} \\
& \leq (\sum_{i=1}^N \frac{1}{\varepsilon^2} \max_x |\frac{\partial}{\partial x_i} \omega(x)|^2)^{\frac{1}{2}} \|u_h\|_{L^\infty}. \qquad (3.6)
\end{aligned}
$$

(3.5) and (3.6) yield (3.3).

Proposition 2.(Weak* continuity of grid function.) Suppose h is characterizing size of finite volume mesh, $u_h(t,x)$ is uniformly bounded sequence of functions satisfying (2.3), $g(t,x)$ is smooth and compactly supported in $\mathrm{IR}_\mathrm{t}^+ \times \mathrm{IR}^\mathrm{N}$. If

$$
k(j) : \mathrm{IN} \to \mathrm{IN}, \quad |\overrightarrow{x}_{\mathrm{k(j)}} - \overrightarrow{x}_\mathrm{j}| \leq \mathrm{K_k} \cdot \mathrm{h}, \qquad (3.7)
$$

K_k is independent of h, then there exist subsequence of u_h such that

$$\lim_{h\to 0} \sum_{n\geq 0} \sum_j \Delta t(u_j^n - u_{k(j)}^n) g_j^n |C_j| = 0. \qquad (3.8)$$

Proof: We set

$$v_h(t,x) = u_{k(j)}^n, \ (t,x) \in (t_n, t_{n+1}) \times C_j. \qquad (3.9)$$

Since u_h, v_h are uniformly bounded in L^∞ one can extract weak* convergent subsequences denoted again via u_h and v_h for the simplicity. Let u and v be weak* limits of u_h and v_h respectively, ε be such that

$$0 < \varepsilon, \ \ \varepsilon = \varepsilon(h), \ \ \lim_{h\to 0} \varepsilon(h) = 0, \ \ \lim_{h\to 0} h/\varepsilon(h) = 0. \qquad (3.10)$$

We regularize u_h and v_h exactly in the same way as in the regularization lemma and denote regularized functions as $u_{h\varepsilon}$ and $v_{h\varepsilon}$ respectively. Notice that according to regularization lemma we have

$$\begin{array}{l} (u_{h\varepsilon(h)} - u_h) \longrightarrow 0, \text{ in } L^\infty \text{ weak* as } h \longrightarrow 0, \\ (v_{h\varepsilon(h)} - v_h) \longrightarrow 0, \text{ in } L^\infty \text{ weak* as } h \longrightarrow 0. \end{array} \qquad (3.11)$$

The expression under limit in (3.8) equivalently writes:

$$\begin{aligned} & \sum_{n\geq 0} \sum_j \Delta t(u_j^n - u_{k(j)}^n) g_j^n |C_j| \\ = & \int_{\mathbb{R}_+^t \times \mathbb{R}^N} (u_h(t,x) - v_h(t,x)) g_h(t,x) dx dt \\ = & \int_{\mathbb{R}_+^t \times \mathbb{R}^N} (u_h(t,x) - v_h(t,x)) g(t,x) dx dt + O(h) \\ = & \int_{\mathbb{R}_+^t \times \mathbb{R}^N} (u_{h\varepsilon(h)}(t,x) - v_{h\varepsilon(h)}(t,x)) g(t,x) dx dt \\ + & \int_{\mathbb{R}_+^t \times \mathbb{R}^N} (u_h(t,x) - u_{h\varepsilon(h)}(t,x)) g(t,x) dx dt \\ - & \int_{\mathbb{R}_+^t \times \mathbb{R}^N} (v_h(t,x) - v_{h\varepsilon(h)}(t,x)) g(t,x) dx dt + O(h). \end{aligned} \qquad (3.12)$$

On account of (3.11) the second and third terms in the right hand side of (3.12) vanish together with h. Thus in order to get (3.8) proven it remains to show that the first term in the right hand side of (3.12) converges to zero as $h \to 0$. The term under consideration writes:

$$\int_{\mathbb{R}_+^t \times \mathbb{R}^N} (u_{h\varepsilon(h)}(t,x) - v_{h\varepsilon(h)}(t,x)) g(t,x) dx dt$$

$$
\begin{aligned}
&= \sum_j \int_{\mathbb{R}_+^t} \int_{C_j} (u_{h\varepsilon(h)}(t,x) - v_{h\varepsilon(h)}(t,x)) g(t,x) dxdt \\
&= \sum_j \int_{\mathbb{R}_+^t} \int_{C_j} (u_{h\varepsilon(h)}(t,x) - u_{h\varepsilon(h)}(t,x_j)) g(t,x) dxdt \\
&- \sum_j \int_{\mathbb{R}_+^t} \int_{C_j} (v_{h\varepsilon(h)}(t,x) - v_{h\varepsilon(h)}(t,x_j)) g(t,x) dxdt \\
&+ \sum_j \int_{\mathbb{R}_+^t} \int_{C_j} (u_{h\varepsilon(h)}(t,x_j) - v_{h\varepsilon(h)}(t,x_j)) g(t,x) dxdt. \qquad (3.13)
\end{aligned}
$$

Notice that if $x \in C_j$ then $|x - x_j| \leq h$. On account of regularization lemma, in particular (3.3), and on account of the supposition (3.7) we have:

$$
\begin{aligned}
&\left| \int_{\mathbb{R}_+^t} \int_{C_j} (u_{h\varepsilon(h)}(t,x) - u_{h\varepsilon(h)}(t,x_j)) g(t,x) dxdt \right| \\
&\qquad \leq \frac{h}{\varepsilon(h)} K_k K_\omega |u_h|_{L^\infty} \int_{\mathbb{R}_+^t} \int_{C_j} |g(t,x)| dxdt, \\
&\left| \int_{\mathbb{R}_+^t} \int_{C_j} (v_{h\varepsilon(h)}(t,x) - v_{h\varepsilon(h)}(t,x_j)) g(t,x) dxdt \right| \\
&\qquad \leq \frac{h}{\varepsilon(h)} K_k K_\omega |v_h|_{L^\infty} \int_{\mathbb{R}_+^t} \int_{C_j} |g(t,x)| dxdt, \\
&\left| \int_{\mathbb{R}_+^t} \int_{C_j} (u_{h\varepsilon(h)}(t,x_j) - v_{h\varepsilon(h)}(t,x_j)) g(t,x) dxdt \right| = \\
&\left| \int_{\mathbb{R}_+^t} \int_{C_j} (u_{h\varepsilon(h)}(t,x_j) - u_{h\varepsilon(h)}(t,x_{k(j)})) g(t,x) dxdt \right| \\
&\qquad \leq \frac{h}{\varepsilon(h)} K_k K_\omega |u_h|_{L^\infty} \int_{\mathbb{R}_+^t} \int_{C_j} |g(t,x)| dxdt. \qquad (3.14)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}_+^t \times \mathbb{R}^N} (u_{h\varepsilon(h)}(t,x) - v_{h\varepsilon(h)}(t,x)) g(t,x) dxdt \right| \\
&\qquad \leq 3 \frac{h}{\varepsilon(h)} K_k K_\omega |u_h|_{L^\infty} \int_{\mathbb{R}_+^t \times \mathbb{R}^N} |g(t,x)| dxdt. \qquad (3.15)
\end{aligned}
$$

On account of (3.10) we get from (3.15) that $(u_{h\varepsilon(h)} - v_{h\varepsilon(h)}) \to 0$ in L^∞ weak* as $h \to 0$. Proof is completed.

Proposition 3.(Weak* continuity of vector valued grid function.) Suppose h is characterizing size of finite volume mesh, $\overrightarrow{u}_h = (\overrightarrow{u}_{h1}, ..., \overrightarrow{u}_{hN})^T$,

$u_{ih}(t,x), 1 \le i \le N$, are uniformly bounded sequences of functions defined according to (2.3), $g(t,x)$ is smooth and compactly supported in $\mathbb{R}_t^+ \times \mathbb{R}^N$. If (3.7) is valid and

$$|\Gamma_{jk}^l| \cdot |\overrightarrow{x}_{k(j)} - \overrightarrow{x}_j| \le K_c \cdot |C_{jk}^l|, \quad C_{jk}^l = co\{\overrightarrow{x}_j \bigcup \Gamma_{jk}^l\}, \tag{3.16}$$

K_c is independent of h, then there exist subsequence of $\overrightarrow{u}_h$ such that

$$\lim_{h\to 0} \sum_{n\ge 0} \sum_{i_{jkl}\in\mathcal{I}} \Delta t < \overrightarrow{u}_j^n - \overrightarrow{u}_{k(j)}^n, \overrightarrow{n}_{jk}^l > (g_j^n - g_{k(j)}^n)|\Gamma_{jk}^l| = 0, \tag{3.17}$$

$\|\overrightarrow{n}_{jk}^l\| = 1$, $\mathcal{I}$ is a set of reference numbers to cell interfaces.

Proof: The expression under limit in (3.17) equivalently writes:

$$\begin{aligned}
&\sum_{n\ge 0} \sum_{i_{jkl}\in\mathcal{I}} \Delta t < \overrightarrow{u}_j^n - \overrightarrow{u}_{k(j)}^n, \overrightarrow{n}_{jk}^l > (g_j^n - g_{k(j)}^n)|\Gamma_{jk}^l| = \\
&\sum_{n\ge 0} \sum_{j} \sum_{kl\in\mathcal{I}_|} \frac{1}{2}\Delta t < \overrightarrow{u}_j^n - \overrightarrow{u}_{k(j)}^n, \overrightarrow{n}_{jk}^l > (g_j^n - g_{k(j)}^n)|\Gamma_{jk}^l| = \\
&\sum_{n\ge 0} \sum_{j} \sum_{kl\in\mathcal{I}_|} < \overrightarrow{u}_j^n - \overrightarrow{u}_{k(j)}^n, \overrightarrow{\varphi}_{jk}^{ln} > \Delta t |C_{jk}^l|,
\end{aligned} \tag{3.18}$$

where

$$\overrightarrow{\varphi}_{jk}^{ln} = \frac{1}{2}\overrightarrow{n}_{jk}^l (g_j^n - g_{k(j)}^n)\frac{|\Gamma_{jk}^l|}{|C_{jk}^l|}.$$

Notice that $C_j = \bigcup_{kl\in\mathcal{I}_|} \bar{C}_{jk}^l$, $\mathcal{I}_|$ is a set of reference numbers to surrounding cell C_j interfaces. We set:

$$\overrightarrow{\varphi}_h(t,x) = \overrightarrow{\varphi}_{jk}^{ln}, \qquad (t,x) \in (t_n, t_{n+1}) \times C_{jk}^l.$$

On account of smoothness and compact support of g we have:

$$\begin{aligned}
\int_{\mathbb{R}_t^+ \times \mathbb{R}^N} \|\overrightarrow{\varphi}_h(t,x)\| dxdt &= \sum_{n\ge 0} \sum_{j} \sum_{kl\in\mathcal{I}_|} \frac{1}{2}\Delta t |g_j^n - g_{k(j)}^n||\Gamma_{jk}^l| \\
&\le \sum_{n\ge 0} \sum_{j} \sum_{kl\in\mathcal{I}_|} \frac{1}{2}\Delta t |\nabla g_{kj}^n||\overrightarrow{x}_j - \overrightarrow{x}_{k(j)}||\Gamma_{jk}^l| \\
&\le \sum_{n\ge 0} \sum_{j} \sum_{kl\in\mathcal{I}_|} \frac{1}{2}\Delta t |\nabla g_{kj}^n| K_c |C_{jk}^l| < +\infty,
\end{aligned} \tag{3.19}$$

where $\nabla g_{kj}^n = g(t_n, \overrightarrow{x}_j\theta + \overrightarrow{x}_{k(j)}(1-\theta))$, $\quad 0 \le \theta \le 1$. Thus $\varphi_h(t,x)$ is L^1 function. Application of the proposition 3 to (3.18) accomplishes the proof.

3.2. Simple technique in one space dimension

Here we apply weak* continuity of grid functions for evaluating space derivative and corresponding residual in finite volume schemes in one space dimension. The case of numerical flux function defined by formula (2.11) will be investigated. After multiplying (2.1) on $\Delta t h_j g_j^n$ the term corresponding to space derivative writes:

$$\begin{aligned}
& \sum_n \sum_j \Delta t[\frac{1}{2}(A_{j+1}^n + A_j^n) - \frac{1}{2}(\tilde{A}_{j+1}^n - \tilde{A}_j^n) \\
- & \frac{1}{2}(Aj^n + A_{j-1}^n) + \frac{1}{2}(\tilde{A}_j^n - \tilde{A}_{j-1}^n)]g_j^n \\
= & \sum_n \sum_j \Delta t[\frac{1}{2}(A_{j+1}^n + A_j^n) - \frac{1}{2}(\tilde{A}_{j+1}^n - \tilde{A}_j^n)](g_j^n - g_{j+1}^n) \\
= & -\sum_n \sum_j \Delta t A_j^n g_{xj}^n h_j + \psi_h, \qquad (3.20)
\end{aligned}$$

where

$$\psi_h = O(h) + \psi_{1h} + \psi_{2h},$$

$$\begin{aligned}
\psi_{1h} &= -\sum_n \sum_j \Delta t[\frac{1}{2}(A_{j+1}^n + A_j^n)(g_{j+1}^n - g_j^n) - A_j^n(g_{j+1/2}^n - g_{j-1/2}^n)] \\
&= \sum_n \sum_j \Delta t[A_j^n h_j - \frac{1}{2}(A_{j+1}^n + A_j^n)h_{j+1/2}]g_{xj} + 0(h) \\
&= \sum_n \sum_j \Delta t[(h_j - \frac{1}{2}h_{j+1/2})A_j^n - \frac{1}{2}h_{j+1/2}A_{j+1}^n]g_{xj} + 0(h) \\
&= \frac{1}{2}\sum_n \sum_j \Delta t[h_{j-1/2}A_j^n - h_{j+1/2}A_{j+1}^n]g_{xj} + 0(h) \\
&= \frac{1}{2}\sum_n \sum_j \Delta t h_{j-1/2}A_j^n(g_{xj} - g_{xj+1}) + 0(h), \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
\psi_{2h} &= -\frac{1}{2}\sum_n \sum_j \Delta t(\tilde{A}_{j+1}^n - \tilde{A}_j^n)](g_j^n - g_{j+1}^n) \\
&= \frac{1}{2}\sum_n \sum_j \Delta t(\tilde{A}_{j+1}^n - \tilde{A}_j^n)g_{xj+1/2}^n h_{j+1/2} + O(h). \qquad (3.22)
\end{aligned}$$

Notice that first term in (3.20) is suitable weak form of first order space derivative in scalar conservation law. Evidently the residual term $\psi_{1h} =$

$O(h)$ and it vanishes together with h. Notice that approximate solutions constructed by means of finite volume scheme are uniformly bounded, see e.g. Proposition 1. Therefore ψ_{2h} vanishes together with h because of the proposition on weak* continuity of grid functions. The residual term induced by space derivative in entropy condition (2.6) can be evaluated by means of using the same technique as given above.

4. Application of simple technique to kinetic schemes in several space dimensions

In this section we assume that unit vectors $\overrightarrow{n}^l_{jk}$, that are normal to cell interfaces Γ^l_{jk}, do not depend on the space variable $\overrightarrow{x}$. Here we also consider finite volume scheme (1.5) with numerical flux function defined as follows:

$$\mathbb{A}(u^n_j, u^n_k, \overrightarrow{n}^l_{jk}) = \frac{1}{2} < (\overrightarrow{A}(u^n_j) + \overrightarrow{A}(u^n_k)) - (\overrightarrow{\tilde{A}}(u^n_j) - \overrightarrow{\tilde{A}}(u^n_k)), \overrightarrow{n}^l_{jk} >,$$
$$\overrightarrow{\tilde{A}} = (\tilde{A}_1, ..., \tilde{A}_N)^T, \quad \tilde{a}_i(u) \geq \max\{a_i(u), 0\},$$
$$\tilde{a}_i(u) = \tilde{A}'_i(u), \quad a_i(u) = A'_i(u), \quad i = 1, 2, ..., N. \tag{4.1}$$

We show below that finite volume scheme (1.5) with numerical flux function (4.1) can be interpreted as kinetic scheme. The basis for this is kinetic formulation of nonlinear scalar conservation laws introduced by P.L.Lions, B.Perthame and E.Tadmor[14]. Interpreting finite volume scheme as kinetic scheme means rewriting of it in terms of the following kinetic "density" function

$$\chi(\xi; u) = \begin{cases} +1, & 0 < \xi \leq u, \\ -1, & u \leq \xi < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

The following lemma, in fact, is a collection of well known properties of the kinetic finite volume schemes.

Lemma 4. Suppose numerical flux function in (1.5) is defined by (4.1) and the following CFL condition

$$\frac{\Delta t}{C_j} \max_j \max_{|\xi| \leq \|u_0\|_{L^\infty}} \sum_k \sum_l \frac{1}{2} | < \overrightarrow{a}(\xi) + \overrightarrow{\tilde{a}}(\xi), \overrightarrow{n}^l_{jk} > | \leq 1 \tag{4.3}$$

holds true. Then solution of the finite volume scheme (1.5) satisfy:

$$\max_{j,n} |u^n_j| \leq \|u_0\|_{L^\infty}, \quad u^{n+1}_j = \int_{\mathbb{R}_\xi} f^{n+1}_j(\xi) d\xi,$$

$$\begin{aligned}\frac{\chi_j^{n+1}(\xi)-\chi_j^n(\xi)}{\Delta t} &+ \frac{1}{2C_j}\sum_{k,l}[<\overrightarrow{a}(\xi),\overrightarrow{n}^l_{jk}>(\chi_j^n(\xi)+\chi_j^k(\xi))\\ &- <\overrightarrow{\tilde{a}}(\xi),\overrightarrow{n}^l_{jk}>(\chi_j^n(\xi)-\chi_j^k(\xi))]\underline{\underline{\partial}}\partial\xi m_j^{n+1}(\xi),\\ \chi_j^n(\xi) &= \chi(\xi;u_j^n),\quad m_j^{n+1}(\xi)\geq 0,\\ m_j^{n+1}(\xi) &= \frac{1}{\Delta t}\int_{-\infty}^{\xi}(\chi_j^{n+1}(\xi)-f_j^{n+1}(\xi))d\xi. \end{aligned}\tag{4.4}$$

Proof: Derivation of the uniform L^∞ estimate under CFL condition (4.3) is trivial when using standard technique, see e.g. [4]. Nonnegativity of $m_j^{n+1}(\xi)$ is a consequence of Brenier's lemma [5], see e.g. [3]. Notice that $m_j^{n+1}(\xi)$ is compactly supported in ξ because of its definition (4.2) and the uniform boundedness of u_j^n. Integrating of

$$\frac{1}{2}[<\overrightarrow{a}(\xi),\overrightarrow{n}^l_{jk}>(\chi_j^n(\xi)+\chi_j^k(\xi))-<\overrightarrow{\tilde{a}}(\xi),\overrightarrow{n}^l_{jk}>(\chi_j^n(\xi)-\chi_j^k(\xi))]$$

in ξ yields exactly (4.1). On account of this and compact support of $m_j^{n+1}(\xi)$ we obtain that finite volume scheme (1.5) is recovered from (4.4) by means of integration in ξ.

Lemma 5. Suppose the unit vectors normal to cell interfaces Γ^l_{jk} are independent of x. If $g(x)$ is sufficiently smooth function then for each Γ^l_{jk} there exist $x^l_{jk}\in\Gamma^l_{jk}$ such that

$$\int_{C_j}<\overrightarrow{A}(u_j^n),\nabla g(x)>dx=\sum_k\sum_l<\overrightarrow{A}(u_j^n),\overrightarrow{n}^l_{jk}>g(x^l_{jk})|\Gamma^l_{jk}|.\tag{4.5}$$

Proof: First of all notice that because of the continuity of $g(x)$ according to mean value theorem for each Γ^l_{jk} there exist x^l_{jk} such that

$$\int_{\Gamma^l_{jk}}g(x)dx=g(x^l_{jk})|\Gamma^l_{jk}|.\tag{4.6}$$

On account of the latter we get:

$$\begin{aligned}\int_{C_j}<\overrightarrow{A}(u_j^n),\nabla g(x)>dx &=<\overrightarrow{A}(u_j^n),\int_{C_j}\nabla g(x)dx>\\ &=<\overrightarrow{A}(u_j^n),\overrightarrow{n}^l_{jk}\sum_k\sum_l\int_{\Gamma^l_{jk}}g(x)dx>\\ &=<\overrightarrow{A}(u_j^n),\overrightarrow{n}^l_{jk}\sum_k\sum_l g(x^l_{jk})|\Gamma^l_{jk}|>\\ &=\sum_k\sum_l<\overrightarrow{A}(u_j^n),\overrightarrow{n}^l_{jk}>g(x^l_{jk})|\Gamma^l_{jk}|.\end{aligned}$$

Lemma is proven.

Lemma 6. Suppose numerical flux function is defined according to (4.1), CFL condition (4.3) holds true, the unit vectors normal to cell subinterfaces are independent of x and $g(t,x)$ is sufficiently smooth function. Then approximate solution defined by (1.5) satisfy

$$\begin{aligned}&\sum_{n\geq 0}\sum_{j}\sum_{k,l}\overrightarrow{A}(u_j^n,u_k^n,\overrightarrow{n}_{jk}^l)g(t_n,x_j)\Delta t|\Gamma_{jk}^l|\\&-\int_{\mathbb{R}_t^+\times\mathbb{R}^N}<\overrightarrow{A}(u_h),\nabla_x g(t,x)>dx+\Psi_h,\end{aligned}\qquad(4.7)$$

where u_h is defined according to (2.3) and $\lim_{h\to 0}\Psi_h=0$.
Proof: Notice that

$$\sum_{k,l}<\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>|\Gamma_{jk}^l|=0.\qquad(4.8)$$

On account of this formula we can rewrite the left hand side of (4.7) as follows:

$$\begin{aligned}&\sum_{n\geq 0}\sum_{j}\sum_{k,l}\overrightarrow{A}(u_j^n,u_k^n,\overrightarrow{n}_{jk}^l)g(t_n,x_j)\Delta t|\Gamma_{jk}^l|\\=\;&-\sum_{n\geq 0}\sum_{j}\sum_{k,l}<\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>g(t_n,x_{jk}^l)\Delta t|\Gamma_{jk}^l|+\Psi_h,\end{aligned}\qquad(4.9)$$

where x_{jk}^l are defined in accordance with (4.5), $\Psi_h=\Psi_{1h}+\Psi_{2h}$,

$$\begin{aligned}\Psi_{1h}\;=\;&\sum_{n\geq 0}\sum_{j}\sum_{k,l}(\frac{1}{2}<\overrightarrow{A}(u_j^n)+\overrightarrow{A}(u_k^n),\overrightarrow{n}_{jk}^l>g(t_n,x_j)\\+\;&<\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>g(t_n,x_{jk}^l))\Delta t|\Gamma_{jk}^l|\\=\;&\sum_{n\geq 0}\sum_{j}\sum_{k,l}(\frac{1}{2}<\overrightarrow{A}(u_k^n)-\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>g(t_n,x_j)\\+\;&<\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>g(t_n,x_{jk}^l))\Delta t|\Gamma_{jk}^l|\\=\;&\sum_{n\geq 0}\sum_{i_{jkl}\in\mathcal{I}}([\frac{1}{2}<\overrightarrow{A}(u_k^n)-\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>g(t_n,x_j)\\+\;&<\overrightarrow{A}(u_j^n),\overrightarrow{n}_{jk}^l>g(t_n,x_{jk}^l)]\\+\;&[\frac{1}{2}<\overrightarrow{A}(u_j^n)-\overrightarrow{A}(u_k^n),\overrightarrow{n}_{kj}^l>g(t_n,x_j)\\+\;&<\overrightarrow{A}(u_k^n),\overrightarrow{n}_{kj}^l>g(t_n,x_{jk}^l)])\Delta t|\Gamma_{jk}^l|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0}\sum_{i_{jkl}\in\mathcal{I}} < \overrightarrow{A}(u_k^n) - \overrightarrow{A}(u_j^n), \overrightarrow{n}_{jk}^l > [\frac{1}{2}g(t_n, x_j) - g(t_n, x_{jk}^l) \\
&+ \frac{1}{2}g(t_n, x_j)], \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{2h} &= \frac{1}{2}\sum_{n\geq 0}\sum_{j}\sum_{k,l} < \overrightarrow{A}(u_j^n) - \overrightarrow{A}(u_k^n), \overrightarrow{n}_{jk}^l > g(t_n, x_j)\Delta t|\Gamma_{jk}^l| \\
&= \frac{1}{2}\sum_{n\geq 0}\sum_{j}\sum_{k,l} < \overrightarrow{A}(u_j^n) - \overrightarrow{A}(u_k^n), \overrightarrow{n}_{jk}^l > \\
&\cdot \quad (g(t_n, x_j) - g(t_n, x_k))\Delta t|\Gamma_{jk}^l|. \qquad (4.11)
\end{aligned}
$$

On account of the lemma 5 and smoothness and compact support of the function $g(t, x)$ the sum in the right hand side of (4.12) equivalently writes

$$
\begin{aligned}
&\sum_{n\geq 0}\sum_{j}\sum_{k,l} < \overrightarrow{A}(u_j^n), \overrightarrow{n}_{jk}^l > g(t_n, x_{jk}^l)\Delta t|\Gamma_{jk}^l| \\
&= \sum_{n\geq 0}\sum_{j}\int_{C_j} < \overrightarrow{A}(u_j^n), \nabla_x g(t_n, x) > dx\Delta t \\
&= \sum_{n\geq 0}\int_{\mathbb{R}^{\mathrm{N}}} < \overrightarrow{A}(u_h(t_n, x)), \nabla_x g(t_n, x) > dx\Delta t \\
&= \int_{\mathbb{R}_t^+\times\mathbb{R}^{\mathrm{N}}} < \overrightarrow{A}(u_h(t, x)), \nabla_x g(t, x) > dx\Delta t + O(\Delta t). \qquad (4.12)
\end{aligned}
$$

Notice that $|\overrightarrow{x}_j - \overrightarrow{x}_k| \leq 2h$, if $\overrightarrow{x}_j$ and $\overrightarrow{x}_k$ have common interface and h is characterizing size of finite volume mesh. Therefore application of the proposition 3 to (4.10) and (4.11) accomplishes the proof.

Theorem 7. Suppose numerical flux function is defined by (4.1), CFL condition (4.3) and the inequality (3.16) hold true. Then approximate solutions constructed by means of the finite volume scheme (1.5) converges in L^1_{loc} to the unique entropy solution of the problem (1.1),(1.2).

Proof: The proof is based on the coupling of the abstract convergence theorem from [3] and the simple technique developed in the previous section. In particular the simple technique provides sufficient framework for the estimating behavior of residuals due to approximation of space derivatives in finite volume schemes and thus it is important for verifying consistency of finite volume schemes that is required by the abstract convergence theorem. It should be emphasized that abstract convergence theorem provides general necessary and sufficient framework for the convergence of approximate solutions to nonlinear scalar conservation laws, see e.g. [3]. Multidimensional version of this theorem is given in [4].

Notice that suppositions of the theorem under consideration comprise suppositions of the lemma 4 and lemma 6. The proof consists in verifying suppositions of the abstract convergence theorem and is divided into following steps.

- First the problem (1.1),(1.2) with compactly supported initial value function is investigated. If for this case convergence is proved in L^1 then L^1_{loc} convergence is obtained for general case by means of using standard diagonalization process of approximate solutions corresponding to suitably selected different compactly supported initial data, see e.g. [3].
- The requirement (C.16)[4] of the abstract convergence theorem, i.e. uniform L^∞ and L^1 bounds on approximate solutions, is provided by the lemma 4 and the compact support of the initial value function.
- Kinetic interpretation of the finite volume scheme under consideration(C.16)[4], the nonnegativity and boundedness of the measure (C.15)[4] in the right hand side of the kinetic scheme(C.13)[4] are provided by the lemma 4.
- Let multiply (4.4) on $\Delta t|C_j|g(t_n, x_j, \xi)$, g is nonnegative, sufficiently smooth and compactly supported, and sum the result with respect to n and j and integrate in ξ. Then vanishing of the residual(C.14)[4] together with the characterizing size of the finite volume mesh is easily obtained by means of application of the lemma 6.
- The suppositions (C.17),(C.18)[4] are verified simply by means of using the technique similar to the one applied in the previous step.

Thus all the suppositions of the abstract convergence theorem are verified and the proof is completed.

5. Conclusion

We have developed here simple technique for studying the behavior of the residuals in finite volume schemes. It is based on the weak* continuity of uniformly bounded sequences of approximate solutions. Using of the developed simple technique for verifying weak consistency of kinetic finite volume schemes yields possibility of proving the convergence of these scheme by means of using abstract convergence theorem [3]. It should be emphasized that though we have considered just first order explicit kinetic finite volume schemes, the developed approach seems to be suitable for the proof of convergence of schemes with more complicated space and time discretizations, e.g. Runge-Kutta type, for fractional step schemes on arbitrary meshes and even for conventional consistent and stable finite volume schemes. All these will be addressed in separate paper.

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