# ON SAINT-VENANT'S PROBLEMS FOR AN ISOTROPIC TWO-LAYERED ELLIPTIC TUBE

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*Abstract*

In this paper the Saint-Venants problems for the homogeneous isotropic two-layered elliptic tube and for the composed isotropic elliptic tube with an anisotropic kernel are studied.

**Key words and phrases**: Saint-Venant's problems, torsion extension, bending, composed elliptic ring, Faber's polynomials.

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## Introduction

Many works are dedicated to the Saint-Venants (S-V) problem for the cylindrical (prismatic) bodies with several cross-sections and variable elastic constants [1,2,3,7,9,11,12,13,15,16,17,19], including nonlinear cases [8].

It would be interesting to consider the (S-V) problems for the elliptic tubes. By means of this problem, a problem of crack for the body with an elliptic split may be investigated.

The torsion's function for the homogeneous isotropic elliptic tube was constructed by A. G. Greenhill [14] (this article is quoted from the book [14], p. 335). The solution to the problem of bending of an elliptic tube by the shear force is obtained by A. Love [14]. The torsion problem for elliptic beams and tube is considered in [11,15]. The bibliography of the papers, concerning to the indicated problems are given in the books [1,3,12,14,15,17,19]. The Saint-Venant principle for the composed bodies is studied in the article [20].

It should be noted that the results for the elliptic tube, given in the above mentioned works, are difficult for the calculations of a torsion's rigidity, displacement and stress in the case of an elliptic tube.

In this article the S-V problem for the homogeneous or composed two-layered elliptic tube is solved by means of the Faber's polynomials.

These problems have not been considered yet. The Faber's polynomials for the simply-connected area was introduced by G. Faber in 1903 [6]. The further generalization of the Faber's polynomials are given in the works of many researchers.

Some of these works are given in the references of [4,5]. The detailed review of the works on the Faber's polynomials is given in the monography of P. Suetin [18].

## 1. Basic Equations

Let us consider an elliptic tube composed of the different elastic materials (obey Hook's law) occupying the cylindrical domain $\Omega = \Omega_0 + \Omega_1 + \Omega_2$ (the domain $\Omega$ occupies an anisotropic kernel). We assume that this materials are glued at each other without cracks.

Consider the cartesian coordinates system $Ox_1x_2x_3$ at the end of the cylindrical body. Each of the domains $\Omega_1$ and $\Omega_2$ of the two-layered elliptic tube are bounded by two planes

$$x_3 = 0, \quad x_3 = l \quad (l > 0) \tag{1.1}$$

and by the elliptic surfaces $\Gamma_0, \Gamma_1$ and $\Gamma_1, \Gamma_2$, respectively. The cross-section of these surfaces are the ellipses $\gamma_e$, $(e = 0, 1, 2)$, given by the equations

$$(x_1)_{\gamma_e} = a_e \cos\theta, \quad (x_2)_{\gamma_e} = b_e \sin\theta, \tag{1.2}$$

$$(0 \le \theta \le 2\pi, \quad a_e > b_e, \quad a_{e+1} > a_e; \quad e = 0, 1, 2), \tag{1.3}$$

where $a_e$ and $b_e$ are semi-axes of the ellipses $\gamma_e$.

A cross-section of the domain $\Omega_m$ enclosed between the surfaces $\Gamma_m$ and $\Gamma_{m-1}$ is denoted by $\omega_m$, which is bounded by the ellipses $\gamma_m$ and $\gamma_{m-1}$.

Let us consider an element of a lateral of the elliptic cylindrical body with the outwards normal $n(n_1, n_2)$

$$\begin{aligned} (n_1 + in_2)_{\gamma_e} &= \Theta_e^{-1}(b_j \cos\theta + ia_e \sin\theta), \\ (x_2 n_1 - x_1 n_2)_{\gamma_e} &= -\frac{c^2}{2\Theta_e} \sin 2\theta, \\ \Theta_e &= \sqrt{a_e^2 \sin^2\theta + b_e^2 \cos^2\theta}, \end{aligned} \tag{1.4}$$

where $i^2 = -1$.

While considering an elliptic tube composed by two different materials, occupying the domains $\Omega_1$ and $\Omega_2$ bounded by the planes (1) and the elliptic surfaces $\Gamma_0, \Gamma_1$ and $\Gamma_1, \Gamma_2$, respectively, it will be proposed that this materials are glued to each other without a split along the surface $\Gamma_1$ (an interface). Cross-section of such composed tube will be represented as two confocal elliptic rings $\omega_1$ and $\omega_2$ bounded by the confocal ellipses $\gamma_0, \gamma_1$ and $\gamma_1, \gamma_2$ (see equalities (2)) glued together along an ellipse $\gamma_1$ (an interface).

In the problem of torsion there will be also considered the two-layer isotropic elliptic tube strengthened by an anisotropic elliptic shift, occupying an elliptic domain $\Omega_0$ bounded by the planes (1) and the surface $\Gamma_0$ and glued with the indicated tubes along the surface $\Gamma_0$. Cross-section of the domain $\Omega_0$ will be the solid elliptic domain $\omega_0$ bounded by the ellipse $\gamma_0$ with semi-axis $a_0$ and $b_0$. It will be noted that an interface of adjacent domains $\omega_e$ and $\omega_{e+1}$ of the composed elliptic domains will be the ellipse $\gamma_e$. An interface of the adjacent domains $\omega_e$ and $\omega_{e+1}$ of the composed elliptic ring is the ellipse $\gamma_e$.

Some auxiliary relationships for the isotropic and anisotropic media, which we will use in the sequel are given below

a) The isotropic medium.

In this case the torsion function $f_j$ and the bending functions $F_k^{(j)}$ (in bending of cantilever) are the solutions of the following boundary value problems [4,5,6]:

$$\Delta f_j = 0, \ \ \Delta F_k^{(j)} = 0, \tag{1.5}$$

in each of the domain $\omega_j$ of the cross-section we have

$$(D_n f_j)_e = (h_j)_e, \ \ (D_n F_k^{(j)})_e = (H_k^{(j)})_e, \tag{1.6}$$

At the points of an exterior boundaries $\gamma e$ of the domains $\omega_j$ we have

$$\begin{aligned} [\mu D_n f]_e - [\mu D_n f]_{e+1} &= [\mu h]_e - [\mu h]_{e+1}, \\ [\mu D_n F_k]_e - [\mu D_n F_k]_{e+1} &= [\mu H_k]_e - [\mu H_k]_{e+1}, \\ [f]_e &= [f]_{e+1}, \quad [F_k]_e = [F_k]_{e+1}, \end{aligned} \tag{1.7}$$

where the symbol $[\ ]_m$ denotes the limiting values of the expressions in the brackets taken from the domains $\omega_m$ and $\omega_{m+1}$, respectively. The operators $\Delta, D_n$ and the functions $h, H$ are given by

$$\begin{aligned} \Delta &= D_1^2 + D_2^2, \quad D_n = n_1 D_1 + n_2 D_2, \\ D_j &= \frac{\partial}{\partial x_j}. \ h = x_2 n_1 - x_1 n_2, \\ H_k &= \frac{1}{2}[(2+\nu)x_k^2 - \nu x_{3-k}^2] n_k + \nu x_1 x_2 n_{3-k}, \end{aligned} \tag{1.8}$$

where $\mu = E[2(1+\nu)]^{-1}$ is the modulus of rigidity, $\nu$ is the Poison's ratio and $E$ is the modulus of elasticity.

For the elliptic tube, cross-section of which is an elliptic ring, bounded by the confocal ellipses (with $a_j^2 - b_j^2 = c^2$, where $2c$ is the focal length), we will use the following formulas

$$
\begin{aligned}
t_1 &= z + w, \quad t_2 = z - w, \quad w = \sqrt{z^2 - c^2}, && (1.9)\\
t_1 t_2 &= c^2, \quad p_e = a_e + b_e, \quad q_e = a_e - b_e, && (1.10)\\
(t_1)_{\gamma_e} &= p_e \exp(i\theta), \quad (t_2)_{\gamma_e} = q_e \exp(-i\theta), && (1.11)\\
2(w)_{\gamma_e} &= 2(b_e \cos\theta + i a_e \sin\theta) \\
&= p_e \exp(i\theta) - q_e \exp(-i\theta), && (1.12)\\
D_1 t_j &= -(-1)^j w^{-1} t_j, \quad D_2 t_j = -i(-1)^j w^{-1} t_j, && (1.13)\\
\lim_{z\to\infty} z^{-1} w &= 1 \quad (j = 1, 2). && (1.14)
\end{aligned}
$$

b) The anisotropic medium.

In this case we will consider only the homogeneous body and the torsion's problem.As it is known from ([2,3]), the torsion's function $f^*$ satisfies the following boundary condition:

$$\Delta^* f^* = 0, \tag{1.15}$$

in the domain $\omega_0$, and

$$(D_n^* f^*)_j = (h^*)_j, \tag{1.16}$$

at the each point of the ellipse $\gamma_0$, where

$$
\begin{aligned}
\Delta^* &= A_{55} D_1^2 + 2A_{45} D_1 D_2 + A_{44} D_2^2, \\
D_n^* &= n_1(A_{55} D_1 + A_{45} D_2) + n_2(A_{45} D_1 + A_{44} D_2), && (1.17)\\
h^* &= h_1^* n_1 + h_2^* n_2, \quad h_1^* = A_{55} x_2 - A_{45} x_1, \quad h_2^* = A_{45} x_2 - A_{44} x_1,
\end{aligned}
$$

$A_{jk} = A_{kj}$ are the coefficients of the stiffness in the Hook's low.

Let us consider the characteristic equation of the equation (9)

$$A_{44} v^2 + 2A_{45} v + A_{55} = 0 \quad (A^2 = A_{44} A_{55} - A_{45}^2 > 0). \tag{1.18}$$

The complex root of this equation is

$$v_* = A_{44}^{-1}(iA - A_{45}) \quad (i^2 = -1). \tag{1.19}$$

Let us introduce the complex variable $z_* = x_1 + v_* x_2$, by means of which, for the solid elliptic domain $\omega_0$, bounded by the ellipse $\gamma_0$ with the semi-axis $a_0$ and $b_0$, will be used the expressions

$$
\begin{aligned}
t_1^* &= (a_0 - i v_* b_0)^{-1}(z_* + w_0^*), \\
t_2^* &= (a_0 - i v_* b_0)^{-1}(z_* + w_0^*),
\end{aligned}
$$

$$\begin{aligned} w_j^* &= (z_*^2 - a_j^2 - v_*^2 b_j^2)^{0.5},\ (w_j^*)_{\gamma_j} \\ &= i(a_j \sin\theta - v_* b_j \cos\theta),\ (t_1^*)_{\gamma_0} = \exp i\theta, \\ (t_2^*)_{\gamma_0} &= \lambda_0 \exp(-i\theta),\ \ \lambda_j = \frac{a_j + iv_* b_j}{a_j - iv_* b_j} \\ (|\lambda_j| &< 1;\ 0 \le \theta \le 2\pi), \quad \lim_{z_* \to \infty} (z_*^{-1} w_j^*) = 1. \end{aligned} \tag{1.20}$$

Below are given the well-known formulas [2,4,5], which will be used for the isotropic as well as for the anisotropic bodies.

At first for the construction of the "ends conditions" of the cylindrical bodies in the S-V problems, let us introduce the notations for the projections on axis $Ox_j$ (j=1,2,3) of the resultant forces by $P(P_1, P_2, P_3)$ and of the resultant moments by $M(M_1, M_2, M_3)$, obtained due to the action of the exterior forces. Therefore, components of the stresses $\tau_{jk}$ in the each cross-section $\omega$ (with the outward normal $n(0,0,+1)$) of the body must satisfy the following equalities:

$$\int\!\!\int_\omega \tau_{j3} d\omega = P_j \ \ (j=1,2,3); \ \int\!\!\int_\omega (x_2\tau_{33} - x_3\tau_{23}) d\omega = M_1, \tag{1.21}$$

$$\int\!\!\int_\omega (x_3\tau_{13} - x_1\tau_{33}) d\omega = M_2, \ \int\!\!\int_\omega (x_1\tau_{23} - x_2\tau_{13}) d\omega = M_3. \tag{1.22}$$

For the homogeneous cylindrical body with the cross-section $\omega$, bounded by the curve $\gamma$, we have

$$\int\!\!\int_\omega \tau_{j3} d\omega = \int_\gamma x_j \tau_{n3} d\gamma + \int\!\!\int_\omega x_j D_3 \tau_{33} d\omega \quad (j=1,2). \tag{1.23}$$

For the composed cylindrical body with the cross-section $\omega = \omega_0 + \omega_1 + \omega_2$, bounded by the external curve $\omega_2$ and the interfaces $\omega_0$ and $\omega_1$, we have

$$\begin{aligned} \int\!\!\int_\omega \tau_{j3} d\omega &= \int_{\gamma_2} x_j \tau_{n3} d\omega + \sum_{k=0,1} \int_{\gamma_k} x_j \{[\tau_{n3}]_k - [\tau_{n3}]_{k+1}\} d\gamma + \\ + \int\!\!\int_\omega x_j D_3 \tau_{33} d\omega \ \ (j &= 1,2), \end{aligned} \tag{1.24}$$

where

$$\tau_{nj} = \tau_{1j} n_1 + \tau_{2j} n_2 + \tau_{3j} n_3 \ (j=1,2,3) \tag{1.25}$$

are the projections on the axis $Ox_j$ of the vector of the stresses $\tau_n(\tau_{n1}, \tau_{n2}, \tau_{n3})$.

In addition, we must remark that in the problems, which we consider, it is proposed that the lateral surface of the cylindrical body is free from the acting of the exterior forces. Also for the composed beam, it is necessary to fulfill the conditions of continuity of the displacement vectors and stresses which cross the interfaces from the adjoint domains. Therefore, the following boundary-contact conditions are necessary:

$$\tau_{nj} = 0 \quad (j = 1, 2, 3) \tag{1.26}$$

on the exterior lateral surface and

$$[u_j]_e = [u_j]_{e+1]}, \quad [\tau_{nj}]_e = [\tau_{nj}]_{e+1} \quad (j = 1, 2, 3), \tag{1.27}$$

on the interface between the domains $\Omega_e$ and $\Omega_{e+1}$.

It is obvious that the components of the stresses on each of the domains $\Omega_k$ must satisfy the equations of equilibrium (when the body forces are absent)

$$D_1\tau_{1j} + D_2\tau_{2j} + D_3\tau_{3j} = 0 \quad (J = 1, 2, 3). \tag{1.28}$$

It is well-known that the potential energy $W$ of an elastic body occupying the domain $\Omega$ bounded with the surface $\omega$ under the action of the surface force $\tau_n(\tau_{n1}, \tau_{n2}, \tau_{n3})$ and of the body force $\Psi(\Psi_1, \Psi_2, \Psi_3)$ will be given by

$$2W = \sum_{j=1}^{3} \left[ \int\int\int_{\Omega} \Psi_j u_j d\Omega + \int\int_{\omega} \tau_{nj} u_j d\omega \right] > 0, \tag{1.29}$$

where $u_j$ are the components of the displacements.

Now let us begin solution of the problems.

## 2. Extension by the Longitudinal Force and the Bending Due to Couples of Forces

Let us consider the deformation of the homogeneous elliptic tube, occupying the domain $\Omega_1$ bounded by the planes (1.1) and the elliptic surfaces (1.2), when cross-section of these surfaces

$$\Gamma_0 = 0, \quad \Gamma_1 = 0 \tag{2.1}$$

is the confocal ring $\omega_1$, bounded by two confocal ellipses $\gamma_j$ $(j = 0, 1)$ (see (1.3))

$$(x_1)_j = a_j \cos\theta, \quad (x_2)_j = b_j \sin\theta \quad (j = 0, 1;\ 0 \le \theta \le 2\pi) \tag{2.2}$$

with the focal length $2c$.

It is proposed that the lateral surfaces (1) are free of acting of exterior forces. A system of the forces applied to the "upper" and "base" $x_3 = l$ of the body statically is equivalent to the one force $\phi_3$ producing the extension acting parallel to the axis $Ox_3$, and to two couples of forces producing the flexures of the body in the planes $Ox_2x_3$ and $Ox_1x_3$ by the moments $M_1$ and $M_2$, respectively.

The corresponding components of the stresses and the displacements for this problem in the domain $\Omega_1$ will be given as [1,4,5,6]

$$\tau_{j1} = \tau_{j2} = \tau_{j3} = 0,\ \tau_{33} = E_m \sum_{e=1}^{3} C_e x^{(e)}); \tag{2.3}$$

$$u_j = -\sum_{e=1}^{3} C_e g_j^{(e)} - \frac{1}{2} C_j x_3^2,\ \ u_3 = x_3 \sum_{e=1}^{3} C_e x^{(e)}, \tag{2.4}$$

$$(j = 1,2;\ \ x^{(1)} = x_1,\ \ x^{(2)} = x_2,\ \ \ x^{(3)} = 1), \tag{2.5}$$

where $m = 1$, the constants $C_j$ will be determined and the functions $g_e^{(j)}$ are given by the equalities

$$2g_j^{(j)} = (-1)^j \nu (x_2^2 - x_1^2),\ \ g_j^{(3)} = \nu x_j, \tag{2.6}$$

$$g_1^{(2)} = g_2^{(1)} = \nu x_1 x_2 \quad (j = 1,2). \tag{2.7}$$

It is obvious that after substituting the expressions (3) in the conditions (1.15), the first two and sixth equations of these conditions will be satisfied identically and, from the other equations for the determination of the coefficients $C_j$, the following algebraic equations are obtained

$$C_1 J_{1j}^{(1)} + C_2 J_{2j}^{(1)} + C_3 J_{3j}^{(1)} = N_j \quad (j = 1,2,3), \tag{2.8}$$

where

$$J_{jk}^{(m)} = \int\int_{\omega} E_m x^{(j)} x^{(k)} d\omega \quad (m = 1,2;\ \ j,k = 1,2,3); \tag{2.9}$$

$$N_1 = -M_2, \quad N_2 = M_1, \quad N_3 = P_3. \tag{2.10}$$

As the origin and axis $Ox_1$ and $Ox_2$ of the system $Ox_1x_2x_3$ coincide with the center and the semi-axis of the elliptic domains respectively, from (6) follows

$$C_j = N_j [J_{jj}^{(1)}]_{-1} \quad (j = 1,2,3;\ \ J_{13} = J_{23} = J_{12} = 0), \tag{2.11}$$

where

$$4J_{11}^{(1)} = \pi E_1 (a_1^3 b_1 - a_0^3 b_0), \tag{2.12}$$

$$4J_{22}^{(1)} = \pi E_1 (a_1 b_1^3 - a_0 b_0^3),\ J_{33}^{(1)} = \pi E_1 (a_1 b_1 - a_0 b_0). \tag{2.13}$$

Now we consider the elliptic tube composed by two isotropic elastic materials, occupying the domains $\Omega_1$ and $\Omega_2$, respectively. It will be proposed that the different materials has identical Poisson's ratio ($\nu_1 = \nu_2$) and the different elastic modulus ($E_1 \neq E_2$).

The domain $\Omega_1 + \Omega_2$ of such composed body will be bounded by the planes (1.1) and by the elliptic surfaces $\Gamma_0, \Gamma_1$ and $\Gamma_2$, given by the equalities (1.2) and (1.3), where $\Gamma_0$ is the interior boundary, $\Gamma_2$ is the exterior boundary and $\Gamma_1$ is an interface, i.e., the border of the different materials. At the points of the surfaces $\Gamma_j$ the boundary-contact conditions (1.20) and (1.21) must be satisfied. As it is seen from (4), taking constants $C_j$ with the same values in each domain, the components of the displacements will be continuous at the crossing of the interface $\Gamma_1$ from $\Omega_1$ to $\Omega_2$.

In this case solution of the problems in each of the domains $\Omega_j$ will be represented in the form (3)–(4), in which, for each of the domain $\Omega_1$ and $\Omega_2$, modulus of elasticity $E$ reaches the values $E_1$ and $E_2$, respectively. It is obvious that all boundary-contact conditions are fulfilled and from (1.16), where $P_1 = P_2 = M_3 = 0$, for the coefficients $C_j$ of the composed tube we get

$$C_j = J_{jj}^{-1} N_j \;\; (j = 1, 2, 3), \quad J_{jj} = J_{jj}^{(1)} + J_{jj}^{(2)}, \tag{2.14}$$

where $N_j$ and $J_{jj}^{(m)}$ are determined by the equalities (6) and $J_{jj}$, for this composed elliptic tube, will be given as

$$\begin{aligned} 4J_{11} &= \pi[E_1(a_1^3 b_1 - a_0^3 b_0) + E_2(a_2^3 b_2 - a_1^3 b_1)], \\ 4J_{22} &= \pi[E_1(a_1 b_1^3 - a_0 b_0^3) + E_2(a_2 b_2^3 - a_1 b_1^3)], \\ J_{33} &= \pi[E_1(a_1 b_1 - a_0 b_0) + E_2(a_2 b_2 - a_1 b_1)]. \end{aligned} \tag{2.15}$$

## 3. A Torsion of a Two-Layer Isotropic Elliptic Tube with an Anisotropic Kernel

Let us consider the two-layer tube, bounded by the planes (1.1) and the elliptic surfaces.The three-dimensional domains occupied by jointly glued layers will be denoted by $\Omega_1$ and $\Omega_2$ respectively. A normal cross-section of this tube will be the domain composed by two isotropic layers (with the different physical characteristics), occupying the elliptic rings $\omega_1$ and $\omega_2$, bounded with the confocal ellipses $\gamma_0, \gamma_1$ and $\gamma_1, \gamma_2$, respectively, given by the equalities (1.3).

It is obvious that the rings $\omega_1$and $\omega_2$ will be glued along the ellipse $\gamma_1$ (interface) , $a_j^2 - b_j^2 = c^2$, where $a_j, b_j$ are semi-axis of the ellipses $\gamma_j$ (see (1.3)) and $2c$ is a focal length of this ellipses. Let us suppose that the composed isotropic elliptic tube is strengthened by the solid elliptic shaft, made from an anisotropic material. At an in each point there passes by a plane of an elastic symmetry perpendicular to the axis $Ox_3$. It is assumed that an elliptic shaft, occupying a simple connected domain $\Omega_0$, is bounded by the planes (1.1) and the surface $\Gamma_1$, along which a shaft is glued with the composed elliptic tube. The cross-section of the domain $\Omega_0$, denoted by $\omega_0$, will be glued with the composed two-layered isotropic planar elliptic ring along the ellipse $\gamma_0$.

Thus, there will be considered a beam, composed by two isotropic and one anisotropic materials, occupying the composed domain $\Omega = \Omega_0 + \Omega_1 + \Omega_2$, cross-section of which will be the composed planar elliptic domain $\omega = \omega_0 + \omega_1 + \omega_2$.

We now consider the torsion's problem for the indicated composed elliptic bar, when the exterior forces, acting at its ends, statistically are equivalent to the couple of forces twisting the bar by the moment $M_3$. The components of the displacements and the stresses in each of the domains $\Omega_j$ $(j = 0, 1, 2)$ will be given by [3,4,5]

$$(u_1)_j = -Gx_2x_3, \ \ (u_2)_j = Gx_1x_3, \ \ (u_3)_j = Gf_j(x_1, x_2). \qquad (3.1)$$

The corresponding components of the stresses in the isotropic domains $\Omega_j$ $(j = 1, 2)$ and in the anisotropic domain $\Omega_0$ are of the form

$$(\tau_{13})_j = G\mu_j(D_1f_j - x_2), \ \ (\tau_{23})_j = G\mu_j(D_2f_j + x_1) \ \ (j = 1, 2), \qquad (3.2)$$

$$\begin{aligned} u_j(D_1(\tau^*_{13})_0 &= G[A_{55}(D_1f^*_0 - x_2) + A_{45}(D_2f^*_0 + x_1)], && (3.3)\\ (\tau^*_{23})_0 &= G[A_{45}(D_1f^*_0 - x_2) + A_{44}(D_2f^*_0 + x_2). && (3.4)\end{aligned}$$

All the other components of the stresses $\tau_{jk}$ are equal to zero.

Substituting the expressions (1)–(3) in equations (1.20) and (1.21), where (1.20) must be fulfilled on $\gamma_2$ and (1.21) must be fulfilled on $\gamma_0$ and $\gamma_1$ (see (1.3)), for the functions $f_1, f_2$ and $f^*_0$, we obtain the boundary-value problems (1.9)–(1.11).

We seek the functions $f_J$ and $f^*_0$, satisfying conditions $1^0 - 3^0$, in the form

$$\begin{aligned} f_j &\equiv \Re\Phi_j(z) = \Re\Big[\frac{ic^2}{2}(m^{(j)}_1 t^2_1 + m^{(j)}_2 t^2_2)\Big] \ (j = 1, 2) && (3.5)\\ f^*_0 &\equiv \Re\Phi_*(z_*) = c^2\Re[m_0(t^{*2}_1 + t^{*2}_2)], && (3.6)\end{aligned}$$

where $2c$ is a focal length, variables $t_j$ and $t^*_j$ are given by the equalities (1.9) and (1.15) and complex coefficients $m_0$ and $m^{(j)}_k$ will be determined.

Taking into account equalities (1.8) and (1.12) and expressions (4), we obtain [1,2,5]

$$\begin{aligned}
(D_n f_k)_{\gamma_j} &= \{\Re[(n_1+in_2)\Phi_k'(z)]\}_{\gamma_j} \quad (j=0,1,2;\ k=1,2), &(3.7)\\
(D_n^* f_0^*)_{\gamma_0} &= A\{\Re[i(n_2-v_* n_1)\Phi_*'(z_*)]\}_{\gamma_0}. &(3.8)
\end{aligned}$$

These values are obtained by using the expression (1.4) and the equalities

$$\begin{aligned}
A_{55}+v_*A_{45} &= -iAv_*,\quad A_{45}+v_*A_{44}=iA,\\
\frac{d}{dz}\Phi(z) &= \Phi'(z),\quad \frac{d}{dz_*}\Phi_*(z_*)=\Phi_*'(z_*).
\end{aligned}$$

It is easy to show that the functions $h$ and $h^*$ from (1.7) on the ellipses $\gamma_j$ take the following values:

$$\begin{aligned}
4(h)_j &= ic^2\Theta_j^{-1}[\exp(2i\theta)-\exp(-2i\theta)]\ (j=1,2),\\
4(h^*)_0 &= i\Theta_0^{-1}[(A_{44}a_0^2-A_{55}b_0^2)\exp(2i\theta)\\
&\quad -(A_{44}a_0^2-A_{55}b_0^2+2A_{45}a_0b_0)\exp(-2i\theta)] \qquad (3.9)\\
(0 &\leq \theta\leq 2\pi).
\end{aligned}$$

Taking into account the expressions (1.4)-(1.15), (1.20), (1.21), (6), (7) and substituting the expressions (5) into the boundary-contact conditions (1.6), (1.7), (1.11), (1.12) and (4), after some elementary calculation, we obtain the following equations for the coefficients $m_0, m_1^{(j)}$ and $m_2^{(j)}$ from (5):

$$\begin{aligned}
2(m_1^{(2)}p_2^2-\overline{m}_2^{(2)}q_2^2)=i,\ (m_1^{(1)}-m_1^{(2)})p_1^2+(\overline{m}_1^{(1)}-\overline{m}_2^{(2)})q_1^2=0,\\
2[(\beta_1m_1^{(1)}-\beta_2m_1^{(2)})p_1^2-(\beta_1\overline{m}_2^{(1)}-\beta_2\overline{m}_2^{(2)})q_1^2=i(\beta_1-\beta_2),\\
m_0+\overline{\lambda^2}\overline{m_0-m_1^{(1)}p_0^2}--\overline{m}_2^{(1)}q_0^2=0, \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
&2c^2[\beta_0(m_0-\overline{\lambda^2}\overline{m}_0)-\beta_1(m_1^{(1)p_0^2}-\overline{m}_2^{(1)}q_0^2)]\\
=\ &i(A_{44}a_0^2-A_{55}b_0^2)-2A_{45}a_0b_0-ic^2\beta_1,
\end{aligned}$$

where

$$\beta_0\equiv A=\sqrt{A_{44}A_{55}-A_{45}^2},\quad \beta_1\equiv\mu_1,\quad \beta_2\equiv\mu_2. \qquad (3.11)$$

The barred expression $\overline{M}$ denotes the complex conjugate value of $M$.

After the simple transformation, the equations (8) reduce to the following three equations:

$$\begin{aligned}
2\beta_1p_1^2p_2^2m_1^{(1)}+(\beta_2Q_{21}-\beta_1P_{21}\overline{m}_2^{(2)}) &= T_1^*, &(3.12)\\
p_2^2q_1^2(m_0+\overline{\lambda^2}\overline{m}_0)+p_2^2Q_{10}m_1^{(1)}-q_0^2P_{21}\overline{m}_2^{(2)} &= T_2^*, &(3.13)\\
\beta_0p_2^2q_1^2(m_0-\overline{\lambda^2}\overline{m}_0)+\beta_1p_2^2Q_{01}m_1^{(1)}+\beta_2q_0^2Q_{12}\overline{m}_2^{(2)} &= T_3, &(3.14)
\end{aligned}$$

where

$$\begin{aligned}
P_{jk} &= p_j^2 q_k^2 + p_k^2 q_j^2, \quad Q_{jk} = p_j^2 q_k^2 - p_k^2 q_j^2. && (3.15)\\
2iT_1^* &= (\beta_2 - \beta_1)p_2^2 - p_1^2, \quad 2iT_2^* = -p_1^2 q_0^2, && (3.16)\\
2iT_3 &= (\beta_2 - \beta_1)p_2^2 q_0^2 - \beta_2 p_1^2 q_0^2 + && (3.17)\\
&+ p_2^2 q_1^2 c^{-2}(A_{55}b_0^2 - A_{44}a_0^2 + 2A_{45}a_0 b_0 + \mu_1). && (3.18)
\end{aligned}$$

From (10) we obtain the following equations with respect to $m_0$ and $m_2^{(2)}$ :

$$\begin{aligned}
2\beta_1 p_1^2 q_1^2 p_2^2(m_0 + \overline{\lambda^2}\overline{m_0}) - B_1\overline{m}_2^{(2)} &= T_1, && (3.19)\\
2\beta_0 p_1^2 q_1^2 p_2^2(m_0 - \overline{\lambda^2}\overline{m_0}) + B_2\overline{m}_2^{(2)} &= T_2, && (3.20)
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \beta_2 Q_{21}Q_{20} + \beta_1 P_{12}P_{10} > 0, && (3.21)\\
B_2 &= \beta_2 Q_{21}P_{10} + \beta_1 P_{21}Q_{10} > 0, && (3.22)\\
T_1 &= 2\beta_1 p_1^2 T_2^* - Q_{20}T_1^*, && (3.23)\\
T_2 &= 2p_1^2 T_3 - Q_{10}T_1^*. && (3.24)
\end{aligned}$$

For the solution of the system (12) let us introduce the denotations

$$\begin{aligned}
R_1 &= (\beta_1 B_2 - \beta_0 B_1)(\beta_1 B_2 + \beta_0 B_1)^{-1}, && (3.25)\\
R_2 &= (B_2 T_1 + B - 1T_2)[2p_1^2 p_2^2 q_1^2(\beta_1 B_2 + \beta_0 B - 1)]^{-1}. && (3.26)
\end{aligned}$$

It is obvious that

$$|\beta_1 B_2 + \beta_0 B_1| > 0, \quad 0 < |R_1| < 1. \qquad (3.27)$$

From the equation (12) we obtain

$$m_0 = \Delta_*^{-1}(R_2 - \overline{\lambda}R_1 R_2), \qquad (3.28)$$

where

$$|\lambda| < 1, \quad \Delta_* = 1 - |\lambda|^4 |R_1|^2 > 0. \qquad (3.29)$$

Thus, from the equations (12), (10) and (8), all the coefficients $m_2^{(2)}$, $m_1^{(1)}$, $m_2^{(1)}$ and $m_1^{(2)}$ will be determined directly.

In this case, in the conditions (1.16), $P_1 = P_2 = P_3 = M_1 = M_2 = 0$ and $M_3 \neq 0$ is the given number. It is easy to show that, by using the formulas (1.17) and (1.18), the first five equations will be satisfied identically and for the fulfillment of the sixth equation, from (1.16) the constant $G$ in (1)–(2) must be taken as

$$\begin{aligned} G &= M_3(G^*)^{-1}, \ G^* = G_1^* + G_2^*, & (3.30)\\ G_1^* &= \sum_{j=1,2} \int\int_{\omega_j} \mu_j(x_1^2 + x_2^2 + x_1 D_2 f_j - x_2 D_1 f_j) d\omega_j > 0; & (3.31)\\ G_2^* &= \int\int_{\omega_0} [A_{44}(D_2 f_0 + x_1)^2 + A_{55}(D_1 f_0 - x_2)^2 & (3.32)\\ &\quad +2A_{45}(D_2 f_0 + x_1)(D_1 f_0 - x_1)] d\omega_0 > 0 & (3.33)\\ &\quad (A_{44}A_{55} - A_{45}^2 > 0). & (3.34) \end{aligned}$$

Having the expressions of the coefficients $m_0$ and $m_k^{(j)}$ and substituting their values in the formulas (5), we can calculate the expressions of the functions $f_j$ and $f_0^*$ and the expressions (18) in the explicit form.

To solve the bending problem, the solutions of the torsion's problem for the homogeneous and composed elliptic tubes will be considered.

At first we consider the case when the anisotropic kernel is absent, i.e., the problem of the torsion of a two -layered isotropic elliptic tube will be considered. The torsion functions $f_j$ in the domains $\Omega_j$ $(j = 1, 2)$ will be represented by the expressions (5) and for the coefficients $m_k^{(j)}$ we have the following equations:

$$2(m_1^{(1)} p_0^2 + m_2^{(1)} q_0^2) = 1, \ 2(m_1^{(2)} p_2^2 + m_2^{(2)} q_2^2) = 1,$$

$$\begin{aligned} m_1^{(1)} p_1^2 - m_2^{(1)} q_1^2 - m_1^{(2)} p_1^2 + m_2^{(2)} q_1^2 &= 0, & (3.35)\\ 2\mu_1(m_1^{(1)} p_1^2 + m_2^{(1)} q_1^2) - 2\mu_2(m_1^{(2)} p_1^2 - m_2^{(2)} q_1^2) &= \mu_1 - \mu_2. \end{aligned}$$

It is easy to show that the solution of these equations will be given in the form [2]

$$\begin{aligned} m_1^{(1)} &= (2q_2^2 B_*)^{-1} \{ \mu_2 [p_2^2 (q_2^2 - q_0^2)(q_2^4 - q_1^4) & (3.36)\\ &\quad + q_0^2 (p_1^2 - p_2^2)(q_1^4 + q_2^4)] + \mu_1 p_1^2 (q_1^4 - q_0^4)(q_2^2 - q_0^2) \} & (3.37)\\ m_1^{(2)} &= (2q_0^2 B_*)^{-1} [\mu_1 (p_1^2 - p_0^2)(q_1^4 - q_0^4)(q_2^2 - q_0^2) & (3.38)\\ &\quad - \mu_2 p_1^2 (q_0^4 + q_1^4)(q_2^2 - q_1^2)], & (3.39) \end{aligned}$$

where

$$\begin{aligned} B_* &= p_0^2 p_2^2 [\mu_1 (q_1^4 - q_0^4) + \mu_2 (q_2^4 - q_1^4)] + q_0^2 q_2^2 [\mu_1 (p_0^4 - p_1^4) + \\ +\mu_2 (p_1^4 - p_2^4)] &< 0. & (3.40) \end{aligned}$$

From (19) we obtain

$$m_2^{(1)} = (2q_0^2)^{-1} - m_1^{(1)} p_0^2 q_0^{-2}, \quad m_2^{(2)} = (2q_2^2) - 1 - m_1^{(2)} p_2^2 q_2^{-2}. \qquad (3.41)$$

Let us consider the homogeneous elliptic tube, occupying the domain $\Omega_1$, bounded by the planes (1.1) and the confocal elliptic cylindrical surfaces $\Gamma_0$ and $\Gamma_1$, cross-section of which will be the elliptic ring $\omega_1$ bounded by two confocal ellipses $\gamma_0$ and $\gamma_1$, given by the equalities (1.3).

The solution of the torsion's problem for such tube in the domain $\Omega_1$ is given in the form (1)–(2) for $j = 1$, where the torsion's function $f_1$ is represented by the formula (5), and the function $f_1(z)$ must satisfy the boundary conditions (6) for $j = 0$ and $j = 1$ on the ellipses $\gamma_0$ and $\gamma_1$. From these conditions, for the coefficients $m_j^{(1)} \equiv m_j$, we obtain the equations [2]

$$2(m_1 p_j^2 + m_2 q_j^2) = 1 \quad (j = 0, 1), \qquad (3.42)$$

(3.22) implies

$$2m_1 = m^{-1}(q_1^2 - q_0^2 0, \; 2m_2 = m^{-1}(p_0^2 - p_1^2), \; m = p_0^2 q_1^2 - p_1^2 q_0^2 \neq 0. \quad (3.43)$$

Substituting the values of these coefficients in the expression of the function $f_1(z)$, given by the formula (5), we obtain

$$f_1 = \Re[ig(t_1^2 + ht_2^2)] = -g\Im(t_1^2 + ht_2^2), \qquad (3.44)$$
$$4g = c^2[(a_0 + b_0)^2 + (a_1 + b_1)^2]^{-1}, \quad h = (a_1 + b_1)^2 (a_0 - b_0)^{-2} \qquad (3.45)$$

Let us represent this function by means of the cartesian coordinates $x_j$.

According to (1.9), we have

$$t_1^2 = 2z^2 - c^2 + [z^2(z^2 - c^2)]^{0.5}, \quad t_2^2 = 2z^2 - c^2 - 2[z^2(z^2 - c^2)]^{0.5},$$

At first it should be calculated the expression

$$\Im(w_1 + iw_2)^{0.5} = \Im[z^2(z^2 - c^2)]^{0.5}, \qquad (3.46)$$

where

$$w_1 = (x_1^2 - x_2^2)(x_1^2 - x_2^2 - c^2) - 4x_1^2 x_2^2, \qquad (3.47)$$
$$w_2 = 2x_1^2 x - 2^2[2(x_1^2 - x_2^2) - c^2]. \qquad (3.48)$$

Now we will represent (26) in the trigonometric form $z^2(z^2 - c^2) = \rho \exp(i\phi)$ by means of the expression $\sin(2^{-1} arctg\, u) = u[u^2 + (1 + \sqrt{(1 + u^2)})^2]^{0.5}$. After the elementary calculations we obtain

$$\Im[z^2(z^2 - c^2)]^{0.5} = 2^{-1}\sqrt{2}\,\sqrt{(\rho - w_1)}, \quad \rho = (w_1^2 + w_2^2)^{0.5}$$

and the function $f_1$ will be written in the form

$$f_1 = -4g(1+h)x_1x_2 - 2^{-1}\sqrt{2}g(1-2h)\sqrt{(\rho - w_1)}. \qquad (3.49)$$

From the equalities (18), for the homogeneous isotropic elliptic tube, follows

$$\begin{aligned} GB_1 &= M_3, &(3.50)\\ B_1 &= \int\!\!\int_{\omega_1} [(D_2 f_0 + x_1)^2 + (D_1 f_0 - x_2)^2] d\omega = B_1^* + B_1^{**} > 0, &(3.51)\\ B_1^{**} &= \int\!\!\int_{\omega_1} (x_1^2 + x_2^2) d\omega, &(3.52)\\ B_1^{*} &= \int\!\!\int_{\omega_1} (x_1 D_2 f_1 - x_2 D_1 f_1) d\omega, &(3.53) \end{aligned}$$

where $B_1$ is a torsional rigidity.

Thus, the solution for the shift with the split along the line segment of the focuses $(-c, +c)$ is obtained. This allow us to calculate the intensity of the stresses in the effective form for the crack's problem.

## 4. Bending of a Cantilever Under a Transverse Force

Let it be assumed that the external forces applied to the end of the homogeneous elliptic tube, if $z = l$, are equivalent to two bending forces $P_1$ and $P_2$, parallel to the axis $Ox_1$ and $Ox_2$, respectively, and applied at the point $(x_1^0, x_2^0, l)$, where $(x_1^0, x_2^0)$ is an arbitrary point of the planar domain $\omega_1$. It is obvious that these forces with respect to the indicated axes create the moments $M_1 = P_2 l$ and $M_2 = -P_1 l$, respectively. Therefore, for the equilibrium of the part of the cylinder enclosed between the planes $x_3 = x_3^0$ and $x_3 = l$, where $x_3^0$ is an arbitrary number from the interval $0 \le x_3^0 < l$, it will be sufficient to require that the components of the stresses $\tau_{ek}$ in each cross-section $x_3 = x_3^0$ of the body satisfy the conditions (1.15), where we mean

$$P_3 = M_3 = 0, \;\; P_1 \neq 0, \;\; P_2 \neq 0, \;\; M_1 = P_2 l, \;\; M_2 = -P_1 l. \qquad (*)$$

Analogously to §2, we assume that the origin and the axis $Ox_1$ and $Ox_2$ of the system of the coordinates $Ox_1x_2x_3$ coincide with the center and the principal axis of the inertia of the domain $\omega_0$ - of an "under" base $(x_3 = 0)$ of the tube.

The solution to this problem, components of the displacements $u_e$ and the stresses $\tau_{mk}$, are given in the books [4,5,6]. We will represent the indicated values in the elementary modified form [1]:

$$
\begin{aligned}
(u_e)_j &= \Big\{(l-x_3)(C_1g_e^{(l)})+C_2g_e^{(2)}+\frac{l}{2}C_ex_3^2-\frac{1}{6}C_ex_3^3 && (4.1)\\
&\quad +G(-1)^ex_{3-e}x_3\Big\}_e && (4.2)\\
(e &= 1,2),\\
(u_3)_j &= \Big\{-\Big[x_3\Big(l-\frac{1}{2}x_3\Big)(C_lx_1+C_2x_2)-C_1F_1-C_2F_2 && (4.3)\\
&\quad +\frac{1}{3}(C_1x_1^3+C_2x_2^3)-Gf_1\Big]\Big\}_j, && (4.4)\\
(\tau_{e3})_j &= \Big\{\mu[D_e(C_1F_1+C_2F_2+Gf_1) && (4.5)\\
&\quad -C_ex_e^2-C_lg_e^{(1)}-C_2g_e^{(2)}+G(-1)^ex_{3-e}]\Big\}_j, && (4.6)\\
(\tau_{33})_j &= \{(x_3-l)(C_lx_1+C_2x_2)\}_j \quad (e=1,2), && (4.7)
\end{aligned}
$$

where $g_e^{(k)}$ are given by the equalities (2.4) and the constants $C_e$, $G$ will be determined. Here the components $u_e$ and $\tau_{ek}$ are supplied by the index $j$, indicating the number of the domain $\Omega_j$.

In the case of the homogeneous body the index $j$ will be omitted. Also, it will be noted that, in the expressions (1), the terms with the multiplier $l$ are the components of the displacements and the stresses multiplying by $l$ in the problem of the bending by the couple forces given by equalities (2.3).

Substituting the components of the stresses (1) in the equations of the equilibrium (1.22) and in the boundary conditions (1.20), we obtain that the torsion function $f$ and the bending functions $F_1$ and $F_2$ should satisfy the equations (1.5) in the elliptic ring $\omega_1$ and the boundary conditions (1.6) on the exterior boundaries $\gamma_0$ and $\gamma_1$, in which the functions $h$ and $H$ are given by the equalities (1.8).

The torsion's function $f$ was determined in the previous paragraph and the functions of bending $F_k^{(j)}$ in the domain $\omega_j$ may be represented by

$$F_k^{(j)}=\Re\Phi_k^{(j)}(z)\equiv\Re\sum_{r=1,3}[(m_r^{(k)})_jt_1^r+(m_r^{*(k)})_jt_2^r] \quad (k=1,2), \tag{4.8}$$

where the variables $t_j$ are given by the equalities (1.4) and the real coefficients $m_r^{(k)}j$ and $m_r^{*(k)}$ will be determined.

As for the torsion's function $f$ (see 3.6), let us represent the conditions for the operator $D_nF_k$ on the ellipse $\gamma_e$ in the form

$$[D_nF_k^{(j)}]_e\equiv\{\Re[(n_1+in_2)\Phi_k'^{(j)}(z)]\}_e=(H_k^{(j)})_{\gamma_e} \quad (k=1,2), \tag{4.9}$$

where $H^{(k)}$ is given by the equality (1.8) and by (see (1.3))

$$(n_1+in_2)_{\gamma_e}=\Theta_e^{-1}(b_e\cos\theta+a_e\sin\theta).$$

From the expressions (1.4), (1.8), (1.9), (4) we get

$$
\begin{aligned}
(D_n F_k^{(j)})_{\gamma_e} &= \Theta_e^{-1} \Re \sum_{r=1,3} r[p_e^r (m_r^{(k)})_j \exp(ir\theta) && (4.10)\\
&\quad - q_e^r (m_r^{*(k)})_j \exp(-ir\theta)] && (4.11)\\
(0 &\le \theta \le 2\pi),\\
[H_k^{(j)}]_{\gamma_e} &= \Theta_e^{-1} \sum_{r=1,3} [A_{re}^{(k)} \exp(ir\theta) && (4.12)\\
&\quad + (A_{-re}^{(k)})_j \exp(-ir\theta)] && (4.13)\\
(e &= 0, 1),
\end{aligned}
$$

where $\Theta_e$, $p_e$ and $q_e$ are given by the equalities (1.4) and (1.9).

In accordance with (1.8), the coefficients $A_k^{(j)}$ may be represented in the form

$$
\begin{aligned}
16A_{1e}^{(1)} &= 16A_{-1e}^{(1)} = 2\nu b_e c^2 + 3a_e^2 b_e (2+\nu), && (4.14)\\
8A_{3e}^{(1)} &= 8A_{-3e}^{(1)} = \Big(1 + \frac{1}{2}\nu\Big) a_e^2 b_e - \nu b_e c^2; && (4.15)\\
16A_{1e}^{(2)} &= 16A_{-1e}^{(2)} = 2i\nu a_j c^2 - 3i a_j b_j^2 (2+\nu), && (4.16)\\
8A_{3e}^{(2)} &= -8A_{-3e}^{(2)} = i\Big(1 + \frac{1}{2}\nu\Big) a_e b_e^2 - i\nu a_e c^2, && (4.17)
\end{aligned}
$$

where $c^2 = a_e^2 - b_e^2$ and $i^2 = -1$.

As in the previous paragraph, after some elementary calculations (what in detail will be given below for a two-layered tube), for determination of the coefficients $m_r^{(k)}$ and $m_r^{*(k)}$ in the expressions of the functions $F_k$ (for the homogeneous tube), given by the equalities (2), the following algebraic equations are obtained:

$$
\begin{aligned}
m_1^{(k)} p_e - m_1^{*(k)} q_e &= A_{1e}^{(k)}, && (4.18)\\
3(m_3^{(k)} p_e^3 - m_3^{*(k)} q_e^3) &= A_{3e}^{(k)} \ \ (e = 0, 1; k = 1, 2). && (4.19)
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
m_1^{(k)} &= B_1^{-1}(A_{11}^{(k)} q_0 - A_{10}^{(k)} q_1), && (4.20)\\
m_1^{*(k)} &= B_1^{-1}(A_{11}^{(k)} p_0 - A_{10}^{(k)} p_1); && (4.21)\\
3m_3^{(k)} &= B_3^{-1}(A_{31}^{(k)} q_0^3 - A_{30}^{(k)} q_1^3), && (4.22)\\
3m_3^{*(k)} &= B_3^{-1}(A_{31}^{(k)} p_0^3 - A_{30}^{(k)} p_1^3) \ \ (k = 1, 2), && (4.23)
\end{aligned}
$$

where

$$
B_1 = p_1 q_0 - p_0 q_1 > 0, \quad B_3 = p_1^3 q_0^3 - p_0^3 q_1^3 > 0, \quad p_1 > p_0, q_0 > q_1. \quad (4.24)
$$

Now let us consider the elliptic tube composed by two isotropic materials, occupying two domains $\Omega_1$ and $\Omega_2$, glued along the interface $\Gamma_1$ bounded by the planes (1.1) and the interior and exterior elliptic surfaces $\Gamma_0$ and $\Gamma_2$.The constants of elasticity in each of the elliptic ring $\Omega_e$ $(e = 1, 2)$ will be supplied with the index $e$. We assume that the different elastic materials have the same Poisson's ratio $\nu_1 = \nu_2$ and the different modulus of the elasticity $E_1 \neq E_2$.

As in the problem for the homogeneous tube, it is proposed that the resultant bending force $P(P_1, P_2, 0)$, of the exterior forces acting at the end $x_3 = l$ of the composed elliptic tube, is applied at the point $(x_1^0, x_2^0, l)$ of this end.

The solution of this problem, components of the displacements and the stresses in each of domains $\Omega_e$ we seek in the form (1). Substituting these expressions in the equations (1.20)–(1.22) and taking into account that the Poisson's ratio in the domains $\Omega_1$ and $\Omega_2$ has the same values $\nu_1 = \nu_2$), for the functions $(f)_e \equiv f_e$ and $(F_k)_e \equiv F_k^{(e)}$ we obtain the boundary value problem (1.5)–(1.7). We assume $\nu_1 = \nu_2 \equiv \nu$ and, therefore, according to (1.8) and (2.4), on the interface $\gamma_1$ we have

$$[g_k^{(j)}]_1 = [g_k^{(j)}]_2, \quad [H_k^{(j)}]_1 = [H_k^{(j)}]_2 \quad (j, k = 1, 2).$$

Therefore, the boundary conditions (1.6)–(1.7) for the harmonic functions $F_k$ we obtain in the form

$$[D_n F_k^{(2)}]_2 = [H_k^{(2)}]_2$$

on $\gamma_2$,

$$[D_n F_k^{(1)}]_0 = [H_k^{(1)}]_0$$

on $\gamma_0$

$$\begin{aligned} [\mu_1 D_n F_k^{(1)}]_1 - [\mu_2 D_n F_k^{(2)}]_2 &= (\mu_1 - \mu_2)[g_1(k) n_1 + g_2^{(k)} n_2], & (4.25) \\ [F_k^{(1)}]_1 - [F_k^{(2)}]_2 &= 0 & (4.26) \end{aligned}$$

on $\gamma_1$.

We recall that the torsion's functions $f_j$ in the domains $\Omega_j$ for the two-layered elliptic tube was determined in §3 by the equalities (3.5) and (3.19)–(3.22). The functions $F_k^{(j)}$ in each of domains $\Omega_j$ will be represented in the form (2), where the variables $t_1$ and $t_2$ are given in §1 and the coefficients $(m_e^{(k)})_j$ , $(m_e^{*(k)})_j$ will be determined.

Taking into account the expressions (3)–(4) and the conditions (9), we obtain

$$\Re \sum_{r=1,3} r[p_e^r (m_r^{(k)})_j \exp(ir\theta) - q_e^r (m_r^{*(k)})_j \exp(-ir\theta)] \qquad (4.27)$$

$$= \sum_{r=1,3} [A_{re}^{(k)} \exp(ir\theta) + A_{-re}^{(k)} \exp(-ir\theta)] \quad (4.28)$$

$$(e = 0, 1;\ k, j = 1, 2) \quad (4.29)$$

$$\Re \sum_{r=1,3} r\{p_1^r[\mu_1(m_r^{(k)})_1 - \mu_2(m_r^{(k)})_2] \exp(ir\theta) \quad (4.30)$$

$$-q_1^r[\mu_1(m_r^{*(k)})_1 - \mu_2(m_r^{*(k)})_2] \exp(-ir\theta)\} \quad (4.31)$$

$$= (\mu_1 - \mu_2) \sum_{r=1,3} [A_{re}^{(k)} \exp(ir\theta) + A_{re}^{(k)} \exp(-ir\theta)], \quad (4.32)$$

$$\Re \sum_{r=1,3} \{ p_1^r[(m_r^{(k)})_1 - (m_r^{(k)})_2] \exp(ir\theta) \quad (4.33)$$

$$-q_1^r[(m_r^{*(k)})_1 - (m_r^{*(k)})_2] \exp(-ir\theta)\}$$

$$= 0 \quad (4.34)$$

Comparing both sides of these equations, the members with the same powers of exp, for the coefficients we obtain the following algebraic equations

$$r[p_0^r(m_r^{(k)})_1 - q_0^r(m_r^{*(k)})_1] = 2A_{r0}^{(k)},\ \ r[p_2^r(m_r^{(k)})_2 - q_2^r(m_r^{*(k)})_2] = 2A_{r2}^{(k)},$$

$$\begin{aligned} rp_1^r[\mu_1(m_r^{(k)})_1 - \mu_2(m_r^{(k)})_2] - rq_1^r[\mu_1(m_r^{*(k)})_1 - \mu_2(m_r^{*(k)})_2] &= 2A_{r1}^{(k)}, \\ p_1^r[(m_r^{(k)})_1 - (m_r^{(k)})_2] + q_1^r[(m_r^{*(k)})_1 - (m_r^{*(k)})_2] &= 0, \quad (4.35) \end{aligned}$$

$$(k = 1, 2; r = 1, 3),$$

where $p_e$, $q_e$ and $A_{re}^{(k)}$ are given by the equalities (1.8) and (5), respectively. These expressions depend on the index $r = 1, 2$. Hence we obtain 8 algebraic equations with respect to 8 coefficients $(m_r^{(k)})_j$ and $(m_r^{*(k)})_j$.

From the first four equations we have

$$(m_r^{*(k)})_j = q_s^{-r}[p_s^r(m_r^{(k)})_j - 2r^{-1}A_{js}^{(k)}] \quad (4.36)$$

$$(j = 1, 2; \quad s = 2j - 2; \quad r = 1, 3).$$

Excluding from the first equation of the system (11) the coefficients $(m_r^{*(k)})_j$, by means of the expressions (12), we obtain

$$\begin{aligned} \mu_1 Q_{r0}^{(1)}(m_r^{(k)})_1 - \mu_2 Q_{r2}^{(1)}(m_r^{(k)})_2 &= B_r^{(k)}, \\ Q_{r0}^{(2)}(m_r^{(k)})_1 - Q_{r2}^{(2)}(m_r^{(k)})_2 &= C_r^{(k)} \ \ (r = 1, 3), \quad (4.37) \end{aligned}$$

where

$$Q_{re}^{(j)} = p_1^r + (-1)^j q_1^r p_e^r q_e^{-r}, \quad (4.38)$$

$$B_r^{(k)} = 2r^{-1}(A_{r1}^{(k)} - \mu_1 q_1^r q_0^{-r} A_{r0}^{(k)} + \mu_2 q_1^r q_2^{-r} A_{r2}^{(k)}), \quad (4.39)$$

$$C_r^{(k)} = 2r^{-1}(q_0^{-r} A_{r0}^{(k)} - q_2^{-r} A_{r2}^{(k)}) \quad (4.40)$$

$$(e = 0, 2;\ j = 1, 2;\ r = 1, 3). \quad (4.41)$$

Taking into account $a_e^2 - b_e^2 = c^2$, where $c = const$ is a focal length, and $a_j > a_{j-1}, b_j > bj - 1$ $(j = 1, 2)$, we conclude $q_{j-1} > q_j, p_j > p_{j-1}$ $(j = 1, 2)$, where (see (1.8)) $p_j = a_j + b_j$, $q_j = a_j - b_j$ $(j = 0, 1, 2)$.

Thus, according to (14), we obtain

$$\begin{aligned} Q_{r0}^{(2)} &> 0, \quad Q_{r2}^{(2)} > 0, \quad Q_{r0}^{(1)} = p_0^r(p_1^r p_0^{-r} - q_1^r q_0^{-r}) > 0, \\ Q_{r2}^{(1)} &= p_2^r(p_1^r p_2^{-r} - q_1^r q_2^{-r}) < 0 \quad (r = 1, 3). \end{aligned} \tag{4.42}$$

From the equations (13) we obtain

$$(m_r^{(k)})_1 = \nabla_r^{-1}(B_r^{(k)} Q_{r2}^{(2)} - \mu_2 C_r^{(k)} Q_{r2}^{(1)}), \tag{4.43}$$

$$(m_r^{(k)})_2 = \nabla_r^{-1}(B_r^{(k)} Q_{r0}^{(2)} - \mu_1 C_r^{(k)} Q_{r0}^{1)}) \tag{4.44}$$

$$(r = 1, 3; k = 1, 2), \tag{4.45}$$

where

$$\nabla_r = \mu_1 Q_{r0}^{(1)} Q_{r2}^{(2)} - \mu_2 Q_{r0}^{(2)} Q_{r2}^{(1)} \quad (r = 1, 3). \tag{4.46}$$

According to (16), $\nabla_r > 0$ and after substituting the values (16) in the right-hand side of the expressions (12), we obtain the values of the coefficients $(m_r^{*(k)})_j$ directly. Therefore, the bending functions $F_k^{(j)}$ are determined completely.

Now we must consider the equilibrium of the part of the tube as the rigid body, included between the upper end $x_3 = l$ and the plane $x_3 = x_3^0$ $(0 \leq x_3^0 < l)$. We must take into account that the normal of the plane of the upper end is $n_u(0, 0, +1)$ and on the opposite lower end is $n_l(0, 0, -1)$.

At first we must verify whether the components (1) produce the bending's forces $P_1$ and $P_2$ at the end of the tube. Substituting the components $\tau_{jk}$, given by the equalities (1), into the first three equations of (1.15), using the formulas (1.16) and the boundary conditions (9), we have

$$\int\int_\omega \tau_{e3} d\omega = \sum_{j=1,2} \int\int_{\omega_j} x_e D_3 \tau_{33} d\omega_j = P_e \quad (e = 1, 2), \tag{4.47}$$

$$\int\int_\omega \tau_{33} d\omega = 0. \tag{4.48}$$

According to (2.6) and (1), the third equality is satisfied identically and the first two equations will be satisfied if the constants $C_1$ and $C_2$ from (1) will be chosen in the following form:

$$C_1 = P_1 (J_{11}^{(1)} + J_{11}^{(2)})^{-1}, \quad C_2 = P_2 (J_{22}^{(1)} + J_{22}^{(2)})^{-1}, \tag{4.49}$$

where $J_{jj}^{(m)}$ are determined by the equalities (2.6)–(2.10).

Now we consider the sixth equation from (1.15), which is related to the equilibrium of the indicated part of the tube under the acting of the torsion's forces. On the upper end ($x_3 = l$), there are acting two forces $P_1$ and $P_2$ applied at the point $(x_1^0, x_2^0, l)$ and parallel to the axis $Ox_1$ and $Ox_2$, respectively. Therefore, these forces create with respect of the axis $Ox_3$ the torsion moment $M_\tau^p = x_1^0 P_2 - x_2^0 P_1$ and the stresses $\tau_{13}$ and $\tau_{23}$, acting on the lower end $x_3^0 = l - x_3 (0 \leq x_3 < l)$ with the norm n(0,0,-1), creating the torsion's moment $M_\tau^s = \int\int_\omega (x_2\tau_{13} - x_1\tau_{23}) d\omega$. Thus, for the equilibrium of the indicated part by acting of the torsion's factors, it must be satisfied the equation

$$\int\int_\omega (x_1\tau_{23} - x_2\tau_{13}) d\omega + x_1^0 P_2 - x_2^0 P_1 = 0. \qquad (**).$$

Substituting in this expression the values of the components $\tau_{j3}$ from (1), we obtain that the constants $G$ must be determined by the equalities

$$\begin{aligned} G &= M_\tau G_1, \\ M_\tau &= -\sum_{j=1,2} \int\int_{\omega_j} \sum_{l=1,2} (-1)^l x_{3-l} [P_l + C_l D_l F_l^{(j)} \\ &\quad - C_l x_l^2 - C_1 (g_l^{(1)})_j - C_2 (g_l^{(2)})_j] d\omega_j, \end{aligned} \qquad (4.50)$$

where $g_l^{(m)}$ and $G_1^*$ are given by the equalities (2.4) and (3.18), respectively. The bending functions are represented by the equalities (2) and (12)–(17).

Thus, the problem of the bending of the composed two-layered isotropic elliptic tube is solved completely.

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