FOURTH ORDER OF ACCURACY KRANC-NICKOLSON TYPE DECOMPOSITION SCHEME FOR EVOLUTION PROBLEM

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Abstract

In the present work symmetrized sequential-parallel type decomposition difference scheme of the fourth degree precision for the solution of Cauchy abstract problem is offered. The fourth degree precision is reached by introducing the complex parameter $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ and by the approximation of the semigroup through the rational approximation. For the considered scheme the explicit a priori estimation is obtained.

Key words and phrases: Decomposition method, Operator split, Semigroup, Trotter formula, Cauchy abstract problem, Rational approximation.

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Introduction

One of the most effective methods to solve multi-dimensional evolution problems is a decomposition method. Decomposition schemes with first and second order accuracy were constructed in the sixties of the XX century (see [7], [11] and references therein). Q. Sheng has proved that in the real number field there do not exist automatically stable decomposition schemes with an accuracy order higher than two (see [12]). Decomposition schemes are called automatically stable if a sum of the absolute values of its split coefficients (coefficients of exponentials' products) equals to one, and the real parts of exponential powers are positive. In the work [1] there is constructed decomposition schemes with the higher order accuracy, but their corresponding decomposition formulas are not automatically stable. In the works [2]-[5] introducing the complex parameter, we have constructed automatically stable decomposition schemes with third order accuracy for two- and multi-dimensional evolution problems and with fourth order accuracy for two-dimensional evolution problem (evolution problem with the operator A is called m-dimensional, if it can be represented as a sum of

m summands $A = A_1 + ... + A_m$). The new idea is an introduction of a complex parameter, which allows us to break the order 2 barrier.

Decomposition formulas constructed in the above mentioned works represent formulas of exponential splitting. Exponential splitting is called a splitting which approximates a semigroup by a combination of semigroups generated by the summands of the operator generating the given semigroup. In view of numerical computations, it is important a rational splitting of the multi-dimensional problem (We call rational splitting such a splitting of the evolution problem that is obtained from the exponential splitting by replacing the semigroups generated by the summands of its main operators with the corresponding rational approximations). Hence, if we have an exponential splitting with some order precision and the same order rational approximation of a semigroup, we can construct a rational splitting of the evolution problem. In the work [6] we have constructed the rational splitting with the third order precision.

In the present work, we have constructed the fourth order precision rational splitting for an evolution problem. We say that the rational approximation of the semigroup used in the work is of Kranc-Nickolson type, as if we replace the parameter α with 1, we obtain the classic approximation of Kranc-Nickolson type. In addition, let us note that in the scalar case, the considered rational approximation represents a Pade classic approximation (see [14]). For the rational approximation constructed in the work, there is obtained the explicit *a priori* estimate.

1. Statement of the problem and main result

Let us consider the Cauchy abstract problem in the Banach space X :

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi, \tag{1.1}$$

where A is a closed linear operator with the definition domain $D[A]$, which is everywhere dense in X, φ is a given element from X, $f(t) \in C^1([0;\infty);X)$.

Let the operator $(-A)$ generate the strongly continuous semigroup $\{\exp(-tA)\}_{t\geq 0}$, then the solution of the problem (1.1) is given by the following formula ([8],[10]):

$$u(t) = U(t,A)\varphi + \int_0^t U(t-s,A)f(s)ds, \tag{1.2}$$

where $U(t,A) = exp(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are compactly defined, closed, linear operators in X.

As it is well-known, the essence of decomposition method consists in splitting the semigroup $U(t, A)$ by means of the semigroups $U(t, A_j)$ $(j = 1, 2)$. In [5] there is constructed the following decomposition formula with the local precision of Fifth order:

$$\begin{aligned} T(\tau) &= \frac{1}{2}\left[T_1(\tau) + T_2(\tau)\right], \qquad (1.3)\\ T_1(\tau) &= U\left(\tau, \frac{\alpha}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U(\tau, \overline{\alpha}A_1)\, U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right),\\ T_2(\tau) &= U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) U(\tau, \overline{\alpha}A_2)\, U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right) \end{aligned}$$

where $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$.

In the above-mentioned work it is shown that:

$$U(\tau, A) - T(\tau) = O_p\left(\tau^5\right),$$

where $O_p\left(\tau^5\right)$ is the operator, norm of which is of the fifth order with respect to τ (more precisely, in the case of the unbounded operator $\left\|O_p\left(\tau^5\right)\varphi\right\| = O\left(\tau^5\right)$ for any φ from the definition domain of $O_p\left(\tau^5\right)$). In the present work (see Section 2) we construct the semigroup approximations with the local precision of the fifth order using the following rational approximation:

$$W(\tau, A) = \left(I - \frac{\alpha}{2}\tau A\right)\left(I + \frac{\overline{\alpha}}{2}\tau A\right)^{-1}\left(I - \frac{\overline{\alpha}}{2}\tau A\right)\left(I + \frac{\alpha}{2}\tau A\right)^{-1}. \qquad (1.4)$$

The approximation defined by formula (1.4) in the scalar case represent the Pade approximations for exponential functions [14].

On the basis of formulas (1.3) and (1.4) we can construct the following decomposition formula:

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left[V_1(\tau) + V_2(\tau)\right], \qquad (1.5)\\ V_1(\tau) &= W\left(\tau, \frac{\alpha}{2}A_1\right) W\left(\tau, \frac{1}{2}A_2\right) W(\tau, \overline{\alpha}A_1)\, W\left(\tau, \frac{1}{2}A_2\right) W\left(\tau, \frac{\alpha}{2}A_1\right),\\ V_2(\tau) &= W\left(\tau, \frac{\alpha}{2}A_2\right) W\left(\tau, \frac{1}{2}A_1\right) W(\tau, \overline{\alpha}A_2)\, W\left(\tau, \frac{1}{2}A_1\right) W\left(\tau, \frac{\alpha}{2}A_2\right) \end{aligned}$$

Below we shall show that this formula has the precision of the fifth order:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^5\right).$$

In the present work, on the basis of formula (1.5), a decomposition scheme with the fourth order precision will be constructed for the solution of problem (1.1).

Let us introduce the following net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 0, 1, ..., \tau > 0\}.$$

According to formula (1.2), we have:

$$u(t_k) = U(\tau, A)u\left(t_{k-1}\right) + \int_{t_{k-1}}^{t_k} U(t_k - s, A)f(s)ds.$$

Let us use Simpson's formula and rewrite this formula in the following form:

$$\begin{aligned} u\left(t_k\right) &= U\left(\tau, A\right) u\left(t_{k-1}\right) + \frac{\tau}{6}\left(f\left(t_k\right) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right)\right. \\ &\quad \left.+U\left(\tau, A\right) f\left(t_{k-1}\right)\right) + R_{5,k}\left(\tau\right), \\ u\left(t_0\right) &= \varphi, \quad k = 1, 2, ... \end{aligned} \tag{1.6}$$

For the sufficiently smooth function f the following estimate is true (see. Lemma 2.3):

$$\left\|R_{k,5}\left(\tau\right)\right\| = O\left(\tau^5\right). \tag{1.7}$$

On the basis of formula (1.6) let us construct the following scheme:

$$\begin{aligned} u_k &= V\left(\tau\right) u_{k-1} \\ &\quad +\frac{\tau}{6}\left(f\left(t_k\right) + 4V\left(\frac{\tau}{2}\right) f\left(t_{k-1/2}\right) + V\left(\tau\right) f\left(t_{k-1}\right)\right), \\ u_0 &= \varphi, \quad k = 1, 2, ... \end{aligned} \tag{1.8}$$

Let us perform the computation of the scheme (1.8) by the following algorithm:

$$u_k = u_{k,0} + \frac{2\tau}{3}u_{k,1} + \frac{\tau}{6}f\left(t_k\right),$$

where $u_{k,0}$ is calculated by the scheme:

$$\begin{aligned} v_{k-4/5,0} &= W\left(\tau, \frac{\alpha}{2}A_1\right)\left(u_{k-1}\right. \quad w_{k-4/5,0} = W\left(\tau, \frac{\alpha}{2}A_2\right) u_{k-1}, \\ &\quad \left.+\frac{\tau}{6}f\left(t_{k-1}\right)\right), \\ v_{k-3/5,0} &= W\left(\tau, \frac{1}{2}A_2\right) v_{k-4/5,0}, \quad w_{k-3/5,0} = W\left(\tau, \frac{1}{2}A_1\right) w_{k-4/5,0}, \\ v_{k-2/5,0} &= W\left(\tau, \overline{\alpha}A_1\right) v_{k-3/5,0}, \qquad w_{k-2.5,0} = W\left(\tau, \overline{\alpha}A_2\right) w_{k-3/5,0}, \end{aligned}$$

$$
\begin{aligned}
v_{k-1/5,0} &= W\left(\tau, \frac{1}{2}A_2\right) v_{k-2/5,0}, \qquad w_{k-1/5,0} = W\left(\tau, \frac{1}{2}A_1\right) w_{k-2/5,0}, \\
v_{k,0} &= W\left(\tau, \frac{\alpha}{2}A_1\right) v_{k-1/5,0}, \qquad w_{k,0} = W\left(\tau, \frac{\alpha}{2}A_2\right) w_{k-1/5,0}, \\
u_{k,0} &= \frac{1}{2}[v_{k,0} + w_{k,0}], \quad u_0 = \varphi + \frac{\tau}{6} f(0), \qquad (1.9)
\end{aligned}
$$

and $u_{k,1}$ - by the scheme:

$$
\begin{aligned}
v_{k-4/5,1} &= W\left(\frac{\tau}{2}, \frac{\alpha}{2}A_1\right) f\left(t_{k-1/2}\right), \quad w_{k-4/5} = W\left(\frac{\tau}{2}, \frac{\alpha}{2}A_2\right) f\left(t_{k-1/2}\right), \\
v_{k-3/5,1} &= W\left(\frac{\tau}{2}, \frac{1}{2}A_2\right) v_{k-2/3,1}, \quad w_{k-3/5} = W\left(\frac{\tau}{2}, \frac{1}{2}A_1\right) w_{k-4/5,1}, \\
v_{k-2/5,1} &= W\left(\frac{\tau}{2}, \overline{\alpha}A_1\right) v_{k-1/3,1}, \qquad w_{k-2.5} = W\left(\frac{\tau}{2}, \overline{\alpha}A_2\right) w_{k-3/5,1}, \\
v_{k-1/5,1} &= W\left(\frac{\tau}{2}, \frac{1}{2}A_2\right) v_{k-1/3,1}, \qquad w_{k-1/5} = W\left(\frac{\tau}{2}, \frac{1}{2}A_1\right) w_{k-2/5,1}, \\
v_{k,1} &= W\left(\frac{\tau}{2}, \frac{\alpha}{2}A_1\right) v_{k-1/3,1}, \qquad w_k = W\left(\frac{\tau}{2}, \frac{\alpha}{2}A_2\right) w_{k-1/5,1}, \\
u_{k,1} &= \frac{1}{2}[v_{k,1} + w_{k,1}], \qquad (1.10)
\end{aligned}
$$

To estimate an error of approximate solution we need the natural powers (A^s, $\quad s = 2, 3, 4, 5$) of the operator $A = A_1 + A_2$. They are usually defined as follows:

$$
\begin{aligned}
A^2 &= \left(A_1^2 + A_2^2\right) + \left(A_1A_2 + A_2A_1\right), \\
A^3 &= \left(A_1^3 + A_2^3\right) + \left(A_1^2A_2 + ... + A_2^2A_1\right) + \left(A_1A_2A_1 + A_2A_1A_2\right),
\end{aligned}
$$

Analogously are defined A^s, $\quad s = 4, 5$.

It is obvious that the definition domain $D\left(A^s\right)$ of the operator A^s represents an intersection of definition domains of its addends.

Let us introduce the following notations:

$$
\begin{aligned}
\|\varphi\|_A &= \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D(A); \\
\|\varphi\|_{A^2} &= \left\|A_1^2\varphi\right\| + \left\|A_2^2\varphi\right\| + \|A_1A_2\varphi\| + \|A_2A_1\varphi\|, \quad \varphi \in D\left(A^2\right),
\end{aligned}
$$

where $\|\cdot\|$ is a norm in X. $\|\varphi\|_{A^s}$, $\quad (s = 3, 4, 5)$ is defined analogously.

The following theorem takes place:

Theorem. *Let the following conditions be satisfied:*

(a) There exists such $\tau_0 > 0$ *that for any* $0 < \tau \leq \tau_0$ *there exist operators* $\left(I + \tau\lambda\gamma A_j\right)^{-1}$, $j = 1, 2, \quad \gamma = 1, \alpha, \overline{\alpha}, \quad \lambda = \alpha, \overline{\alpha}$ *and they are bounded. Besides, the following inequalities are true:*

$$\|W(\tau, \gamma A_j)\| \leq e^{\omega\tau}, \quad \omega = const > 0;$$

(b) The operator $(-A)$ *generates the strongly continuous semigroup* $U(t,A) = \exp(-tA)$*, for which the following inequality is true:*

$$\|U(t,A)\| \leq Me^{\omega t}, \quad M, \omega = const > 0;$$

(c) $U(s,A)\varphi \in D\left(A^5\right)$ *for any* $s \geq 0$*;*

(d) $f(t) \in C^4([0,\infty);X)$*;* $f(t) \in D\left(A^4\right)$, $f'(t) \in D\left(A^3\right)$, $f''(t) \in D\left(A^2\right)$, $f'''(t) \in D(A)$ *and* $U(s,A)f(t) \in D\left(A^4\right)$ *for any fixed* t *and* s $(t,s \geq 0)$.

Then the following estimate holds:

$$\begin{aligned} \|u(t_k) - u_k\| \leq\ & ce^{\omega_0 t_k} t_k \tau^5 \left(\sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^5} \right. \\ & + t_k \sup_{s,t\in[0,t_k]} \|U(s,A)f(t)\|_{A^5} + \sup_{t\in[0,t_k]} \|f(t)\|_{A^4} \\ & + \sup_{t\in[0,t_k]} \left\|f'(t)\right\|_{A^3} + \sup_{t\in[0,t_k]} \left\|f''(t)\right\|_{A^2} \\ & \left. + \sup_{t\in[0,t_k]} \left\|f'''(t)\right\|_{A} + \sup_{t\in[0,t_k]} \left\|f^{(IV)}(t)\right\| \right), \end{aligned} \tag{1.11}$$

where c *and* ω_0 *are positive constants.*

2. Auxiliary Lemmas

Let us prove the auxiliary lemmas on which the proof of the Theorem 1.1 is based.

Lemma 2.1. *If the condition (a) of the Theorem 1.1 is satisfied, then for the operator* $W(t,A)$ *the following decomposition is true:*

$$W(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_{W,k}(t,A), \quad k = 1,...,5, \tag{2.1}$$

where, for the residual member, the following estimate holds:

$$\begin{aligned} \|R_{W,k}(t,A)\varphi\| &\leq c_0 e^{\omega_0 t} t^k \left\|A^k \varphi\right\|, \quad \varphi \in D\left(A^k\right), \\ c_0,\ \omega_0 &= const > 0. \end{aligned} \tag{2.2}$$

proof. We obviously have:

$$\begin{aligned} (I+\gamma A)^{-1} &= I - I + (I+\gamma A)^{-1} = I - (I+\gamma A)^{-1}(I+\gamma A - I) \\ &= I - \gamma A (I+A)^{-1}. \end{aligned}$$

From this for any natural k we can get the following expansion:

$$(I+\gamma A)^{-1}=\sum_{i=0}^{k-1}(-1)^i\gamma^iA^i+\gamma^kA^k(I+\gamma A)^{-1}. \tag{2.3}$$

Let us rewrite $W(\tau,A)$ in the following form:

$$W(\tau,A)=S(\tau,A)-\frac{1}{2}\tau AS(\tau,A)+\frac{1}{12}\tau^2A^2S(\tau,A)$$

where

$$S(\tau,A)=\left(I+\frac{\overline{\alpha}}{2}\tau A\right)^{-1}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}.$$

Let us decompose $S(\tau,A)$ by means of the formula (2.3), we obtain the following recurrent relation:

$$S(\tau,A)=I-\frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}}{2}\tau AS(\tau,A). \tag{2.4}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (2.4) up to the first order, we obtain:

$$W(\tau,A)=I-R_{W,1}(\tau,A), \tag{2.5}$$

where

$$\begin{aligned}R_{W,1}(\tau,A) &= \tau A\left(\frac{\alpha}{2}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}+1}{2}S(\tau,A)\right)\\ &\quad+\frac{1}{12}\tau^2A^2S(\tau,A).\end{aligned}$$

Since $(I+\lambda\tau A)^{-1}$ is bounded according to the condition *(a)* of the Theorem 1.1, therefore:

$$\|R_{W,1}(\tau,A)\varphi\|\le c_0e^{\omega_0\tau}\tau\|A\varphi\|,\quad \varphi\in D(A). \tag{2.6}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (2.4) up to the second order:

$$\begin{aligned}W(\tau,A) &= I-\tau A\left(\frac{\alpha}{2}I-\frac{\alpha^2}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{1+\overline{\alpha}}{2}I\right.\\ &\quad\left.-\frac{\alpha+\alpha\overline{\alpha}}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}+\overline{\alpha}^2}{4}\tau AS(\tau,A)\right)\\ &\quad+\frac{1}{12}\tau^2A^2S(\tau,A)\\ &= I-\tau A+R_{W,2}(\tau,A)\end{aligned}$$

where

$$\begin{aligned} R_{W,2}(\tau, A) &= \frac{\alpha^2+\alpha+\alpha\overline{\alpha}}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1} \\ &\quad +\frac{3\overline{\alpha}+3\overline{\alpha}^2+1}{12}S(\tau, A) \\ &= \frac{\alpha-\frac{1}{3}+\alpha+\frac{1}{3}}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1} \\ &\quad +\frac{3\overline{\alpha}+3\overline{\alpha}-1+1}{12}S(\tau, A) \\ &\quad = \tau^2A^2\left(\frac{\alpha}{2}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{\overline{\alpha}}{2}S(\tau, A)\right). \end{aligned}$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\|R_{W,2}(\tau, A)\varphi\| \le c_0e^{\omega_0\tau}\tau^2\left\|A^2\varphi\right\|, \quad \varphi\in D\left(A^2\right). \tag{2.7}$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.4) up to the third order:

$$\begin{aligned} W(\tau, A) &= I-\tau A+\tau^2A^2\left(\frac{\alpha}{2}I-\frac{\alpha^2}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\right. \\ &\quad \left.+\frac{\overline{\alpha}}{2}\left(I-\frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}}{2}\tau AS(\tau, A)\right)\right) \\ &= I-\tau A+\frac{1}{2}\tau^2A^2+R_{W,3}(\tau, A), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} R_{W,3}(\tau, A) &= -\tau^3A^3\left(\frac{1+3\alpha^2}{12}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{\overline{\alpha}^2}{4}R(\tau, A)\right) \\ &= -\tau^3A^3\left(\frac{\alpha}{4}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{\overline{\alpha}^2}{4}R(\tau, A)\right). \end{aligned}$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\|R_{W,3}(\tau, A)\varphi\| \le c_0e^{\omega_0\tau}\tau^3\left\|A^3\varphi\right\|, \quad \varphi\in D\left(A^3\right). \tag{2.9}$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.4) up to the fourth order:

$$\begin{aligned} W(\tau, A) &= I-\tau A+\frac{1}{2}\tau^2A^2-\tau^3A^3\left(\frac{\alpha}{4}I-\frac{\alpha^2}{8}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\right. \\ &\quad \left.+\frac{\overline{\alpha}^2}{4}\left(I-\frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}}{2}\tau AS(\tau, A)\right)\right) \\ &= I-\tau A+\frac{1}{2}\tau^2A^2-\frac{1}{6}\tau^3A^3+R_{W,4}(\tau, A), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} R_{W,4}(\tau, A) &= \tau^4 A^4 \left(\frac{\alpha^2 + \alpha\overline{\alpha}^2}{8} \left(I + \frac{\alpha}{2}\tau A \right)^{-1} + \frac{\overline{\alpha}^3}{8} S(\tau, A) \right) \\ &= \tau^4 A^4 \left(\frac{\alpha}{12} \left(I + \frac{\alpha}{2}\tau A \right)^{-1} + \frac{\overline{\alpha}^3}{8} S(\tau, A) \right) \end{aligned}$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\| R_{W,4}(\tau, A)\varphi \| \leq c_0 e^{\omega_0 \tau} \tau^4 \left\| A^4 \varphi \right\|, \quad \varphi \in D\left(A^4\right). \tag{2.11}$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.4) up to the fifth order:

$$\begin{aligned} W(\tau, A) &= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 \\ &\quad -\tau^4 A^4 \left(\frac{\alpha}{12} - \frac{\alpha^2}{24}\tau A \left(I + \frac{\alpha}{2}\tau A \right)^{-1} \right. \\ &\quad \left. + \frac{\overline{\alpha}^3}{8} \left(I - \frac{\alpha}{2}\tau A \left(I + \frac{\alpha}{2}\tau A \right)^{-1} - \frac{\overline{\alpha}}{2}\tau A S(\tau, A) \right) \right) \\ &= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 \\ &\quad - \frac{1}{24}\tau^4 A^4 + R_{W,5}(\tau, A), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} R_{W,5}(\tau, A) &= \tau^5 A^5 \left(\frac{2\alpha^2 + 3\overline{\alpha}^3\alpha}{48} \left(I + \frac{\alpha}{2}\tau A \right)^{-1} + \frac{\overline{\alpha}^4}{16} S(\tau, A) \right) \\ &= \tau^5 A^5 \left(\frac{\alpha}{24} \left(I + \frac{\alpha}{2}\tau A \right)^{-1} + \frac{\overline{\alpha}^4}{16} S(\tau, A) \right) \end{aligned}$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\| R_{W,5}(\tau, A)\varphi \| \leq c_0 e^{\omega_0 \tau} \tau^5 \left\| A^5 \varphi \right\|, \quad \varphi \in D\left(A^5\right) \qquad \blacksquare \tag{2.13}$$

Lemma 2.2. *If the conditions (a), (b) and (c) of the Theorem 1.1 are satisfied, then the following estimate holds:*

$$\left\| \left[U^k(\tau, A) - V^k(\tau) \right] \varphi \right\| \leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [0, t_k]} \| U(s, A)\varphi \|_{A^5}, \tag{2.14}$$

where c and ω_0 are positive constants.

Proof. The following formula is true (see Kato. T. [9], p. 603):

$$A\int_{r}^{t} U(s,A)\,ds = U(r,A) - U(t,A)\,, \quad 0 \le r \le t. \tag{2.15}$$

Hence we get the following expansion:

$$U(t,A) = \sum_{i=0}^{k-1}(-1)^i \frac{t^i}{i!}A^i + R_k(t,A), \tag{2.16}$$

where

$$R_k(t,A) = (-A)^k \int_0^t \int_0^{s_1} ... \int_0^{s_{k-1}} U(s,A)ds ds_{k-1}...ds_1. \tag{2.17}$$

Let us decompose W operators in the expression of $V(\tau)$ according to the formula (2.1) from right to left, so that each residual member be of the fifth order. We shall have:

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_{V,5}(\tau)\,, \tag{2.18}$$

where for the residual member according to the condition *(a)* of the Theorem 1.1 we have the following estimate:

$$\|R_{V,5}(\tau)\varphi\| \le c e^{\omega_0 \tau}\tau^5 \|\varphi\|_{A^5}\,, \quad \varphi \in D\left(A^5\right). \tag{2.19}$$

From the (2.16) and (2.18) it follows:

$$U(\tau,A) - V(\tau) = R_5(\tau,A) - R_{V,5}(\tau)\,.$$

From here according to inequalities (2.17) and (2.19) we obtain the following estimate:

$$\|[U(\tau,A) - V(\tau)]\varphi\| \le c e^{\omega_0 \tau}\tau^5 \|\varphi\|_{A^5}\,, \quad \varphi \in D\left(A^5\right). \tag{2.20}$$

The following representation is obvious:

$$[U^k(\tau,A) - V^k(\tau)]\varphi = \sum_{i=1}^{k} V^{k-i}(\tau)[U(\tau,A) - V(\tau)]U^{i-1}(\tau,A)\varphi.$$

Hence, according to the conditions *(a), (b), (c)* of the Theorem 1.1 and inequality (2.20), we have the sought estimate ■

Lemma 2.3. *Let the following conditions be satisfied:*

(a) The operator A satisfies the conditions of the Theorem 1.1;

(b) $f(t) \in C^4([0,\infty);X)$, and $f(t) \in D\left(A^4\right)$, t, $f^{(k)}(t) \in D\left(A^{4-k}\right)$ $(k=1,2,3)$ for every fixed $t \geq 0$.

Then the following estimate holds

$$\left\|R_{5,k}\left(\tau\right)\right\| \leq c e^{\omega_0 \tau} \tau^5 \sum_{i=0}^{4} \max_{s \in [t_{k-1}, t_k]} \left\|f^{(i)}\left(s\right)\right\|_{A^{4-i}}, \tag{2.21}$$

where

$$\begin{aligned} R_{5,k}\left(\tau\right) &= \int_{t_{k-1}}^{t_k} U\left(t_k - s, A\right) f\left(s\right) ds \\ &\quad -\frac{\tau}{6}\left(f\left(t_k\right) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right)\right. \\ &\quad \left.+ U\left(\tau, A\right) f\left(t_{k-1}\right)\right) \end{aligned} \tag{2.22}$$

and where c and ω_0 are positive constants, and $f^{(0)}\left(s\right) = f\left(s\right)$.

Proof. By means of cvladTa gardaqmna, the integral in the equality (2.22) takes the following form:

$$\int_{t_{k-1}}^{t_k} U\left(t_k - s, A\right) f\left(s\right) ds = \int_{0}^{\tau} U\left(\tau - s, A\right) f\left(t_{k-1} + s\right) ds.$$

If we decompose the function $f\left(t_{k-1}+s\right)$ into the Taylor series, and expand the semigroup $U\left(\tau - s, A\right)$ according to formula (2.16), we obtain:

$$U\left(\tau - s, A\right) f\left(t_{k-1} + s\right) = P_{3,k}\left(s\right) + \widetilde{R}_{4,k}\left(\tau, s\right), \tag{2.23}$$

where

$$\begin{aligned} P_{3,k}\left(s\right) &= \left(I - \left(\tau - s\right) A + \frac{\left(\tau - s\right)^2}{2} A^2 - \frac{\left(\tau - s\right)^3}{6} A^3\right) f\left(t_{k-1}\right) \\ &\quad + s\left(I - \left(\tau - s\right) A + \frac{\left(\tau - s\right)^2}{2} A^2\right) f'\left(t_{k-1}\right) \\ &\quad + \frac{s^2}{2}\left(I - \left(\tau - s\right) A\right) f''\left(t_{k-1}\right) + \frac{s^3}{6} f'''\left(t_{k-1}\right), \\ \widetilde{R}_{4,k}\left(\tau, s\right) &= \frac{1}{6} U\left(\tau - s, A\right) \int_0^s \left(s - \xi\right)^3 f^{(IV)}\left(t_{k-1} + \xi\right) d\xi \\ &\quad + R_4\left(\tau - s, A\right) f\left(t_{k-1}\right) \end{aligned}$$

$$
\begin{aligned}
&+(\tau-s)\,AR_3(\tau-s,A)\,f'(t_{k-1})\\
&+\frac{(\tau-s)^2}{2}A^2R_2(\tau-s,A)\,f''(t_{k-1})\\
&+\frac{(\tau-s)^3}{6}A^3R_1(\tau-s,A)\,f'''(t_{k-1})\,.
\end{aligned}
$$

Hence according condition *b)* and *d)* of the Theorem 1.1 we obtain the following estimate:

$$
\widetilde{R}_{4,k}(\tau,s)\le ce^{\omega_0\tau}\tau^4\sum_{i=0}^{4}\max_{s\in[t_{k-1},t_k]}\left\|f^{(i)}(s)\right\|_{A^{4-i}}. \tag{2.24}
$$

From equality (2.22) with account of formula (2.23), we have:

$$
\begin{aligned}
R_{5,k}(\tau) &= \int_0^{\tau}U(\tau-s,A)\,f(t_{k-1}+s)\,ds\\
&\quad-\frac{\tau}{6}\left(f(t_k)+4U\left(\frac{\tau}{2},A\right)f\left(t_{k-1/2}\right)+U(\tau,A)\,f(t_{k-1})\right)\\
&= \int_0^{\tau}P_{3,k}(s)\,ds+\int_0^{\tau}\widetilde{R}_{4,k}(\tau,s)\,ds\\
&\quad-\frac{\tau}{6}\left(P_{3,k}(\tau)+4P_{3,k}\left(\frac{\tau}{2}\right)+P_{3,k}(0)\right)\\
&\quad-\frac{\tau}{6}\widetilde{R}_{4,k}(\tau,0)+4\widetilde{R}_{4,k}\left(\tau,\frac{\tau}{2}\right)+\widetilde{R}_{4,k}(\tau,\tau)\,,
\end{aligned} \tag{2.25}
$$

radgan simpsonis formula zustia mesame rigis polinomisaTvis, amitom $R_{5,k}(\tau)$-sTvis gvaqvs:

$$
R_{5,k}(\tau)=\int_0^{\tau}\widetilde{R}_{4,k}(\tau,s)\,ds-\frac{\tau}{6}\left(\widetilde{R}_{4,k}(\tau,0)+4\widetilde{R}_{4,k}\left(\tau,\frac{\tau}{2}\right)+\widetilde{R}_{4,k}(\tau,\tau)\right).
$$

hence according to inequality (2.23), we have:

$$
\left\|R_{k,5}(\tau)\right\|\le ce^{\omega_0\tau}\tau^5\sum_{i=0}^{4}\max_{s\in[t_{k-1},t_k]}\left\|f^{(i)}(s)\right\|_{A^{4-i}}\qquad\blacksquare \tag{2.26}
$$

3. Proof of the theorem

Let us return to the proof of the Theorem 1.1.

Let us write formula (1.6) in the following form:

$$u(t_k) = U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A)\left(F_i^{(1)} + R_{5,k}(\tau)\right), \tag{3.1}$$

where

$$F_k^{(1)} = \frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + U(\tau, A) f(t_{k-1})\right). \tag{3.2}$$

Analogously let us present u_k as follows:

$$u_k = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau) F_i^{(2)}, \tag{3.3}$$

where

$$F_i^{(2)} = \frac{\tau}{6}\left(f(t_k) + 4V\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + V(\tau, A) f(t_{k-1})\right). \tag{3.4}$$

From equalities (3.1) and (3.3) it follows:

$$\begin{aligned} u(t_k) - u_k &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ &+ \sum_{i=0}^{k}\left[U^{k-i}(\tau, A)F_i^{(1)} - V^{k-i}(\tau)F_i^{(2)}\right] \\ &+ \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{k,5}(\tau) \\ &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi + \sum_{i=1}^{k}\left[\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right)F_i^{(1)}\right. \\ &\left. + V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right] + \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{5,k}(\tau). \end{aligned} \tag{3.5}$$

From formulas (3.2) and (3.4) we have:

$$\begin{aligned} F_k^{(1)} - F_k^{(2)} &= \frac{\tau}{6}\left(4\left(U\left(\frac{\tau}{2}, A\right) - V\right) f\left(t_{k-1/2}\right)\right. \\ &\left. + \left(U(\tau, A) - V(\tau, A)\right) f(t_{k-1})\right) \end{aligned} \tag{3.6}$$

From here, according to inequality (2.18) and Lemma 2.1 we obtain the following estimate:

$$\left\|F_k^{(1)} - F_k^{(2)}\right\| \le c e^{\omega_0 \tau}\tau^5 \sup_{t\in[t_{k-1}, t_k]} \|f(t)\|_{A^4}. \tag{3.7}$$

According to the Lemma 2.1 we have:

$$\begin{aligned} &\left\| \sum_{i=1}^{k} \left(U^{k-i}(\tau, A) - V^{k-i}(\tau) \right) F_i^{(1)} \right\| \\ \leq \quad & c e^{\omega_0 t_k} t_k^2 \tau^4 \sup_{s,t \in [0,t_k]} \| U(s,A) f(t) \|_{A^5} . \end{aligned} \tag{3.8}$$

From equality (3.5) according to inequalities (3.7), (3.8), (2.21) and the condition *(b)* of the Theorem 1.1 we obtain sought estimation ■

Remark 1. *The operator $V^k(\tau)$ is the solution operator of the above-considered decomposed problem. It is obvious that, according to the condition of the Theorem 1.1 $\left(\|W(t,\gamma A_j)\| \leq e^{\omega t}\right)$, the norm of the operator $V^k(\tau)$ is less than or equal to $e^{\omega_0 t_k}$. From this follows the stability of the above-stated decomposition scheme on each finite time interval.*

Remark 2. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint non negative operators, in estimate (1.10) ω_0 will be replaced by 0. Alongside with this, for the transition operator of the split problem, the estimate $\left\|V^k(\tau)\right\| \leq 1$ will be true.*

Remark 3. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint, positive definite operators, in estimate (1.10) ω_0 will be replaced by $-\alpha_0$, $\alpha_0 > 0$. Alongside with this, for the transition operator of the split problem, the estimate $\left\|V^k(\tau)\right\| \leq e^{-\alpha_1 t_k}$, $\alpha_1 > 0$ will be true.*

Remark 4. *According to the classical theorem of Hille-Philips-Iosida (see [13]), if the operator $(-A)$ generates a strongly continuous semigroup, then the inequality in the condition (b) of the Theorem 1.1 is automatically satisfied. The proof of this inequality is based on the uniform boundedness principle, according to which the constants M and ω exist, but generally can not be explicitly constructed (according to the method of the proof). That is why we demand satisfying of the inequality in the condition (b) of the Theorem 1.1.*

4. Conclusion

In the case when the operators A_1, A_2 are matrices, it is obvious that the conditions of the Theorem 1.1 are automatically satisfied. The conditions of Theorem 1.1 are also satisfied if A_1, A_2 and A are self-adjoint, positive definite operators. Moreover, the conditions of the Theorem 1.1 are automatically satisfied if the operators A_1, A_2 and A are normal operators. However, in this case, certain restrictions are imposed on the spectrums of this operators: the spectrum of the operator A have to be included in the right half-plane and the spectrums of the operators A_1 and A_2 have to be included in the sector with angle of $120°$, in order the spectrums of the operators A_1 and A_2 to remain in the right half-plane after turning by $\pm 30^{\circ}$ (this is caused by multiplication of the operators A_1 and A_2 on the parameters α and $\overline{\alpha}$).

The fourth order precision is reached by introducing a complex parameter. For this reason, each equation of the given decomposed system is replaced by a pair of real equations, unlike the lower order precision schemes. To solve the specific problem, (for example) the matrix factorization may be used, where the coefficients are the matrices of the second order, unlike the lower order precision schemes, where the common factorization may be used.

It must be noted that, unlike the high order precision decomposition schemes considered in [1], the sum of absolute values of coefficients of the addends of the transition operator $V(\tau)$ equals to one. Hence the considered scheme is stable for any bounded operators A_1, A_2.

References

1. M. Schatzman *Higher order alternate directions methods.* Comput. Methods Appl. Mech. Engineering., **116** (1992), No 1-4, pp. 219-225, ICOSAHOM'92 (Montpiller, 1992).

2. Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High-degree precision decomposition method for an evolution problem*, Tbilisi, Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics, **14** (1999), No. 3, pp. 45-48.

3. Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *Sequention-Parallel method of high degree precision for Cauchy abstract problem solution* , Minsk, Comput. Methods in Appl. Math., **1** (2001), No. 2, pp. 173-187.

4. Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High degree precision decomposition method for the evolution problem with an operator under a split form*, Paris, M2AN Math. Model. Numer. Anal., **36** (2002), No. 4, pp. 693-704.

5. Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *The Fourth Order Accuracy Decomposition Scheme for an Evolution Problem*, Paris, M2AN Math. Model. Numer. Anal., **38** (2004), No. 4, pp. 707-722.

6. Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High degree precision decomposition formulas of semigroup approximation*, Tbilisi, Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics, **16** (2001), No. 1-3, pp. 89-92.

7. N. N. Ianenko, *Fractional steps method of solving for multidimensional problems of mathematical physics*, Novosibirsk, Nauka, 1967.

8. T. Kato, *The theory of perturbations of linear operators*, M.:Mir, 1972.

9. S. G. Krein, *Linear equations in Banach space*, M.: Nauka, 1971.

10. G. I. Marchuk, *Split methods*, M.: Nauka, 1988.

11. Q. Sheng, *Solving linear partial differential equation by exponential spliting*, IMA J. Numerical Anal., **9** 1989, pp. 199-212.

12. M. Reed, B. Simon *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, New York-London, Academic Press [Harcourt Brace Jovanovich, Publishers], 1975.

13. G. A. Baker Jr, P. Graves-Morris, *Pade Approximations*, M.: Mir, 1986.