REGULARIZATION FOR INTEGRAL EQUATIONS OF THE FIRST KIND IN THE THEORY OF THERMOELASTIC PSEUDO-OSCILLATIONS

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Abstract

In this paper integral equations of the first kind arising in homogeneous isotropic linear pseudo-oscillations thermoelastic theory are regularized. As a byproduct several integral representations for the solutions of the four basic boundary value problems of pseudo-oscillations thermoelastic theory are obtained. These representations are different from the classical ones [11].

Key words and phrases: Thermoelasticity, potentials, boundary integral equations.

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1 Introduction

In this paper we study the four basic boundary value problems of the theory of thermoelastic pseudo-oscillations for an isotropic elastic body. To solve these problems we are inspired by a Fichera's idea [8] developed by Cialdea in [1] where he used this idea in order to solve the Dirichlet problem for Laplace equation in any number of variables representing the solution by means of a simple layer potential. The method does not use the theory of pseudo-differential operators, but it hinges on the theory of reducing operators ([5], [7], [13]) and the theory of differential forms [6]. This approach was applied also for solving the Neumann problem for Laplace equation by means of a double layer potential [4] and it was generalized to the biharmonic equation ([2], [3]) in any number of variables, to the Dirichlet problem for Lamé and Stokes systems in [4] and to the traction problem for Lamé and Stokes systems [12].

The aim of the present paper is to extend this method in order to have some representation theorems for the solutions of the four basic boundary value problems of thermoelastic pseudo-oscillation's theory. These integral representations are obtained by means of thermoelasto-potentials and they are different from the usual ones [11].

In the last section we show that the solutions of each of the four basic boundary value problems of thermoelastic pseudo-oscillation's theory can be represented by any of the four thermoelasto-potentials provided that the data are given in suitable spaces.

We note that the case of pseudo-oscillations is an essential element for the dynamic problems of thermoelasticity theory (see [11], p. 572-591).

2 Formulation of the basic problems

The pseudo-oscillation equations of thermoelastic theory are [11]:

$$\begin{cases} \mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad\,div}\ u(x) - \gamma\,\mathrm{grad}\,\vartheta(x) + \varrho\omega^2 u(x) = 0 \\ \Delta\vartheta(x) + \dfrac{i\omega}{k}\vartheta(x) + i\omega\eta\mathrm{div}\ u(x) = 0, \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ denotes the spatial variable; $u = (u_1, u_2, u_3)$ is the displacement vector; ϑ is the temperature; $\omega = i\tau$, with $\Re e\,\tau > 0$; ϱ, λ, μ, k, η, γ are constants satisfying the natural restrictions [9], [11]:

$$\varrho > 0, \quad \mu > 0, \quad 3\lambda + 2\mu > 0, \quad k > 0, \quad \gamma/\eta > 0.$$

In particular, ϱ is the density, λ and μ are the *Lamé constants* and k is the conductivity.

The equations of thermoelasto-static state are the following:

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\mathrm{grad\,div}\ u - \gamma\,\mathrm{grad}\,\vartheta = 0 \\ \Delta\vartheta = 0. \end{cases} \tag{2.2}$$

It is convenient to write the equations (2.1) in the following matrix form:

$$B(\partial_x, \omega)U(x) = 0, \tag{2.3}$$

where

$$U = (u, \vartheta), \qquad u = (u_1, u_2, u_3),$$
$$B(\partial_x, \omega) = \|B_{jk}(\partial_x, \omega)\|_{4\times 4},$$
$$B_{jk}(\partial_x, \omega) = A_{jk}(\partial_x, \omega), \qquad B_{j4}(\partial_x, \omega) = -\gamma\frac{\partial}{\partial x_j}$$

$$A_{jk}(\partial_x,\omega) = \delta_{jk}(\mu\Delta + \varrho\omega^2) + (\lambda+\mu)\frac{\partial^2}{\partial x_j \partial x_k},$$
$$B_{4k}(\partial_x,\omega) = i\omega\eta\frac{\partial}{\partial x_k} \qquad j,k = 1,2,3, \qquad B_{44}(\partial_x,\omega) = \Delta + \frac{i\omega}{k}.$$

The matrix $\tilde{B}(\partial_x,\omega) = \|\tilde{B}_{jk}(\partial_x,\omega)\|_{4\times 4}$ denotes the matrix whose elements are:

$$\tilde{B}_{jk}(\partial_x,\omega) = B_{kj}(-\partial_x,\omega).$$

We remark that the operator $\tilde{B}(\partial_x,\omega)$ can be obtained from the operator $B(\partial_x,\omega)$ by replacing γ by $i\omega\eta$.
The matrix of fundamental solutions of the homogeneous equation (2.3) is:

$$\Phi(x,\omega) = \|\Phi_{jk}(x,\omega)\|_{4\times 4},$$

where

$$\begin{aligned}\Phi_{kj}(x,\omega) \quad &= \sum_{l=1}^{3}\Big\{(1-\delta_{k4})(1-\delta_{j4})\Big(\frac{\delta_{kj}}{2\pi\mu}\delta_{3l} - \alpha_l\frac{\partial^2}{\partial x_j\partial x_k}\Big)\\ &+ \beta_l\Big[i\omega\eta\delta_{k4}(1-\delta_{j4})\frac{\partial}{\partial x_j} - \gamma\delta_{j4}(1-\delta_{k4})\frac{\partial}{\partial x_k}\Big]\\ &+ \delta_{k4}\delta_{j4}\gamma_l\Big\}\frac{e^{i\lambda_l|x|}}{|x|},\end{aligned}$$

δ_{kj} denotes Kronecker's symbol and

$$\alpha_l = \frac{(-1)^l(1-i\omega k^{-1}\lambda_l^{-2})(\delta_{1l}+\delta_{2l})}{2\pi(\lambda+2\mu)(\lambda_2^2-\lambda_l^2)} - \frac{\delta_{3l}}{2\pi\varrho\omega^2}, \qquad \sum_{l=1}^{3}\alpha_l = 0,$$
$$\beta_l = \frac{(-1)^l(\delta_{1l}+\delta_{2l})}{2\pi(\lambda+2\mu)(\lambda_2^2-\lambda_l^2)}, \qquad \sum_{l=1}^{3}\beta_l = 0,$$

$$\gamma_l = \frac{(-1)^l(\lambda_l^2 - k_1^l)(\delta_{1l}+\delta_{2l})}{2\pi(\lambda_2^2-\lambda_1^2)}, \qquad \sum_{l=1}^{3}\gamma_l = 1,$$
$$k_1^2 = \lambda_2^2 = \frac{\varrho\omega^2}{\lambda+2\mu}, \quad \lambda_1^2 = \frac{i\omega}{k}, \quad \lambda_3^2 = \frac{\varrho\omega^2}{\mu}.$$

We note that each column vector in the matrix $\Phi(x,\omega)$ has a unique singularity at the point $x=0$ of order not higher than $1/|x|$ and that each column of the matrix $\Phi(x,\omega)$ satisfies system (2.3) everywhere in the space except the origin ([11], Theorem 3.1, p. 96). Moreover, the matrix $\Phi(x,\omega)$ is unsymmetrical and its rows considered as vectors do not satisfy (2.3).

We denote by $\tilde{\Phi}(x,\omega) = \|\tilde{\Phi}_{jk}(x,\omega)\|_{4\times 4}$ the matrix

$$\tilde{\Phi}_{jk}(x,\omega) = \Phi_{kj}(-x,\omega). \tag{2.4}$$

From the matrices $\Phi(x,\omega)$ and $\tilde{\Phi}(x,\omega)$ with $\omega = 0$ we obtain

$$\Phi(x) = \|\Phi_{kj}(x)\|_{4\times 4}, \qquad \tilde{\Phi}(x) = \|\tilde{\Phi}_{kj}(x)\|_{4\times 4} = \|\Phi_{jk}(-x)\|_{4\times 4}.$$

In this case, $\Phi_{kj}(x)$ has the following simple form:

$$\Phi_{kj}(x) = \left\| \begin{matrix} \Gamma_{11}(x) & \Gamma_{12}(x) & \Gamma_{13}(x) & \widetilde{\gamma}\,\frac{x_1}{|x|} \\ \Gamma_{21}(x) & \Gamma_{22}(x) & \Gamma_{23}(x) & \widetilde{\gamma}\,\frac{x_2}{|x|} \\ \Gamma_{31}(x) & \Gamma_{32}(x) & \Gamma_{33}(x) & \widetilde{\gamma}\,\frac{x_3}{|x|} \\ 0 & 0 & 0 & \frac{1}{2\pi}\frac{1}{|x|} \end{matrix} \right\|$$

where

$$\widetilde{\gamma} = \frac{\gamma}{4\pi}\frac{1}{\lambda + 2\mu} \tag{2.5}$$

and $\Gamma = \|\Gamma_{kj}(x)\|_{3\times 3}$ is the *Kelvin matrix* whose entries are ([11], p. 84):

$$\Gamma_{kj}(x) = \frac{1}{2\pi\mu}\Big(\frac{\delta_{kj}}{|x|} - \frac{(\lambda+\mu)}{2(\lambda+2\mu)}\frac{\partial^2}{\partial x_k \partial x_j}|x|\Big). \tag{2.6}$$

We note that the matrices $\Phi(x)$ and $\tilde{\Phi}(x)$ satisfy the homogeneous static equation (2.2).

We recall the following theorems ([11], p. 97) which we shall use in the sequel:

Theorem 2.1 *The elements of the matrices* $[\Phi(x,\omega)-\Phi(x)]$ *and* $[\tilde{\Phi}(x,\omega)-\tilde{\Phi}(x)]$ *are bounded at* $x = 0$*, while the first derivatives have isolated singularity of the kind* $|x|^{-1}$*.*

Theorem 2.2 *The second derivatives of the elements of* $[\Phi_{kj}(x,\omega)-\Phi_{kj}(x)]$ *and* $[\tilde{\Phi}_{kj}(x,\omega)-\tilde{\Phi}_{kj}(x)]$ $(k,j = 1,2,3)$*,* $[\Phi_{44}(x,\omega)-\Phi_{44}(x)]$ *and* $[\tilde{\Phi}_{44}(x,\omega)-\tilde{\Phi}_{44}(x)]$ *have a singularity of the kind* $|x|^{-1}$*, while the second derivatives of the elements* $[\Phi_{k4}(x,\omega)-\Phi_{k4}(x)]$ *and* $[\tilde{\Phi}_{4j}(x,\omega)-\tilde{\Phi}_{4j}(x)]$ $(k,j = 1,2,3)$ *are bounded. Moreover* $\Phi_{44}(x,\omega) = \dfrac{1}{2\pi}\dfrac{1}{|x|} + \mathcal{O}(|x|)$*.*

From now on, Ω is assumed to be a bounded domain of $\mathbb{R}^{\nVdash}$ such that its boundary $\partial\Omega$ is a surface Σ of class $C^{2,\lambda}$, $\lambda \in (0,1]$ and such that $\mathbb{R}^{\nVdash} - \overline{\nleq}$ is connected; $\nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x))$ denotes the unit normal vector at the point $x = (x_1,x_2,x_3) \in \Sigma$ directed outside Ω.

Further we recall the *thermoelastic stress operator* ([11], p. 60):

$$P(\partial_x, \nu(x))U = T(\partial_x, \nu(x))u - \gamma\nu(x)\vartheta \tag{2.7}$$

where $U = (u, \vartheta)$ is a four-components vector and T is the following operator ([11], p. 57):

$$T(\partial_x, \nu(x))u = \lambda\nu(x)\text{div } u + 2\mu\frac{\partial u}{\partial\nu(x)} + \mu(\nu(x) \times \text{curl } u). \tag{2.8}$$

For the later, we shall use the following matrix differential operators:

$$(\partial_x, \nu, \gamma) = \|\mathcal{Q}_{kj}(\partial_x, \nu, \gamma)\|_{4\times4} =$$

$$= \left\| \begin{matrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial\nu} \end{matrix} \right\|;$$

$$(\partial_x, \nu, \gamma) = \|\mathcal{R}_{kj}(\partial_x, \nu, \gamma)\|_{4\times4} =$$

$$= \left\| \begin{matrix} T_{11}(x) & T_{12}(x) & T_{13}(x) & -\gamma\nu_1 \\ T_{21}(x) & T_{22}(x) & T_{23}(x) & -\gamma\nu_2 \\ T_{31}(x) & T_{32}(x) & T_{33}(x) & -\gamma\nu_3 \\ 0 & 0 & 0 & \frac{\partial}{\partial\nu} \end{matrix} \right\|; \tag{2.9}$$

$$(\partial_x, \nu, \gamma) = \|\mathcal{P}_{kj}(\partial_x, \nu, \gamma)\|_{4\times4} =$$

$$= \left\| \begin{matrix} T_{11}(x) & T_{12}(x) & T_{13}(x) & -\gamma\nu_1 \\ T_{21}(x) & T_{22}(x) & T_{23}(x) & -\gamma\nu_2 \\ T_{31}(x) & T_{32}(x) & T_{33}(x) & -\gamma\nu_3 \\ 0 & 0 & 0 & -1 \end{matrix} \right\|, \tag{2.10}$$

where $T_{kj}(x)$ are the elements of (2.8).

In the following we denote by $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{P}}$ the matrices obtained from (2.9) and (2.10) substituting γ by $i\omega\eta$.

If $A = \|a_{kj}\|_{4\times4}$ and $B = \|b_{kj}\|_{4\times4}$ are 4×4 matrices and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is a vector, then

$$AB = \|c_{kj}\|_{4\times4} \quad \text{where} \quad c_{kj} = \sum_i a_{ki}b_{ij};$$

$$A\varphi = \|d_k\|_{4\times 1} \quad \text{where} \quad d_k = \sum_j a_{ki}\varphi_j;$$

$$(AB')'\varphi = BA'\varphi = \| \sum_{ij} a_{kj} b_{ij} \varphi_j \|_{4\times 4};$$

the symbol $'$ denotes the transposition operation.

Now we introduce the *simple layer potential*:

$$V(x) = \int_\Sigma \Phi(x-y,\omega)\varphi(y)d\sigma_y; \tag{2.11}$$

the *double layer potential*:

$$W(x) = \int_\Sigma \left[\widetilde{\mathcal{R}}(\partial_y,\nu)\Phi'(x-y,\omega)\right]'\varphi(y)d\sigma_y; \tag{2.12}$$

the following *mixed potentials*:

$$Z(x) = \int_\Sigma \left[\widetilde{\mathcal{P}}(\partial_y,\nu)\Phi'(x-y,\omega)\right]'\varphi(y)d\sigma_y; \tag{2.13}$$

and

$$Y(x) = \int_\Sigma \left[\mathcal{Q}(\partial_y,\nu)\Phi'(x-y,\omega)\right]'\varphi(y)d\sigma_y. \tag{2.14}$$

Using the four potentials (2.11), (2.12), (2.13), (2.14) presented above, one may develop the general theory for the basic boundary value problems [11]. The first problem consists in finding the four-component vector $U = (u,\vartheta)$ which in Ω satisfies the equations (2.1) by the boundary conditions:

$$u = f, \quad \vartheta = g. \tag{2.15}$$

In second problem the boundary conditions are:

$$PU = f, \quad \frac{\partial\vartheta}{\partial\nu} = g, \tag{2.16}$$

in the third problem

$$u = f, \quad \frac{\partial\vartheta}{\partial\nu} = g, \tag{2.17}$$

and in the fourth problem

$$PU = f, \quad \vartheta = g. \tag{2.18}$$

In the classical thermoelasticity theory (see [11]) the solution of the basic boundary value problems (2.15), (2.16), (2.17), (2.18) are represented in the form of (2.12), (2.11), (2.13), (2.14) respectively.

3 First problem

In this section, we study the Dirichlet problem for the system of thermoelastic pseudo-oscillations:

$$\begin{cases} B(\partial_x,\omega)U = 0 & \text{in } \Omega, \\ u = f, & \text{on } \Sigma, \\ \vartheta = f_4, & \text{on } \Sigma, \end{cases} \tag{3.1}$$

where $\omega = i\tau$, $\Re e\tau > 0$. The data $F = (f, f_4) = (f_1, f_2, f_3, f_4)$ is assumed to be in the space $[W^{1,p}(\Sigma)]^4$, $1 < p < \infty$.

Preliminarily, we consider the following lemmas.

Lemma 3.1 *The singular integral operator*

$$J : L^p(\Sigma) \longrightarrow L^p_1(\Sigma),$$

$$J\varphi(x) = -\frac{1}{4\pi}\int_\Sigma \varphi(y)d_x[\frac{1}{|x-y|}]d\sigma_y, \tag{3.2}$$

where d_x denotes the exterior differentiation [6] , can be reduced on the left and

$$J' : L^p_1(\Sigma) \longrightarrow L^p(\Sigma)$$

$$J'\psi(z) = *_\Sigma \int_\Sigma \psi(x) \wedge d_z[S_1(z,x)] \tag{3.3}$$

is a reducing operator for (3.2), where $S_1(z,x)$ is the double 1*-form introduced by Hodge [10]:*

$$S_1(z,x) = -\frac{1}{4\pi|z-x|}\sum_j dz^j dx^j. \tag{3.4}$$

This theorem is proved in [1]. In particular we have that

$$J'J\varphi(x) = -\frac{1}{4}\varphi(x) + \int_\Sigma \varphi(y)L(x,y)d\sigma_y \tag{3.5}$$

where $L(x,y)$ has a weak singularity and thus the integral defines a compact operator from $L^p(\Sigma)$ into itself.

Lemma 3.2 *The singular integral operator*

$$R : [L^p(\Sigma)]^3 \longrightarrow [L^p_1(\Sigma)]^3$$

$$R_j\varphi(x) = \int_\Sigma \varphi_k(y) d_x[\Gamma_{jk}(x,y)] d\sigma_y \tag{3.6}$$

($j = 1,2,3$), where $\Gamma_{jk}(x,y)$ are given by (2.6), can be reduced on the left and

$$R_0^{'} : [L_1^p(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$$

where

$$R_{0i}^{'}\psi = \frac{(\lambda+\mu)(\lambda+2\mu)}{(\lambda+3\mu)}\mathcal{K}_{jj}(\psi)\nu_i \qquad +2\mu\frac{(\lambda+2\mu)}{(\lambda+3\mu)}\mathcal{K}_{ij}(\psi)\nu_j + \\ + \mu\frac{(\lambda+\mu)}{(\lambda+3\mu)}\delta_{sp}^{ij}\nu_j\mathcal{K}_{ps}(\psi) \tag{3.7}$$

is a reducing operator for (3.6). Here $\mathcal{K}_{ij}$ are the operators defined as follows ([4], p. 37):

$$\mathcal{K}_{js}(\varphi) = *_\Sigma \int_\Sigma d_x[S_1(x,y)] \wedge \varphi_j(y) \wedge dx^s - \delta_{ihp}^{123} \int_\Sigma \frac{\partial}{\partial x_s}[K_{ij}(x,y)] \wedge \varphi_h(y) \wedge dy^p \tag{3.8}$$

where $S_1(x,y)$ is given by (3.4) and

$$K_{ij}(x,y) = \frac{1}{4\pi}\Big[\mu\frac{(\lambda+\mu)}{(\lambda+3\mu)}\frac{\partial|x-y|}{\partial y_j}\frac{\partial|x-y|}{\partial y_i}\Big]\frac{1}{|x-y|}.$$

This lemma is proved in [4]. In particular we have that

$$R_0' R\varphi = -\frac{1}{4}\varphi + K^2\varphi, \tag{3.9}$$

where

$$K_j\varphi(x) = \int_\Sigma \varphi_h(y) T_{jx}^0[\Gamma^h(x,y)]\, d\sigma_y,$$

and $T^0 u$ is the *pseudostress* [11].

Lemma 3.3 *The singular integral operator*

$$S_0 : [L^p(\Sigma)]^4 \longrightarrow [L_1^p(\Sigma)]^4$$

defined as

$$S_0(\varphi)_k = (1-\delta_{k4})[R(\varphi)_k + T_k(\varphi_4)] - 2\delta_{k4}J(\varphi_4) \qquad k = 1,2,3,4 \tag{3.10}$$

where R and J are given by (3.6) and (3.2) respectively,

$$T_k\varphi_4(x) = \tilde{\gamma}\int_\Sigma d_x\Big[\frac{x_k - y_k}{|x-y|}\Big]\ \varphi_4(y)\ d\sigma_y, \qquad k = 1,2,3 \tag{3.11}$$

and $\tilde{\gamma}$ *is given by (2.5), can be reduced on the left by*

$$S' : [L_1^p(\Sigma)]^4 \longrightarrow [L^p(\Sigma)]^4$$

defined as

$$S'(\psi)_k = (1-\delta_{k4})R_0'(\psi)_k - \frac{1}{2}\delta_{k4}J'(\psi_4), \qquad k = 1,2,3,4 \tag{3.12}$$

where R_0' *and* J' *are given by (3.3) and (3.7) respectively.*

Proof. We have that

$$\begin{aligned} S'S_0(\psi)_k \quad &= (1-\delta_{k4})R_0'(S_0\psi)_k + \delta_{k4}J'(S_0\psi_4) = \\ &= (1-\delta_{k4})^2((R_0'R\psi)_k + R'T_k(\psi_4)) + \delta_{k4}^2 J'J\psi_4, \end{aligned}$$

and this is a Fredholm operator in view of (3.5), (3.9) and the compactness of operators T_k.

Lemma 3.4 *Let* ω *be a complex constant. The singular integral operator*

$$S : [L^p(\Sigma)]^4 \longrightarrow [L_1^p(\Sigma)]^4$$

defined as

$$S\varphi(x) = \int_\Sigma \varphi(y)d_x[\Phi(x-y,\omega)]d\sigma_y$$

can be reduced on the left by S' *(3.12).*

Proof. Set

$$S = (S - S_0) + S_0.$$

The operator S' reduces S because

$$S'S = S'(S-S_0) + S'S_0$$

is a Fredholm operator, since $(S-S_0)$ is compact as it follows from theorem 2.1.

Theorem 3.1 *Let* ω *be a complex constant. Given* $F \in [W^{1,p}(\Sigma)]^4$*, there exists a solution of the singular integral system:*

$$\int_\Sigma d_x[\Phi(x-y,\omega)]\varphi(y)d\sigma_y = dF(x), \ ([1])$$

[1] dF is the vector (dF_1, dF_2, dF_3, dF_4).

if, and only if,

$$\int_\Sigma \gamma_i \wedge dF_i = 0 \tag{3.13}$$

for any $\gamma \in [L_1^q(\Sigma)]^4$ *solution of the homogeneous adjoint system:*

$$S_j^*\gamma(x) \equiv \int_\Sigma \gamma_i(y) \wedge d_y[\Phi_{ij}(y-x,\omega)] = 0 \qquad \text{a.e. } x \in \Sigma \tag{3.14}$$

$j = 1, 2, 3, 4.$

Proof. Because of lemma 3.4 the range of S is closed in $[L_1^p(\Sigma)]^4$. This implies the result.

In order to apply this general result, it is necessary to determine explicitly the compatibility condition (3.13). We shall prove that (3.13) are automatically satisfied in the thermoelastic pseudo-oscillation's theory. In this case the nonnegative number

$$\sigma_\varepsilon = \frac{\lambda + 2\mu}{\varrho k}(1-\varepsilon), \tag{3.15}$$

where

$$\varepsilon = \frac{k\gamma\eta}{\lambda + 2\mu},$$

plays an important role. In fact in [11], p. 572, it is shown that if $\Re e\ \tau > \sigma_\varepsilon$, the basic boundary value problems of the theory of thermoelasticity are solvable and the solutions can be represented by means of the usual integral representations.

Theorem 3.2 *If* $\Re e\ \tau > \sigma_\varepsilon$ *where* σ_ε *is given by (3.15), the vector* $\gamma \in [L_1^q(\Sigma)]^4$ *is an eigensolution of* S^* *if, and only if,* γ *is a weakly closed form, i.e.*

$$\int_\Sigma \gamma_i \wedge dg_i = 0, \qquad \forall\ g \in [C^\infty(\mathbb{R}^3)]^4.$$

Proof. If γ belongs to kernel of S^*, from (3.14) it follows that for any $p \in [C^\lambda(\Sigma)]^4$ we have

$$0 = \int_\Sigma p_j(x)\, d\sigma_x \int_\Sigma \gamma_i(y) \wedge d_y[\Phi_{ij}(y-x,\omega)] =$$
$$\int_\Sigma \gamma_i(y) \wedge d_y \int_\Sigma p_j(x)\, \Phi_{ij}(y-x,\omega) d\sigma_x.$$

Let u be a smooth solution of $BU = 0$. We can represent u by means of a simple layer potential ([11], p. 544):

$$u_i(y) = \int_\Sigma p_j(x)\, \Phi_{ij}(y-x,\omega) d\sigma_x.$$

Then we obtain that

$$\int_{\Sigma} \gamma_i \wedge du_i = 0.$$

Therefore

$$\int_{\Sigma} \gamma_i(y) \wedge d_y[\Phi_{ij}(y-x,\omega)] = 0, \qquad \forall\, x \in \mathbb{R}^{\nVdash} - \overline{\nleq}. \tag{3.16}$$

Let us denote by $w_j(x)$, $j = 1,2,3,4$, the left-hand side of (3.16). It follows from (2.4) that $\widetilde{B}w = 0$. Then

$$w_j(x) = \int_{\Sigma} \gamma_i(y) \wedge d_y[\widetilde{\Phi}_{ji}(x-y,\omega)]$$

and, for fixed i, $(\widetilde{\Phi}_{1i}, \widetilde{\Phi}_{2i}, \widetilde{\Phi}_{3i}, \widetilde{\Phi}_{4i})(x-y,\omega)$ is solution of the system $\widetilde{B}_x u = 0$.

Now we consider $v \in [C^{\infty}(\mathbb{R}^{\nVdash})]^{\ntrianglerighteq}$ and $\eta \in [C^1(\overline{\Omega})]^4 \cap [C^2(\Omega)]^4$ such that $B\eta = Bv$ in Ω and $\eta = 0$ on Σ. Such a solution does exist because of ([11], Theorem 3.1, p. 572).

We have that

$$\begin{aligned}\int_{\Omega} w_j\, B_j v\, dx \qquad &= \int_{\Omega} w_j\, B_j\eta\, dx = \int_{\Omega} B_j\eta(x)\, dx \int_{\Sigma} \gamma_i(y) \wedge d_y[\widetilde{\Phi}_{ji}(x-y,\omega)] \\ &= \int_{\Sigma} \gamma_i(y) \wedge d_y \int_{\Omega} B_j\eta(x)\, \widetilde{\Phi}_{ji}(x-y,\omega)\, dx.\end{aligned}$$

We remark that, from [11], (2.15) p. 536, it follows that

$$\eta_i(y) = \int_{\Sigma} \widetilde{\Phi}_{ji}(x-y,\omega)\, \mathcal{R}_j\eta(x)\, d\sigma_x - \int_{\Omega} \widetilde{\Phi}_{ji}(x-y,\omega)\, B_j\eta(x)\, dx, \qquad y \in \Omega \tag{3.17}$$

where $\mathcal{R}_j$ are the operators defined by (2.9). Letting $y \to \Sigma$, (3.17) gives

$$\int_{\Sigma} \widetilde{\Phi}_{ji}(x-y,\omega)\, \mathcal{R}_j\eta(x)\, d\sigma_x = \int_{\Omega} \widetilde{\Phi}_{ji}(x-y,\omega)\, B_j\eta(x)\, dx \qquad y \in \Sigma. \tag{3.18}$$

Therefore, from (3.18), (2.4), and (3.14), we get

$$\begin{aligned}\int_{\Omega} w_j\, B_j v\, dx &= \int_{\Sigma} \gamma_i(y) \wedge d_y \int_{\Sigma} \widetilde{\Phi}_{ji}(x-y,\omega)\, \mathcal{R}_j\eta(x)\, d\sigma_x = \\ &= \int_{\Sigma} \mathcal{R}_j\eta(x)\, d\sigma_x \int_{\Sigma} \gamma_i(y) \wedge d_y[\widetilde{\Phi}_{ji}(x-y,\omega)] = 0.\end{aligned} \tag{3.19}$$

Let $\psi \in [C^{\infty}(\mathbb{R}^{\nVdash})]^{\ntrianglerighteq}$ be a function with compact support. It follows from (3.17) that

$$\psi_i(y) = -\int_{\mathbb{R}^{\nVdash}} \widetilde{\Phi}_{ji}(x-y,\omega)\, B_j\psi(x)\, dx \qquad \forall\, y \in \mathbb{R}^{\nVdash}$$

and then, keeping in mind (3.16) and (3.19),

$$0 = \int_{\mathbb{R}^{\nVdash}} w_j B_j \psi \, dx = \int_{\mathbb{R}^{\nVdash}} B_j \psi(x) \, dx \int_\Sigma \gamma_i(y) \wedge d_y [\widetilde{\Phi}_{ji}(x-y,\omega)] =$$
$$= \int_\Sigma \gamma_i(y) \wedge d_y \int_{\mathbb{R}^{\nVdash}} B_j \psi(x) \, \widetilde{\Phi}_{ji}(x-y,\omega) \, dx = -\int_\Sigma \gamma_i \wedge d\psi_i.$$

and this concludes the proof.

Lemma 3.5 *The solution of the following boundary value problem*

$$\begin{cases} B(\partial_x, \omega)V = 0 & \textit{in } \Omega, \\ V = C, & \textit{on } \Sigma \end{cases} \tag{3.20}$$

where $C = (c_1, c_2, c_3, c_4) \in \mathbb{R}^{\ntrianglerighteq}$ *can be represented by a simple layer potential (2.11) with density* $\varphi_0 \in C^{1,\lambda'}(\Sigma)$, $0 < \lambda < \lambda' \leq 1$.

Proof. The solution $V = (v, \tilde{\vartheta})$ of (3.20) can be represented in the form of double layer potential (2.12) ([11], p. 572):

$$v(x) = \int_\Sigma \left[\widetilde{\mathcal{R}}(\partial_y, \nu) \Phi'(z-y,\omega) \right]' \varphi_0(y) \, d\sigma_y.$$

It follows from [11], p. 544, that

$$-\varphi_0(z) + \int_\Sigma \left[\widetilde{\mathcal{R}}(\partial_y, \nu) \Phi'(z-y,\omega) \right]' \varphi_0(y) d\sigma_y = C.$$

Because C is constant, we obtain from theorem 3.3 of [11], p. 359 that $\varphi_0 \in C^{1,\lambda'}(\Sigma)$, $0 < \lambda' < \lambda \leq 1$ and it follows from theorem 7.1 of [11], p. 317, that $v(z) \in C^{1,\lambda}(\overline{\Omega})$. Now we consider the following boundary value problem

$$\begin{cases} B(\partial_x, \omega)U = 0 & \text{in } \Omega, \\ Pu = Pv, & \text{on } \Sigma \\ \dfrac{\partial \vartheta}{\partial \nu} = \dfrac{\partial \tilde{\vartheta}}{\partial \nu} & \text{on } \Sigma \end{cases} \tag{3.21}$$

where $(v, \tilde{\vartheta})$ is the solution of (3.20). Because the solution U of (3.21) can be represented by a simple layer potential (2.11) ([11], p. 572) and since (3.21) has only one solution we obtain $U = V$.

Now we show the following representation's theorem of Dirichlet problem for pseudo-oscillation's theory.

Theorem 3.3 *Given $F \in [W^{1,p}(\Sigma)]^4$, the solution of problem (3.1) where $\omega = i\tau$, $\Re e\ \tau > \sigma_\varepsilon$ and σ_ε is given by (3.15), does exist and can be represented by a simple layer potential (2.11) with density in the space $[L^p(\Sigma)]^4$.*

Proof. We seek a solution of Dirichlet problem for the pseudo-oscillation system (2.1) in the form of a simple layer potential (2.11) $V(x)$. Imposing the boundary condition we obtain the following integral system of the first kind:

$$\int_\Sigma \Phi(x-y,\omega)\,\varphi(y)\,d\sigma_y = F(x), \qquad x \in \Sigma.$$

Taking the differential of both sides on Σ, we have the following system of singular integral equations:

$$\int_\Sigma d_x[\Phi(x-y,\omega)]\,\varphi(y)\,d\sigma_y = dF(x), \qquad x \in \Sigma,$$

(in which the unknown is a vector whose components are scalar function $\varphi_i \in L^p(\Sigma)$, $i = 1,2,3,4$, while the data is a vector whose components are differential forms of degree one). Then we obtain the equation

$$S\varphi = dF. \tag{3.22}$$

It follows from lemma 3.4 that S' reduces S and from theorem 3.1, that there exists a solution of (3.22) if and only if the data dF satisfies (3.13). These compatibility conditions are satisfied in view of theorem 3.2 and then there exists a solution of (3.22).
This means that there exists a solution of the following boundary value problem

$$\begin{cases} B(\partial_x,\omega)W = 0 & \text{in } \Omega, \\ dW = dF, & \text{on } \Sigma \end{cases}$$

and that W can be represented by a simple layer potential (2.11). Since $dW = dF$ on Σ, we have $W = F + C$ on Σ, where $C \in \mathbb{R}^{\ntrianglerighteq}$. Let V be the solution of the boundary value problem (3.20). It is clear that $W - V$ solves the boundary value problem (3.1) and it can be represented by a simple layer potential in view of lemma 3.5.

We observe that the hypothesis $\Re e\ \tau > \sigma_\varepsilon$ assures that the solution of the first problem is unique as shown in [11], Theorem 3.1, p. 572.

4 Third problem

In this section, we study the third boundary value problem of equations of thermoelastic pseudo-oscillations (2.1) with boundary conditions (2.16):

$$\begin{cases} B(\partial_x, \omega) = 0, & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \\ \dfrac{\partial \vartheta}{\partial \nu} = f_4 & \text{on } \Sigma, \end{cases} \tag{4.1}$$

where $\omega = i\tau$, $\Re e\tau > 0$, $f = (f_1, f_2, f_3) \in [W^{1,p}(\Sigma)]^3$ and $f_4 \in L^p(\Sigma)$, $1 < p < \infty$, satisfies the condition $\int_\Sigma f_4 \, d\sigma = 0$.
First we prove the following results.

Lemma 4.1 *The operator*

$$S_0 : [L^p(\Sigma)]^3 \longrightarrow [L_1^p(\Sigma)]^3$$

defined as

$$S_{0k}\varphi = R_k(\varphi) + T_k(\varphi_4) \qquad k = 1, 2, 3 \tag{4.2}$$

where R is given by (3.6) and T_k are given by (3.11), can be reduced on the left by

$$S' : [L_1^p(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$$

defined as

$$S'_k(\psi) = R'_{0k}(\psi), \qquad k = 1, 2, 3 \tag{4.3}$$

where R'_{0k} is given by (3.7).

Proof. We have that

$$S'S_0(\psi)_k = R'_0(S_0\psi)_k = (R'_0 R\psi)_k + R'T_k(\psi_4)$$

and this is a Fredholm operator, as it follows from lemma 3.2 and from the compactness of the operators T_k (3.11).

Lemma 4.2 *Let ω be a complex constant. The singular integral operator*

$$S_k : L^p(\Sigma) \longrightarrow L_1^p(\Sigma), \qquad k = 1, 2, 3$$

given by

$$(S\varphi)_k(x) = \sum_{j=1}^{4} \int_\Sigma d_x[\Phi_{kj}(x - y, \omega)] \, \varphi_j(y) \, d\sigma_y \qquad k = 1, 2, 3$$

can be reduced on the left by $S'_k : L_1^p(\Sigma) \longrightarrow L^p(\Sigma)$ $(k = 1, 2, 3)$ where S'_k is given by (4.3).

Proof. Set

$$S_k = (S_k - (S_0)_k) + (S_0)_k, \qquad k = 1, 2, 3$$

we have that S'_k reduces S_k. In fact

$$(S'S)_k = S'_k(S_k - (S_0)_k) + (S'S_0)_k, \qquad k = 1, 2, 3$$

is a Fredholm operator, because $(S - S_0)_k$ is a compact operator as it follows from the theorem 2.1.

Theorem 4.1 *Let ω be a complex constant. Given $f \in [W^{1,p}(\Sigma)]^3$, there exists a solution of the singular integral system*

$$\sum_{j=1}^{4} \int_\Sigma d_x[\Phi_{kj}(x - y, \omega)]\, \varphi_j(y)\, d\sigma_y = df_k(x), \qquad k = 1, 2, 3$$

if, and only if,

$$\int_\Sigma \gamma_i \wedge df_i = 0, \qquad i = 1, 2, 3 \tag{4.4}$$

for any $\gamma \in [L^q_1(\Sigma)]^3$ solution of the homogeneous adjoint system:

$$S^*_j \gamma(x) \equiv \int_\Sigma \gamma_i(y) \wedge d_y[\Phi_{ij}(y - x, \omega)] = 0 \qquad \text{a.e. } x \in \Sigma \tag{4.5}$$

$j = 1, 2, 3.$

Proof. Because of lemma 4.2 the range of S_k is closed in $L^p_1(\Sigma)$, and we have the result.

Now we recall the following theorem proved in [4] for any number of variables.

Theorem 4.2 *For any $g \in L^p(\Sigma)$ such that $\int_\Sigma g\, d\sigma = 0$, the solution of Neumann problem for Laplace equation*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \Sigma, \end{cases}$$

can be represented in the form of a double layer potential

$$u(x) = -\frac{1}{4\pi} \int_\Sigma \psi(y) \Big[\frac{\partial}{\partial \nu_y} \frac{1}{|x - y|}\Big]\, d\sigma_y, \qquad x \in \Omega, \tag{4.6}$$

with $\psi \in W^{1,p}(\Sigma)$.

Let us remark that the proof of this theorem hinges on the following identity:

$$J'(d\psi) = -g, \tag{4.7}$$

where J' is the operator previously introduced in (3.3).
Now we give the main result of this section.

Theorem 4.3 *Given $F = (f, f_4) \in [W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$ such that $\int_\Sigma f_4 \, d\sigma = 0$, the solution of the problem (4.1), where $\omega = i\tau$, $\Re e\ \tau > \sigma_\varepsilon$ and σ_ε is given by (3.15), does exist and can be represented by a mixed potential (2.14) with density in $[L^p(\Sigma)]^3 \times W^{1,p}(\Sigma)$.*

Proof. We seek a solution of (4.1) in the form of a mixed potential (2.14): Y. According to definition of Y, we have that

$$\begin{cases} Y_k(x) = \displaystyle\sum_{i,j=1}^{4} \int_\Sigma \Phi_{kj}(x-y,\omega)\varphi_j(y)\ d\sigma_y, \qquad k = 1,2,3, \\ Y_4(x) = \displaystyle\sum_{j=1}^{4} \int_\Sigma \frac{\partial}{\partial \nu_y} \Phi_{4j}(x-y,\omega)\varphi_j(y)\, d\sigma_y. \end{cases} \tag{4.8}$$

For $k = 1, 2, 3$, imposing the boundary condition $u = f$ in $(4.8)_1$, we have an integral system of the first kind and, taking the differential, we obtain the following system of singular integral equations:

$$\sum_{j=1}^{4} \int_\Sigma d_x[\Phi_{kj}(x-y,\omega)]\, \varphi_j(y)\, d\sigma_y = df_k(x), \qquad x \in \Sigma, \quad k = 1,2,3.$$

Thus we have the equation:

$$(S\varphi)_k = df_k, \qquad k = 1,2,3. \tag{4.9}$$

It follows from lemma 4.2 that S'_k reduces S_k. Therefore, from theorem 4.1 there exists a solution φ of (4.9) if, and only if, df_k satisfy the compatibility conditions (4.4) for $k = 1, 2, 3$. The (4.4) are satisfied in view of theorem 3.2. The thesis follows like in the theorem 3.3.
With regard to the last component, we consider

$$Y_4(x) = Y_{04}(x) + (Y_4(x) - Y_{04}(x)),$$

where Y_{04} is given by

$$\frac{1}{2\pi} \int_\Sigma \frac{\partial}{\partial \nu_y} \Big[\frac{1}{|x-y|} \Big] \varphi_4(y)\, d\sigma_y. \tag{4.10}$$

In view of (4.7) and theorem 2.2, imposing the boundary condition $(4.1)_4$ we obtain a Fredholm equation.

5 Second and fourth problem

We consider the following representation theorem related to the *traction three-dimensional problem* (5.2) of elasticity theory for an isotropic homogeneous body.

Theorem 5.1 *For any* $f \in [L^p(\Sigma)]^3$, $1 < p < \infty$, *such that*

$$\int_\Sigma f \cdot (a + b \wedge x) d\sigma = 0, \quad \forall\ a, b \in \mathbb{R}^{\nvDash} \tag{5.1}$$

any solution of the following boundary value problem

$$\begin{cases} \mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div}\ u = 0 & \textit{in } \Omega, \\ Tu = f & \textit{on } \Sigma, \end{cases} \tag{5.2}$$

where T *is given by (2.8), can be represented in the form of an elastic double layer potential*

$$w(x) = \int_\Sigma [T(\partial_y, \nu)\Gamma(y - x)]' \varphi(y)\, d\sigma_y, \tag{5.3}$$

with the density $\varphi \in [W^{1,p}(\Sigma)]^3$. *Moreover (5.3) is a solution of (5.2) if, and only if, its density* φ *is given by*

$$\varphi(x) = \int_\Sigma \psi(y)\Gamma(x, y)\, d\sigma_y, \qquad x \in \Sigma,$$

$\psi \in [L^p(\Sigma)]^3$ *being a solution of the singular integral equation*

$$-\psi + V^2\psi = f$$

where V *is given by*

$$V\psi(x) = \int_\Sigma [T(\partial_y, \nu)\Gamma(x, y)]' \psi(y)\, d\sigma_y.$$

This theorem is proved in [12].

Now we show the following representation's theorem of second boundary value problem (5.4) for system of thermoelastic pseudo-oscillations.

Theorem 5.2 *Given* $(f, f_4) \in [L^p(\Sigma)]^3 \times W^{1,p}(\Sigma)$, $1 < p < \infty$, *such that (5.1) are satisfied, the solution of the fourth boundary value problem*

$$\begin{cases} B(\partial_x, \omega) = 0, & \textit{in } \Omega, \\ PU = f & \textit{on } \Sigma, \\ \vartheta = f_4 & \textit{on } \Sigma \end{cases} \tag{5.4}$$

where $\omega = i\tau$, $\Re e\ \tau > \sigma_\varepsilon$, σ_ε *is given by (3.15) and* P *is the thermoelastic stress (2.7), can be represented by a mixed potential (2.13) with density* $\varphi \in [W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$.

Proof. Let us seek a solution $U = (u, \vartheta)$ of (5.4) by means of a mixed potential (2.13) Z. According to the definition of $Z = (z, Z_4) = (Z_1, Z_2, Z_3, Z_4)$, we have for the components $Z_k(x)$, $k = 1, 2, 3, 4$

$$Z_k(x) = \sum_{i,j=1}^{4} \int_\Sigma \widetilde{\mathcal{P}}_{ji} \Phi_{ki}(x-y,\omega)\varphi_j(y)\, d\sigma_y.$$

Taking into account the definition of the operator $\widetilde{\mathcal{P}}$ (2.10) and the symmetry property of $\Phi_{ki} = \Phi_{ik}$, when $i, k = 1, 2, 3$, we can write for $k = 1, 2, 3, 4$:

$$\begin{aligned} Z_k(x) \quad &= \int_\Sigma \sum_{i,j=1}^{3} \widetilde{\mathcal{P}}_{ji} \Phi_{ki}(x-y,\omega)\varphi_j(y)\, d\sigma_y + \\ &+ \int_\Sigma \sum_{i=1}^{3} \widetilde{\mathcal{P}}_{4i} \Phi_{ki}(x-y,\omega)\varphi_4(y)\, d\sigma_y + \\ &+ \int_\Sigma \sum_{j=1}^{3} \widetilde{\mathcal{P}}_{j4} \Phi_{k4}(x-y,\omega)\varphi_j(y)\, d\sigma_y + \\ &+ \int_\Sigma \widetilde{\mathcal{P}}_{44} \Phi_{k4}(x-y,\omega)\varphi_4(y)\, d\sigma_y = \\ &= \int_\Sigma \sum_{i,j}^{3} T_{ji} \Phi_{ik}(x-y,\omega)\varphi_j(y)\, d\sigma_y - \\ &- i\omega\eta \int_\Sigma \sum_{j=1}^{3} \nu_j \Phi_{k4}(x-y,\omega)\varphi_j(y)\, d\sigma_y - \\ &- \int \Phi_{k4}(x-y,\omega)\varphi_4(y)\, d\sigma_y. \end{aligned}$$

This expression is transformed as follows:

$$\begin{aligned} Z_k(x) \quad &= \int_\Sigma \sum_{i,j=1}^{3} T_{ji} \Phi_{ik}(x-y)\varphi_j(y)\, d\sigma_y + \\ &+ \int_\Sigma \sum_{i,j}^{3} T_{ji} [\Phi_{ik}(x-y,\omega) - \Phi_{ik}(x-y)]\varphi_j(y)\, d\sigma_y - \\ &- i\omega\eta \int_\Sigma [\sum_{j=1}^{3} \nu_j \varphi_j(y) + \frac{1}{i\omega\eta}\varphi_4(y)] \Phi_{k4}(x-y,\omega)\, d\sigma_y, \\ &\qquad k = 1, 2, 3, 4. \end{aligned} \tag{5.5}$$

The first term of the right hand side for $k = 1, 2, 3$ is nothing else but the component $w_k(x)$ of the (elastic) double layer potential (5.3). The second term, that we denote by $E_k(x)$, is a compact operator as it follows from the theorem 2.1. In the last term, which we denote by $L_k(x)$, for $k = 1, 2, 3$, we have that $\Phi_{k4}(x - y, \omega) = \mathcal{O}(1)$ ([11], p. 530). The last component Z_4 is equal to the fourth component V_4 of the simple layer potential (2.11) plus compact components.

Imposing the boundary conditions: $PU = f$, $\vartheta = f_4$ and taking into account the definition of

$$PZ = Tz - \gamma\nu Z_4$$

we have from (5.5)

$$\begin{cases} Tw_k + TE_k + TL_k - \gamma\nu Z_4 = f_k & k = 1, 2, 3 \\ Z_4 = f_4. \end{cases}$$

Thus we obtain a Fredholm equation in view of theorem 5.1 and the previous remarks on E_k, L_k and Z_4.

Finally, we study the so-called second boundary value problem (5.6) of thermoelastic pseudo-oscillation's equations showing the following theorem.

Theorem 5.3 *Given* $(f, f_4) \in [L^p(\Sigma)]^4$,$1 < p < \infty$, *such that (5.1) and* $\int_\Sigma f_4 \, d\sigma = 0$ *are satisfied, the solution of the second boundary value problem*

$$\begin{cases} B(\partial_x, \omega) = 0, & in\ \Omega, \\ PU = f & on\ \Sigma, \\ \dfrac{\partial\vartheta}{\partial\nu} = g & on\ \Sigma. \end{cases} \tag{5.6}$$

where $\omega = i\tau$, $\Re e\ \tau > \sigma_\varepsilon$, σ_ε *is given by (3.15) and* P *is the thermoelastic stress (2.7), can be represented by a double layer potential (2.12) with density in the space* $[W^{1,p}(\Sigma)]^4$.

Proof. We want to represent a solution of this problem (5.6) by means of a double layer potential (2.12) W. Taking into account the definition of the operator $\widetilde{\mathcal{R}}$ (2.9) and the symmetry property of $\Phi_{ki} = \Phi_{ik}$, $i, k = 1, 2, 3$ we can write for $k = 1, 2, 3, 4$:

$$\begin{aligned} W_k(x) \quad &= \int_\Sigma \sum_{i,j=1}^{3} \widetilde{\mathcal{R}}_{ji} \Phi_{ki}(x - y, \omega) \varphi_j(y) \, d\sigma_y + \\ &+ \int_\Sigma \sum_{i=1}^{3} \widetilde{\mathcal{R}}_{4i} \Phi_{ki}(x - y, \omega) \varphi_4(y) \, d\sigma_y + \end{aligned}$$

$$
\begin{aligned}
&+\int_\Sigma \sum_{j=1}^{3} \widetilde{\mathcal{R}}_{j4}\Phi_{k4}(x-y,\omega)\varphi_j(y)\,d\sigma_y +\\
&+\int_\Sigma \widetilde{\mathcal{R}}_{44}\Phi_{k4}(x-y,\omega)\varphi_4(y)\,d\sigma_y =\\
&=\int_\Sigma \sum_{i,j}^{3} T_{ji}\Phi_{ik}(x-y,\omega)\varphi_j(y)\,d\sigma_y -\\
&-i\omega\eta\int_\Sigma \sum_{j=1}^{3} \nu_j\Phi_{k4}(x-y,\omega)\varphi_j(y)\,d\sigma_y +\\
&+\int \frac{\partial}{\partial\nu}\Big[\Phi_{k4}(x-y,\omega)\Big]\varphi_4(y)\,d\sigma_y.
\end{aligned}
$$

This expression is transformed as follows:

$$
\begin{aligned}
W_k(x) \quad &= \int_\Sigma \sum_{i,j=1}^{3} T_{ji}\Phi_{ik}(x-y)\varphi_j(y)\,d\sigma_y +\\
&+\int_\Sigma \sum_{i,j}^{3} T_{ji}[\Phi_{ik}(x-y,\omega)-\Phi_{ik}(x-y)]\varphi_j(y)\,d\sigma_y -\\
&-i\omega\eta\int_\Sigma \sum_{j=1}^{3} \nu_j\varphi_j(y)\Big[\Phi_{k4}(x-y,\omega)-\Phi_{k4}(x-y)\Big]\,d\sigma_y -\\
&-i\omega\eta\int_\Sigma \sum_{j=1}^{3} \nu_j\varphi_j(y)\Phi_{k4}(x-y)\,d\sigma_y +\\
&+\int_\Sigma \frac{\partial}{\partial\nu}\Big[\Phi_{k4}(x-y,\omega)-\Phi_{k4}(x-y)\Big]\varphi_4(y)\,d\sigma_y +\\
&+\int_\Sigma \frac{\partial}{\partial\nu}\Big[\Phi_{k4}(x-y)\Big]\varphi_4(y)\,d\sigma_y,\\
&\qquad\qquad\qquad\qquad k=1,2,3,4.
\end{aligned}
$$

Imposing the boundary condition $PW = f$, $\dfrac{\partial w_4}{\partial\nu} = f_4$ we obtain a system of Fredholm equations in view of theorems 2.1, 2.2, 5.1, 4.2.

6 Representation theorems

In this section, we combine the classical representation theorems [11] with the results obtained in this paper. This leads to show that the solutions of each of the four basic boundary value problems (3.1), (4.1), (5.4), (5.6) can be represented by any of the thermoelasto-potentials (2.11), (2.12), (2.13), (2.14). The space where the data are given has to be chosen according to the used representation.

Theorem 6.1 *The solution U of the first boundary value problem (3.1) can be represented by:*

1. *a simple layer potential (2.11) with density in $[L^p(\Sigma)]^4$ provided that the data (f, g) is assumed to be in the space $[W^{1,p}(\Sigma)]^4$, $1 < p < \infty$;*
2. *a double layer potential (2.12) with density in $[L^p(\Sigma)]^4$ provided that the data (f, g) is assumed to be in the space $[L^p(\Sigma)]^4$, $1 < p < \infty$;*
3. *a mixed potential (2.14) with density in $[L^p(\Sigma)]^4$ provided that the data (f, g) is assumed to be in the space $[W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$, $1 < p < \infty$;*
4. *a mixed potential (2.13) with density in $[L^p(\Sigma)]^4$ provided that the data (f, g) is assumed to be in the space $W^{1,p}(\Sigma) \times [L^p(\Sigma)]^3$, $1 < p < \infty$.*

Proof. The first statement is proved in § 3. The second one is considered in [11]. If $(f, g) \in [W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$ we can represent a solution of (3.1) in the form of mixed potential (2.14):

$$\begin{cases} Y_k(x) = \displaystyle\sum_{i,j=1}^{4} \int_\Sigma \Phi_{kj}(x-y,\omega)\varphi_j(y)\, d\sigma_y, \qquad k = 1,2,3, \\ Y_4(x) = \displaystyle\sum_{j=1}^{4} \int_\Sigma \frac{\partial}{\partial \nu_y}\Phi_{4j}(x-y,\omega)\varphi_j(y)\, d\sigma_y. \end{cases}$$

For $k = 1,2,3$ we proceed as in theorem 4.3 and for the last component we apply the classical method [11]. Finally, if $(f, g) \in W^{1,p}(\Sigma) \times [L^p(\Sigma)]^3$ we can represent a solution of (3.1) in the form of mixed potential (2.13):

$$Z_k(x) = \sum_{i,j=1}^{4} \int_\Sigma \widetilde{\mathcal{P}}_{ji}\Phi_{ki}(x-y,\omega)\varphi_j(y)\, d\sigma_y.$$

In fact, we apply the classical method [11] to the first three components of Z_k, and the method used in theorem 5.2 to the last component of Z_4.

The following results corresponding to the boundary value problems (5.6), (4.1) and (5.4) can be proved arguing as in theorem 6.1.

Theorem 6.2 *The solution U of the second boundary value problem (5.6) can be represented by:*

1. *a simple layer potential (2.11) with density in* $[L^p(\Sigma)]^4$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*;*

2. *a double layer potential (2.12) with density in* $[W^{1,p}(\Sigma)]^4$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*;*

3. *a mixed potential (2.14) with density in* $[W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*;*

4. *a mixed potential (2.13) with density in* $W^{1,p}(\Sigma) \times [L^p(\Sigma)]^3$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*.*

Theorem 6.3 *The solution* U *of the third boundary value problem (4.1) can be represented by:*

1. *a simple layer potential (2.11) with density in* $[L^p(\Sigma)]^4$ *if the data* (f, g) *is assumed to be in the space* $[W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$, $1 < p < \infty$*;*

2. *a double layer potential (2.12) with density in* $[L^p(\Sigma)]^3 \times W^{1,p}(\Sigma)$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*;*

3. *a mixed potential (2.14) with density in* $[L^p(\Sigma)]^3 \times W^{1,p}(\Sigma)$ *if the data* (f, g) *is assumed to be in the space* $[W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$, $1 < p < \infty$*;*

4. *a mixed potential (2.13) with density in* $[L^p(\Sigma)]^4$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*.*

Theorem 6.4 *The solution* U *of the fourth boundary value problem (5.4) can be represented by:*

1. *a simple layer potential (2.11) with density in* $[L^p(\Sigma)]^4$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^3 \times W^{1,p}(\Sigma)$, $1 < p < \infty$*;*

2. *a double layer potential (2.12) with density in* $[W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^4$, $1 < p < \infty$*;*

3. *a mixed potential (2.14) with density in* $[L^p(\Sigma)]^4$ *if the data* (f, g) *is assumed to be in the space* $[L^{1,p}(\Sigma)]^4$, $1 < p < \infty$*;*

4. *a mixed potential (2.13) with density in* $[W^{1,p}(\Sigma)]^3 \times L^p(\Sigma)$ *if the data* (f, g) *is assumed to be in the space* $[L^p(\Sigma)]^3 \times W^{1,p}(\Sigma)$, $1 < p < \infty$*.*

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