THE THIRD ORDER OF ACCURACY OPERATOR SPLIT OF THE EVOLUTION PROBLEM USING PADE APPROXIMATION

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Abstract

In the present work symmetrized sequential-parallel decomposition difference scheme of the third degree precision for the solution of Cauchy abstract problem is offered. Third degree precision is reached by introducing $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ complex parameter. For the error of the considered scheme the explicit a priori estimation is obtained.

Key words and phrases: Decomposition Method, Splitting Operator, Semigroup, Trotter formula, Cauchy abstract problem, Resolvent Polynomials.

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Introduction

The study of the approximated schemes of the solution of evolution problems leads to the conclusion that to each approximated scheme corresponds a certain operator (the solution operator of a discrete problem), which approximates the solution operator (semigroup) of a continuous problem. The inverse statement is also true: constructing approximation of a continuous semigroup, thereby we build an approximated scheme of the solution of an evolution problem. For example, if we apply the Rotte's method for the solution of an evolution problem, the solution operator of the obtained semidiscrete problem will be a discrete semigroup. Thus we arrive at the problem of approximation of a continuous semigroup by means of discrete semigroups (see T. Kato [35], Ch. IX).

If a decomposition method is applied, the corresponding solution operator generates the Trotter formula, (see Trotter H. [53]) or the Chernoff formula, (see Chernoff P. R. [8], [9]) or a formula, which is the combination of these formulas. Therefore, the error estimate of a decomposition method is equivalent to the problem of approximation of a continuous semigroup using Trotter and Chernoff type formulas. The works of T. Ichinose and

S. Takanobu [31], T. Ichinose and H. Tamura [32], J. Rogava [45] (see also [46] Ch. II) are dedicated to the error estimate of the Chernoff and Trotter type formulas.

There exist decomposition schemes of two types: differential and difference. Trotter type formulas correspond to differential schemes, and Chernoff type formulas - to difference schemes.

We call Trotter type formulas the formulas, which give an approximation of a semigroup by the combination of semigroups generated by the addends of its generating operator.

We call Chernoff type formulas the formulas, which can be obtained from Trotter type formulas if we replace the semigroups by the corresponding resolvents.

The decomposition scheme, associated with the Trotter formula, allows us to split the Cauchy problem for an evolution equation with the operator $A = A_1 + A_2$ into two problems corresponding to the operators A_1 and A_2. These problems are solved sequentially on each time interval of the length t/n.

The decomposition scheme, associated with the Chernoff formula, is known as the method of fractional steps (see N. N. Ianenko [29]).

The first works devoted to the construction and investigation of decomposition schemes for nonstationary problems were published in the fifties and sixties of the 20-th century. (see V. B. Andreev [2], G. A. Baker [3], G. A. Baker, T. A. Oliphant [4], G. Birkhoff, R. S. Varga [6], G. Birkhoff, R. S. Varga, D. Young [7], J. Douglas [14], J. Douglas, H. Rachford [15], E. G. Diakonov [11],[12], M. Dryja [16], G. Fairweather, A. R. Gourlay, A. R. Mitchell [17], I. V. Fryazinov [18]), D. G. Gordeziani [23], A. R. Gourlay, A. R. Mitchell [27], N. N. Ianenko [28], [29], N. N. Ianenko, G. V. Demidov [30], A. N. Konovalov [36], G. I. Marchuk [39], G. I. Marchuk, N. N. Ianenko [41], G. I. Marchuk, U. M. Sultangazin [42], D. Peaceman, H. Rachford [43], V. P. Ilin [33], A. A. Samarskii [47]-[49], R. Temam [52]). The works of these authors were the basis of the further investigations of decomposition schemes.

From the point of view of computation, decomposition schemes can be divided into two groups: schemes of sequential account (for example see G. I. Marchuk [40]) and schemes of parallel account (D. G. Gordeziani, H. V. Meladze [25], [26], D. G. Gordeziani, A. A. Samarskii [24], A. M. Kuzyk, V. L. Makarov [38]). In [46] (see Ch. II) are obtained the explicit estimates for decomposition schemes of parallel account, which were considered in [25]. At present, there exist many papers dedicated to decomposition method (see [29], [40], [50] and their references).

In the above-stated works the considered schemes has the first or second order precision. As far as we know, high order precision decomposition formulas in the case of two addends ($A = A_1 + A_2$) for the first time were obtained by B. O. Dia and M. Schatzman (see [13]). Let us note that the formulas constructed in this paper are not automatically stable. Decomposition formulas are called automatically stable if the sum of the absolute values of split coefficients is equal to one. Q. Sheng proved (see [51]) that on the real number field there does not exist an automatically stable decomposition of the semigroup $exp\,(-tA)$ with the precision order higher than two. In the works [19]-[21], by introducing a complex parameter, there are constructed the third order decomposition differential schemes, the corresponding formulas of which represent automatically stable decomposition formulas.

In the present work there are constructed the third order decomposition difference schemes. These schemes can be obtained on the basis of the decomposition formulas constructed in [21], if we replace semigroups by the corresponding resolvent polynomials of high order precision. For the considered schemes there are obtained explicit a priori estimates. Under explicit estimates we imply such a priori estimates for the error of solution, where the constants in the right-hand side do not depend on the solution of an initial continuous problem, i.e. are absolute constants.

1. Statement of the problem and main result

Let us consider the Cauchy abstract problem in the Banach space X :

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi, \tag{1.1}$$

where A is a closed linear operator with the definition domain $D\,[A]$, which is everywhere dense in X, φ is a given element from X, $f(t) \in C^1\left([0;\infty)\,;X\right)$.

Let the operator $(-A)$ generate the strongly continuous semigroup $\{\exp(-tA)\}_{t\geq 0}$, then the solution of the problem (1.1) is given by the following formula ([34], [37]):

$$u(t) = U(t,A)\varphi + \int\limits_0^t U(t-s,A)f(s)ds, \tag{1.2}$$

where $U(t,A) = exp\,(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are compactly defined, closed, linear operators in X.

As it is well-known, the essence of decomposition method consists in splitting the semigroup $U\,(t,A)$ by means of the semigroups $U\,(t,A_j)$ $(j = 1, 2)$.

In [21] (see also [22],[23]) there is constructed the following decomposition formula with the local precision of fourth order:

$$V(\tau) = \frac{1}{2}\left[U(\tau, \overline{\alpha}A_1) U(\tau, A_2) U(\tau, \alpha A_1) + U(\tau, \overline{\alpha}A_2) U(\tau, A_1) U(\tau, \alpha A_2)\right], \tag{1.3}$$

where $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$.

In the above-mentioned work it is shown that:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^4\right),$$

where $O_p\left(\tau^4\right)$ is the operator, norm of which is of the fourth order with respect to τ (more precisely, in the case of the unbounded operator $\left\|O_p\left(\tau^4\right)\varphi\right\| = O\left(\tau^4\right)$ for any φ from the definition domain of $O_p\left(\tau^4\right)$). At the same time, in [22], we constructed the semigroup approximations with the local precision of the fourth order using the following resolvent polynomials:

$$\begin{aligned} W(\tau, A) &= aI + b\,(I + \lambda\tau A)^{-1} + c\,(I + \lambda\tau A)^{-2}, \qquad (1.4)\\ W(\tau, A) &= \left(I - \frac{1}{3}\tau A\right)(I + \lambda\tau A)^{-1}\left(I + \overline{\lambda}\tau A\right)^{-1}, \end{aligned}$$

where in the first formula $\lambda = \frac{1}{2} + \frac{1}{2\sqrt{3}}, \;\; a = 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}, \;\; b = \frac{3}{\lambda} - \frac{1}{\lambda^2}, \;\; c = \frac{1}{2\lambda^2} - \frac{1}{\lambda}$, and in the second $\lambda = \frac{1}{3} \pm i\frac{1}{3\sqrt{2}} \quad \left(i = \sqrt{-1}\right)$.

The approximations defined by formulas (1.4) in the scalar case represent the Pade approximations for exponential functions (see [5]).

Using simple transformation, we can show that the operator $W(\tau, A)$ defined by formula (1.4) coincides with the transition operator of the Calahan scheme (see [54]). The stability of the Calahan scheme for an abstract parabolic equation is investigated in [1].

On the basis of formulas (1.3) and (1.4) we can construct the following decomposition formula (Analogously we can construct a decomposition formula for another resolvent polynomial):

$$\begin{aligned} V(\tau) = \frac{1}{2}\big[&W(\tau, \overline{\alpha}A_1) W(\tau, A_2) W(\tau, \alpha A_1)\\ &+W(\tau, \overline{\alpha}A_2) W(\tau, A_1) W(\tau, \alpha A_2)\big]. \end{aligned} \tag{1.5}$$

Below we shall show that this formula has the precision of the fourth order:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^4\right).$$

In the present work, on the basis of formula (1.5), a decomposition scheme with the third order precision will be constructed for the solution of problem (1.1).

Let us introduce the following net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 0, 1, 2, ..., \tau > 0\}.$$

According to formula (1.2), we have:

$$u(t_k) = U(\tau, A)u(t_{k-1}) + \int_{t_{k-1}}^{t_k} U(t_k - s, A) f(s) ds.$$

Let us rewritten this formula in the following form:

$$\begin{aligned} u(t_k) &= U(\tau, A)u(t_{k-1}) \\ &+\frac{\tau}{4}\left(3U\left(\tau, \frac{1}{3}A\right) f\left(t_{k-1/3}\right) + U(\tau, A) f(t_{k-1})\right) + R_{k,4}(\tau), \\ u(t_0) &= \varphi \quad (k = 1, 2, ...), \end{aligned} \tag{1.6}$$

where $R_{k,4}(\tau)$ is the residual member of the quadrature formula

$$\begin{aligned} R_{k,4}(\tau) &= \int_{t_{k-1}}^{t_k} U(t_k - s, A) f(s) ds \\ &-\frac{\tau}{4}\left(3U\left(\tau, \frac{1}{3}A\right) f\left(t_{k-1/3}\right) + U(\tau, A) f(t_{k-1})\right) \cdot \end{aligned} \tag{1.7}$$

For the sufficiently smooth function f the following estimate is true (see. Lemma 2.3):

$$\|R_{k,4}(\tau)\| = O\left(\tau^4\right).$$

On the basis of formula (1.6) let us construct the following scheme:

$$\begin{aligned} u_k &= V(\tau)u_{k-1} \\ &+\frac{\tau}{4}\left(3V_{KN}\left(\frac{1}{3}\tau\right) f\left(t_{k-1/3}\right) + V_{KN}(\tau) f(t_{k-1})\right), \\ u_0 &= \varphi \quad (k = 1, 2, ...), \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left[W(\tau, \overline{\alpha}A_1) W(\tau, A_2) W(\tau, \alpha A_1)\right. \\ &\left.+W(\tau, \overline{\alpha}A_2) W(\tau, A_1) W(\tau, \alpha A_2)\right], \\ V_{KN}(\tau) &= KN\left(\tau, \frac{1}{2}A_1\right) KN(\tau, A_2) KN\left(\tau, \frac{1}{2}A_1\right), \\ KN(\tau, A) &= \left(I - \frac{1}{2}\tau A\right)\left(I + \frac{1}{2}\tau A\right)^{-1}. \end{aligned}$$

and $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$, $\lambda = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, $a = 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}$, $b = \frac{3}{\lambda} - \frac{1}{\lambda^2}$, $c = \frac{1}{2\lambda^2} - \frac{1}{\lambda}$. Let us note that the operator $KN(\tau, A)$ is the transition operator of the Krank-Nickolson scheme.

Let us perform the computation of the scheme (1.8) by the following algorithm:

$$u_k = u_{k,0} + \frac{\tau}{4}\left(3u_{k,1} + u_{k,2}\right),$$

where $u_{k,0}$ is calculated by the scheme:

$$\begin{aligned} v_{k-2/3} &= W(\tau, \alpha A_1) u_{k-1}, \quad w_{k-2/3} = W(\tau, \alpha A_2) u_{k-1}, \\ v_{k-1/3} &= W(\tau, A_2) v_{k-2/3}, \quad w_{k-1/3} = W(\tau, A_1) w_{k-2/3}, \\ v_k &= W(\tau, \overline{\alpha} A_1) v_{k-1/3}, \quad w_k = W(\tau, \overline{\alpha} A_2) w_{k-1/3}, \\ u_{k,0} &= \frac{1}{2}[v_k + w_k], \quad u_0 = \varphi, \end{aligned} \tag{1.9}$$

and $u_{k,s}$ $(s = 1, 2)$ - by the scheme:

$$\begin{aligned} u_{k-2/3,s} &= KN\left(\tau, \frac{1}{2}\gamma_s A_1\right) f(t_k - \gamma_s \tau), \\ u_{k-1/3,s} &= KN(\tau, \gamma_s A_2) u_{k-2/3,s}, \\ u_{k,s} &= KN\left(\tau, \frac{1}{2}\gamma_s A_1\right) u_{k-1/3,s}, \end{aligned}$$

with $\gamma_1 = \frac{1}{3}$ and $\gamma_2 = 1$.

We need the natural powers of the operator $A = A_1 + A_2 \quad (A^s, \quad s = 2, 3, 4)$. Usually they are defined as follows:

$$\begin{aligned} A^2 &= \left(A_1^2 + A_2^2\right) + \left(A_1A_2 + A_2A_1\right), \\ A^3 &= \left(A_1^3 + A_2^3\right) + \left(A_1^2A_2 + ... + A_2^2A_1\right) + \left(A_1A_2A_1 + A_2A_1A_2\right), \\ A^4 &= \left(A_1^4 + A_2^4\right) + \left(A_1^3A_2 + ... + A_2^3A_1\right) \\ &\quad + \left(A_1^2A_2A_1 + ... + A_2^2A_1A_2\right) + \left(A_1A_2A_1A_2 + A_2A_1A_2A_1\right). \end{aligned}$$

Obviously, the definition domain $D(A^s)$ of the operator A^s is the intersection of the domains of its addends.

Let us introduce the following denotations:

$$\begin{aligned} \|\varphi\|_A &= \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D[A], \\ \|\varphi\|_{A^2} &= \sum_{i,j=1}^{2} \|A_iA_j\varphi\|, \quad \varphi \in D\left[A^2\right], \end{aligned}$$

where $\|\cdot\|$ is a norm in X, $\|\varphi\|_{A^s}$ $(s = 3, 4)$ are defined similarly.

The following theorem takes place:

Theorem 1.1 *Let the following conditions be satisfied:*

(a) There exists such $\tau_0 > 0$ *that for any* $0 < \tau \leq \tau_0$ *there exist operators* $(I + \gamma\lambda_i\tau A_j)^{-1}$, $j = 1, 2,$ $\gamma = 1, \alpha, \overline{\alpha}$ *and they are bounded. Besides, the following inequalities are true:*

$$\|W(\tau, \gamma A_j)\| \leq e^{\omega\tau}, \quad \omega = const > 0;$$

(b) The operator $(-A)$ *generates the strongly continuous semigroup* $U(t, A) = \exp(-tA)$, *for which the following inequality is true:*

$$\|U(t, A)\| \leq Me^{\omega t}, \quad M, \omega = const > 0;$$

(c) $U(s, A)\varphi \in D[A^4]$ *for any* $s \geq 0$*;*

(d) $f(t) \in C^3([0, \infty); X)$; $f(t) \in D[A^3]$, $f'(t) \in D[A^2]$, $f''(t) \in D[A]$ *and* $U(s, A) f(t) \in D[A^4]$ *for any fixed* t *and* s $(t, s \geq 0)$.

Then the following estimate holds:

$$\begin{aligned} \|u(t_k) - u_k\| \leq\ & ce^{\omega_0 t_k} t_k \tau^3 \Bigg(\sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^4} \\ & + t_k \sup_{s,t\in[0,t_k]} \|U(s, A) f(t)\|_{A^4} + \\ & \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} \\ & + \sup_{t\in[0,t_k]} \|f''(t)\|_{A} + \sup_{t\in[0,t_k]} \|f'''(t)\| \Bigg), \end{aligned} \tag{1.10}$$

where c *and* ω_0 *are positive constants.*

2. Auxiliary lemmas

Let us prove the auxiliary lemmas on which the proof of the Theorem 1.1 is based.

Lemma 2.1 *If the condition (a) of the Theorem 1.1 is satisfied, then for the operator* $W(t, A)$ *the following decomposition is true:*

$$W(t, A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_{W,k}(t, A), \quad k = 1, 2, 3, 4, \tag{2.1}$$

where, for the residual member, the following estimate holds:

$$\|R_{W,k}(t,A)\varphi\| \leq c_0 e^{\omega_0 t} t^k \left\|A^k \varphi\right\|, \quad \varphi \in D\left[A^k\right], \quad c_0, \omega_0 = const > 0. \tag{2.2}$$

Proof. We obviously have:

$$\begin{aligned}(I+\gamma A)^{-1} &= I - I + (I+\gamma A)^{-1} = I - (I+\gamma A)^{-1}(I+\gamma A - I) \\ &= I - \gamma A (I+A)^{-1}.\end{aligned}$$

From this for any natural k we can get the following expansion:

$$(I+\gamma A)^{-1} = \sum_{i=0}^{k-1} (-1)^i \gamma^i A^i + \gamma^k A^k (I+\gamma A)^{-1}. \tag{2.3}$$

Let us decompose the resolvent polynomial $W(\tau, A)$ according to the formula (2.3) up to the first order, we obtain:

$$\begin{aligned}W(\tau, A) &= aI + b(I+\lambda\tau A)^{-1} + c(I+\lambda\tau A)^{-2} \\ &= (a+b+c)I + R_{W,1}(\tau, A),\end{aligned} \tag{2.4}$$

where

$$R_{W,1}(\tau,A) = -(b+c)\lambda\tau A(I+\lambda\tau A)^{-1} - c\lambda\tau A(I+\lambda\tau A)^{-2}.$$

Since $(I+\lambda\tau A)^{-1}$ is bounded according to the condition *(a)* of the Theorem 1.1, therefore:

$$\|R_{W,1}(\tau,A)\varphi\| \leq c_0 e^{\omega_0\tau}\tau\|A\varphi\|, \quad \varphi \in D[A]. \tag{2.5}$$

Substituting the values of the parameters a, b and c in (2.4), we obtain:

$$W(\tau,A) = I + R_{W_1,1}(\tau,A). \tag{2.6}$$

Let us decompose the resolvent polynomial $W(\tau, A)$ according to the formula (2.3) up to the second order:

$$W(\tau,A) = (a+b+c)I - (b+2c)\lambda\tau A + R_{W,2}(\tau,A), \tag{2.7}$$

where

$$R_{W,2}(\tau,A) = (b+2c)\lambda^2\tau^2A^2(I+\lambda\tau A)^{-1} + \lambda^2\tau^2(I+\lambda\tau A)^{-2}A^2.$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\|R_{W,2}(\tau,A)\varphi\| \leq c_0 e^{\omega_0\tau}\tau^2\left\|A^2\varphi\right\|, \quad \varphi \in D\left[A^2\right]. \tag{2.8}$$

If we substitute the values of the parameters a, b and c in (2.7), we obtain:

$$W(\tau, A) = I - \tau A + R_{W,2}(\tau, A). \tag{2.9}$$

Let us decompose the resolvent polynomial $W(\tau, A)$ according to the formula (2.3) up to the third order:

$$\begin{aligned} W(\tau, A) &= (a+b+c) I - (b+2c) \lambda\tau A + (b+3c) \lambda^2\tau^2 A^2 \\ &\quad + R_{W,3}(\tau, A), \end{aligned} \tag{2.10}$$

where

$$R_{W,3}(\tau, A) = -(b+3c) \lambda^3\tau^3 (I+\lambda\tau A)^{-1} A^3 - c\lambda^3\tau^3 (I+\lambda\tau A)^{-2} A^3,$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\|R_{W,3}(\tau, A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^3 \left\|A^3\varphi\right\|, \quad \varphi \in D\left[A^3\right]. \tag{2.11}$$

If we substitute the values of the parameters a, b and c in (2.10), we obtain:

$$W(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 + R_{W,3}(\tau, A). \tag{2.12}$$

Finally let us decompose the resolvent polynomial $W(\tau, A)$ according to the formula (2.3) up to the fourth order:

$$\begin{aligned} W(\tau, A) &= (a+b+c) I - (b+2c) \lambda\tau A + (b+3c) \lambda^2\tau^2 A^2 \\ &\quad -(b+4c)\lambda^3\tau^3 A^3 + R_{W,4}(\tau, A), \end{aligned} \tag{2.13}$$

where

$$R_{W,4}(\tau, A) = (b+4c) \lambda^4\tau^4 (I+\lambda\tau A)^{-1} A^4 + c\lambda^4\tau^4 (I+\lambda\tau A)^{-2} A^4.$$

According to the condition *(a)* of the Theorem 1.1 we have:

$$\|R_{W,4}(\tau, A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^4 \left\|A^4\varphi\right\|, \quad \varphi \in D\left[A^4\right]. \tag{2.14}$$

If we substitute the values of the parameters a, b and c in (2.13), we obtain:

$$W(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{W,4}(\tau, A). \tag{2.15}$$

Uniting formulas (2.6),(2.9),(2.12) and (2.15) we obtain formula (2.1), and uniting inequalities (2.5), (2.8), (2.11) and (2.14) we obtain estimate (2.2) ■

Lemma 2.2 *If the conditions (a), (b) and (c) of the Theorem 1.1 are satisfied, then the following estimate holds:*

$$\left\| \left[U^k(\tau, A) - V^k(\tau) \right] \varphi \right\| \leq c e^{\omega_0 t_k} t_k \tau^3 \sup_{s \in [0, t_k]} \| U(s, A) \varphi \|_{A^4}, \tag{2.16}$$

where c *and* ω_0 *are positive constants.*

Proof. The following formula is true (see Kato. T. [35], p. 603):

$$A \int_r^t U(s, A)\, ds = U(r, A) - U(t, A), \quad 0 \leq r \leq t. \tag{2.17}$$

Hence we get the following expansion:

$$U(t, A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t, A), \tag{2.18}$$

where

$$R_k(t, A) = (-A)^k \int_0^t \int_0^{s_1} \ldots \int_0^{s_{k-1}} U(s, A)\, ds\, ds_{k-1} \ldots ds_1. \tag{2.19}$$

Let us decompose all the resolvent polynomials in the operator $V(\tau)$ according to the formula (2.1) from right to left, so that each residual member be of the fourth order. We shall have:

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{V,4}(\tau), \tag{2.20}$$

where

$$R_{V,4}(\tau) = \frac{1}{2}\left[R_{1,2}(\tau) + R_{2,1}(\tau) \right],$$

and

$$\begin{aligned} R_{i,j}(\tau) \;=\; & R_{W,4}(\tau, \overline{\alpha} A_i) - \tau R_{W,3}(\tau, \overline{\alpha} A_i) A_j + \frac{1}{2}\tau^2 R_{W,2}(\tau, \overline{\alpha} A_i) A_j^2 \\ & - \frac{1}{6}\tau^3 R_{W,1}(\tau, \overline{\alpha} A_i) A_j^3 + W(\tau, \overline{\alpha} A_i) R_{W,4}(\tau, A_j) \\ & - \alpha \tau R_{W,3}(\tau, \overline{\alpha} A_i) A_i \\ & + \alpha \tau^2 R_{W,2}(\tau, \overline{\alpha} A_i) A_j A_i - \frac{1}{2}\alpha \tau^3 R_{W,1}(\tau, \overline{\alpha} A_i) A_j^2 A_i \\ & - \alpha \tau W(\tau, \overline{\alpha} A_i) R_{W,3}(\tau, A_j) A_i \\ & + \frac{1}{2}\alpha^2 \tau^2 R_{W,2}(\tau, \overline{\alpha} A_i) A_i^2 - \frac{1}{2}\alpha^2 \tau^3 R_{W,1}(\tau, \overline{\alpha} A_i) A_j A_i^2 \\ & + \frac{1}{2}\alpha^2 \tau^2 W(\tau, \overline{\alpha} A_i) R_{W,2}(\tau, A_j) A_i^2 - \frac{1}{6}\alpha^3 \tau^3 R_{W,1}(t, \overline{\alpha} A_i) A_i^3 \end{aligned}$$

$$-\frac{1}{6}\alpha^3\tau^3 W(\tau,\overline{\alpha}A_i)R_{W,1}(\tau,A_j)A_i^3$$
$$+W(\tau,\overline{\alpha}A_i)W(\tau,A_j)R_{W,4}(\tau,\alpha A_i),$$
$$i,j = 1,2.$$

Hence according to the condition *(a)* of the Theorem 1.1 we have the following estimate:

$$\|R_{V,4}(\tau)\varphi\| \le ce^{\omega_0\tau}\tau^4\|\varphi\|_{A^4}, \quad \varphi\in D\left[A^4\right]. \tag{2.21}$$

From the (2.18) ($k=4$) and (2.20) it follows:

$$U(\tau,A)-V(\tau)=R_4(\tau,A)-R_{V,4}(\tau).$$

From here according to inequalities (2.19) and (2.21) we obtain the following estimate:

$$\|[U(\tau,A)-V(\tau)]\varphi\| \le ce^{\omega_0\tau}\tau^4\|\varphi\|_{A^4}, \quad \varphi\in D\left[A^4\right]. \tag{2.22}$$

The following representation is obvious:

$$[U^k(\tau,A)-V^k(\tau)]\varphi=\sum_{i=1}^{k}V^{k-i}(\tau)[U(\tau,A)-V(\tau)]U^{i-1}(\tau,A)\varphi.$$

Hence, according to the conditions *(a), (b), (c)* of the Theorem 1.1 and inequality (2.22), we have the sought estimate ■

Lemma 2.3 *Let the following conditions be satisfied:*

(a) The operator A satisfies the conditions of the Theorem 1.1;

(b) $f(t)\in C^3([0,\infty);X)$, and $f(t)\in D\left[A^3\right]$ for every fixed t, $f^{(k)}(t)\in D\left[A^{3-k}\right]$, $k=1,2$.

Then the following estimate holds

$$\left\|\int_0^\tau U(\tau-s,A)f(s)\,ds-\frac{\tau}{4}\left[U(\tau,A)f(0)+3U\left(\frac{1}{3}\tau,A\right)f\left(\frac{2}{3}\tau\right)\right]\right\|\le$$

$$\le ce^{\omega_0\tau}\tau^4\left[\left\|A^3f\left(\frac{2}{3}\tau\right)\right\|+\sup_{\xi\in[o,\tau]}\left\|A^2f'(\xi)\right\| + \sup_{\xi\in[o,\tau]}\left\|Af''(\xi)\right\|+\sup_{\xi\in[o,\tau]}\left\|f'''(\xi)\right\|\right], \tag{2.23}$$

where c and ω_0 are positive constants.

Proof. Using the simple transformation, we will obtain the following representation:

$$\int_0^\tau U(\tau-s,A)f\left(s\right)ds-\frac{\tau}{4}\left[U\left(\tau,A\right)f\left(0\right)+3U\left(\frac{1}{3}\tau,A\right)f\left(\frac{2}{3}\tau\right)\right]$$
$$= r\left(\tau\right)-U\left(\tau,A\right)z\left(\tau\right)-R\left(\tau,A\right)f\left(\frac{2}{3}\tau\right). \qquad (2.24)$$

where

$$z\left(\tau\right) = \frac{1}{4}\int_0^\tau f\left(0\right)ds+\frac{3}{4}\int_0^\tau f\left(\frac{2}{3}\tau\right)ds-\int_0^\tau f\left(s\right)ds,$$
$$R\left(\tau,A\right) = \frac{3}{4}\int_0^\tau U\left(\frac{1}{3}\tau,A\right)ds+\frac{1}{4}\int_0^\tau U\left(\tau,A\right)ds-\int_0^\tau U(\tau-s,A)ds$$

and

$$r\left(\tau\right)=\int_0^\tau \left[U(\tau-s,A)-U\left(\tau,A\right)\right]\left[f\left(s\right)-f\left(\frac{2}{3}\tau\right)\right]ds.$$

According to formula (2.17) for $r\left(\tau\right)$ we can obtain the following representation:

$$r\left(\tau\right) = A\int_0^\tau \left[\int_0^s A\int_0^\xi U(\tau-\eta,A)d\eta d\xi\int_{\frac{2}{3}\tau}^s f'\left(\xi\right)d\xi\right]ds$$
$$-A\int_0^\tau \left[\int_0^s U\left(\tau,A\right)d\xi\int_{\frac{2}{3}\tau}^s\int_0^\xi f''\left(\eta\right)d\eta d\xi\right]ds.$$

Hence we obtain the following estimate:

$$\left\|r\left(\tau\right)\right\|\leq ce^{\omega\tau}\tau^4\left[\sup_{\xi\in[o,\tau]}\left\|A^2f'\left(\xi\right)\right\|+\sup_{\xi\in[o,\tau]}\left\|Af''\left(\xi\right)\right\|\right]. \qquad (2.25)$$

For the function $(-z\left(\tau\right))$ the following representation is valid:

$$-z\left(\tau\right)=\frac{1}{4}\int_0^\tau\int_0^s\int_0^\xi\int_0^\eta f'''(\zeta)d\zeta d\eta d\xi ds+\frac{3}{4}\int_0^\tau\int_{\frac{2}{3}\tau}^s\int_0^\xi\int_0^\eta f'''(\zeta)d\zeta d\eta d\xi ds.$$

Hence we obtain the following estimate:

$$\|U(\tau,A)\,z(\tau)\| \le ce^{\omega\tau}\tau^4 \sup_{s\in[o,\tau]}\left\|f'''(s)\right\|. \tag{2.26}$$

And finally let us transform the integral $R(\tau,A)$ according to formula (2.17):

$$\begin{aligned}-R(\tau,A) &= -\frac{3}{4}A^3\int_0^\tau\int_{\frac{2}{3}\tau}^{s}\int_0^\xi\int_0^\eta U(\tau-\zeta,A)d\zeta d\eta d\xi ds\\ &\quad -\frac{1}{4}A^3\int_0^\tau\int_0^{s}\int_0^\xi\int_0^\eta U(\tau-\zeta,A)d\zeta d\eta d\xi ds.\end{aligned}$$

Hence we obtain the following estimate:

$$\left\|R(\tau,A)\,f\left(\frac{2}{3}\tau\right)\right\| \le ce^{\omega\tau}\tau^4\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\|. \tag{2.27}$$

From equality (2.24) according to inequalities (2.25), (2.26) and (2.27) we obtain the sought estimate.

According to the Lemma 2.3 for $R_{k,4}(\tau)$ (see formula (1.7)), the following estimate holds:

$$\begin{aligned}\|R_{k,4}(\tau)\| &\le ce^{\omega_0\tau}\tau^4\left[\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\| + \sup_{\xi\in[t_{k-1},t_k]}\left\|A^2 f'(\xi)\right\|\right.\\ &\quad\left. + \sup_{\xi\in[t_{k-1},t_k]}\left\|Af''(\xi)\right\| + \sup_{\xi\in[t_{k-1},t_k]}\left\|f'''(\xi)\right\|\right].\end{aligned} \tag{2.28}$$

3. Proof of the Theorem 1.1

Let us return to the proof of the Theorem 1.1.

Proof. Let us write formula (1.5) in the following form:

$$u(t_k) = U^k(\tau,A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau,A)\left(F_i^{(1)} + R_{k,4}(\tau)\right), \tag{3.1}$$

where

$$F_i^{(1)} = \frac{\tau}{4}\left(3U\left(\frac{1}{3}\tau,A\right)f\left(t_{i-1/3}\right) + U(\tau,A)\,f\left(t_{i-1}\right)\right). \tag{3.2}$$

Analogously let us present u_k as follows:

$$u_k = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau)F_i^{(2)}, \tag{3.3}$$

where

$$F_i^{(2)} = \frac{\tau}{4}\left(3V_{KN}\left(\frac{1}{3}\tau\right) f\left(t_{i-1/3}\right) + V_{KN}\left(\tau\right) f\left(t_{i-1}\right)\right). \tag{3.4}$$

From equalities (3.1) and (3.3) it follows:

$$\begin{aligned} u(t_k) - u_k &= \left[U^k(\tau,A) - V^k(\tau)\right]\varphi \\ &+ \sum_{i=0}^{k}\left[U^{k-i}(\tau,A)F_i^{(1)} - V^{k-i}(\tau)F_i^{(2)}\right] \\ &+ \sum_{i=0}^{k} U^{k-i}(\tau,A)R_{k,4}\left(\tau\right) = \left[U^k(\tau,A) - V^k(\tau)\right]\varphi \\ &+ \sum_{i=1}^{k}\left[\left(U^{k-i}(\tau,A) - V^{k-i}(\tau)\right)F_i^{(1)}\right. \\ &\left. + V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right] \\ &+ \sum_{i=0}^{k} U^{k-i}(\tau,A)R_{k,4}\left(\tau\right). \end{aligned} \tag{3.5}$$

From formulas (3.2) and (3.4) we have:

$$\begin{aligned} F_i^{(1)} - F_i^{(2)} &= \frac{\tau}{4}\left(3\left(U\left(\frac{1}{3}\tau, A\right) - V_{KN}\left(\frac{1}{3}\tau\right)\right) f\left(t_{i-1/3}\right) + \right. \\ &\left. + \left(U\left(\tau, A\right) - V_{KN}\left(\frac{1}{3}\tau\right)\right) f\left(t_{i-1}\right)\right). \end{aligned} \tag{3.6}$$

The following inequality can be easily obtained:

$$\left\|\left[U\left(\tau,A\right) - KN\left(\tau\right)\right]\varphi\right\| \le ce^{\omega_0\tau}\tau^3\left\|\varphi\right\|_{A^3}, \quad \varphi \in D\left[A^3\right].$$

Hence analogously to estimate (2.22) we obtain:

$$\left\|\left[U\left(\tau,A\right) - V_{KN}\left(\tau\right)\right]\varphi\right\| \le ce^{\omega_0\tau}\tau^3\left\|\varphi\right\|_{A^3}, \quad \varphi \in D\left[A^3\right].$$

According to this inequality, from equality (3.6) we obtain the following estimate:

$$\left\|F_k^{(1)} - F_k^{(2)}\right\| \le ce^{\omega_0\tau}\tau^4 \sup_{t\in[t_{k-1},t_k]}\left\|f(t)\right\|_{A^3}. \tag{3.7}$$

According to the Lemma 2.1 we have:

$$\left\| \sum_{i=1}^{k} \left(U^{k-i}(\tau, A) - V^{k-i}(\tau) \right) F_i^{(1)} \right\| \leq c e^{\omega_0 t_k} t_k^2 \tau^3 \sup_{s,t \in [0,t_k]} \| U(s,A) f(t) \|_{A^4} . \tag{3.8}$$

From equality (3.5) according to inequalities (3.7), (3.8), (2.16), (2.28) and the condition *(b)* of the Theorem 1.1 we obtain:

$$\begin{aligned} \|u(t_k) - u_k\| \leq\ & c e^{\omega_0 t_k} t_k \tau^3 \Bigg(\sup_{s \in [0,t_k]} \|U(s,A)\varphi\|_{A^4} \\ & + t_k \sup_{s,t \in [0,t_k]} \|U(s,A) f(t)\|_{A^4} + \sup_{t \in [0,t_k]} \|f(t)\|_{A^3} \Bigg) \\ & + \sup_{t \in [0,t_k]} \left\| f'(t) \right\|_{A^2} + \sup_{t \in [0,t_k]} \left\| f''(t) \right\|_{A} + \sup_{t \in [0,t_k]} \left\| f'''(t) \right\| \end{aligned}$$

■

Remark 3.1. *The operator $V^k(\tau)$ is the solution operator of the above-considered decomposed problem. It is obvious that, according to the condition of the Theorem 1.1 $\left(\|W(t, \gamma A_j)\| \leq e^{\omega t}\right)$, the norm of the operator $V^k(\tau)$ is less than or equal to $e^{\omega_0 t_k}$. From this follows the stability of the above-stated decomposition scheme on each finite time interval.*

Remark 3.2. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint non negative operators, in estimate (1.10) ω_0 will be replaced by 0. Alongside with this, for the transition operator of the splitted problem, the estimate $\left\|V^k(\tau)\right\| \leq 1$ will be true.*

Remark 3.3. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint, positive definite operators, in estimate (1.10) ω_0 will be replaced by $-\alpha_0$, $\alpha_0 > 0$. Alongside with this, for the transition operator of the splitted problem, the estimate $\left\|V^k(\tau)\right\| \leq e^{-\alpha_1 t_k}$, $\alpha_1 > 0$ will be true.*

Remark 3.4. *According to the classical theorem of Hille-Philips-Iosida (see [44]), if the operator $(-A)$ generates a strongly continuous semigroup, then the inequality in the condition (b) of the Theorem 1.1 is automatically satisfied. The proof of this inequality is based on the uniform boundedness principle, according to which the constants M and ω exist, but generally can not be explicitly constructed (according to the method of the proof). That is why we demand satisfying of the inequality in the condition (b) of the Theorem 1.1.*

4. Stability of the splitted problem

In this paragraph we state the sufficient conditions, from which follows the inequality:

$$\left\| V^k(\tau) \right\| \le c, \quad c = const > 0 \quad (k = 1, 2, ...).$$

Fullfilment of the inequality means the stability of splitted problem.

Let us examine first the stability of non split problem. Below we will prove the theorems, concerning the stability of non split problems with the transition operators given by formulas (1.4). These theorems obviously have an independent value, and the proof of the stability of split problem is based on them.

Theorem 4.1 *Assume that A is a linear, closed, densely defined operator in the Banach space X. Assume the sector $S = \{z : |\arg z| < \varphi_0, \quad z \ne 0,$ $0 < \varphi_0 < \frac{\pi}{2}\}$ completely includes the spectrum of the operator A and for any $z \notin S$ $(z \ne 0)$ the following inequality holds:*

$$\|zI - A\| \le \frac{c}{|z|}, \quad c = const > 0. \tag{4.1}$$

Then, for any $\tau > 0$ and natural k, the following estimate is valid:

$$\left\| W^k(\tau, A) \right\| \le c, \quad c = const > 0,$$

where

$$W(\tau, A) = \left(I - \frac{1}{3}\tau A \right) (I + \lambda \tau A)^{-1} \left(I + \overline{\lambda} \tau A \right)^{-1}, \quad \lambda = \frac{1}{3} \pm i \frac{1}{3\sqrt{2}}.$$

The proof of the Theorem 1.1 is based on the following lemma.

Lemma 4.1 *Assume that the operator A satisfies conditions of the Theorem 1.1.*

Then for any $\tau > 0$ and natural k the following inequality is valid:

$$\left\| (I + \tau A)^{-k} \right\| \le c, \quad c = const > 0.$$

Proof. Let us compare the operator $(I + \tau A)^{-k}$ to the operator $(I + (t_k/2)A)^{-2}$ $(t_k = k\tau)$. With this purpose we present their difference by means of the Danford-Taylor integral (see [51] Ch. VII):

$$\begin{aligned} (I + \tau A)^{-k} - \left(I + \frac{t_k}{2} A \right)^{-2} &= \frac{1}{2\pi i} \int\limits_{\Gamma} \left((1 + \tau z)^{-k} - \left(1 + \frac{t_k}{2} z \right)^{-2} \right) \\ &\times (zI - A)^{-1} dz, \end{aligned} \tag{4.2}$$

where Γ is a bound of the sector $\left\{|\arg z| \leq \varphi, \quad \varphi_0 \leq \varphi < \frac{\pi}{2}\right\}$. Let us estimate the absolute value of the integrand scalar function. With this purpose we use the following representation:

$$\begin{aligned}(1+\tau z)^{-k} - \left(1+\frac{t_k}{2}z\right)^{-2} &= \int_0^{t_k} \frac{d}{ds}\left[\left(1+\frac{t_k-s}{2}z\right)^{-2}\left(1+\frac{s}{k}z\right)^{-k}\right] ds \\ &= z^2 \int_0^{t_k} \left(\frac{s}{k} - \frac{t_k-s}{2}\right) \times \left(1+\frac{t_k-s}{2}z\right)^{-3} \quad (4.3) \\ &\quad \times \left(1+\frac{s}{k}z\right)^{-k-1} ds. \quad (4.4)\end{aligned}$$

Obviously we have:

$$\begin{aligned}\left|1+\frac{s}{k}z\right|^{k+1} &= \left|1+\frac{s}{k}\rho\left(\cos\varphi + i\sin\varphi\right)\right|^{k+1} \\ &= \left(1+2\frac{s}{k}\mu\rho + \frac{s^2}{k^2}\rho^2\right)^{\frac{k+1}{2}}, \\ \mu &= \cos\varphi, \quad \varphi = \arg(z), \quad |z| = \rho.\end{aligned}$$

From here follows the inequality:

$$\begin{aligned}\left|1+\frac{s}{k}z\right|^{k+1} &\geq \left(1+\frac{s}{k}\mu\rho\right)^{k+1} \geq 1 + \frac{k+1}{k}s\mu\rho \\ &\quad + \frac{k+1}{2k}s^2\mu^2\rho^2 + \frac{k^2-1}{6k^2}s^3\mu^3\rho^3 \\ &\geq 1 + s\mu\rho + \frac{1}{2}s^2\mu^2\rho^2 + \frac{1}{8}s^3\mu^3\rho^3 \quad (k \geq 2).\end{aligned}$$

With account of this inequality we have:

$$\begin{aligned}&\left|1+\frac{t_k-s}{2}z\right|^3 \left|1+\frac{s}{k}z\right|^{k+1} \\ \geq\ &\left(1+(t_k-s)\mu\rho + \frac{1}{2}(t_k-s)^2\mu^2\rho^2 + \frac{1}{8}(t_k-s)^3\mu^3\rho^3\right) \\ &\times \left(1+s\mu\rho + \frac{1}{2}s^2\mu^2\rho^2 + \frac{1}{8}s^3\mu^3\rho^3\right) \\ \geq\ &1 + t_k\mu\rho + \frac{1}{2}\left(s^2 + (t_k-s)^2\right)\mu^2\rho^2 \\ &+ \frac{1}{8}\left(s^3 + (t_k-s)^3\right)\mu^3\rho^3 \\ \geq\ &1 + t_k\mu\rho + \frac{1}{4}t_k^2\mu^2\rho^2 + \frac{1}{32}t_k^3\mu^3\rho^3 \quad (4.5) \\ \geq\ &(1+\mu_0 t_k\rho)^3, \quad \mu_0 = \frac{1}{3\sqrt{2}}\mu.\end{aligned}$$

From (4.3), with account of (4.4), it follows:

$$\left|(I+\tau z)^{-k}-\left(I+\frac{t_k}{2}z\right)^{-2}\right| \le \frac{\rho^2}{(1+\mu_0 t_k\rho)^3}\int\limits_0^{t_k}\left(\frac{s}{k}+\frac{t_k-s}{2}\right)ds \le \frac{(t_k\rho)^2}{(1+\mu_0 t_k\rho)^3} \tag{4.6}$$

From (4.2), with account of (4.5) and (4.1), it follows:

$$\left\|(I+\tau A)^{-k}-\left(I+\frac{t_k}{2}A\right)^{-2}\right\| \le ct_k^2\int\limits_0^{\infty}\frac{\rho}{(1+\mu_0 t_k\rho)^3}d\rho = c. \tag{4.7}$$

Due to inequality (4.1) we have:

$$\left\|\left(I+\frac{t_k}{2}A\right)^{-2}\right\| \le c. \tag{4.8}$$

From (4.6) and (4.7), according to the triangle inequality, the sought estimate follows ■

Proof of the Theorem 4.1.

Let us compare the operator $W^k(\tau,A)$ to the corresponding powers of the operator $W_0(\tau,A)=(I+\tau A)^{-1}$. Obviously the representation is valid:

$$W_0^k(\tau,A)-W^k(\tau,A)=(W_0(\tau,A)-W(\tau,A))\sum_{i=0}^{k-1}W_0^i(\tau,A)\,W^{k-i-1}(\tau,A), \tag{4.9}$$

In order to estimate the norm of the operator in the right hand-side of this equality let us estimate the absolute values of the scalar functions $W(\tau,z), W_0(\tau,z)$, and $W_0(\tau,z)-W(\tau,z)$ $(z\in\Gamma)$. We obtain:

$$W(\tau,z)=\frac{P_1(\tau z)}{P_2(\tau z)},$$

where

$$P_1(z) = 1-\frac{1}{3}z,$$
$$P_2(z) = 1+\frac{2}{3}z+\frac{1}{6}z^2.$$

Let us calculate the squares of the modules of the polynomials $P_1(\tau z)$ and $P_2(\tau z)$:

$$|P_1(\tau z)|^2 = \left|1-\frac{1}{3}\tau\rho(\cos\varphi+i\sin\varphi)\right|^2 = 1-\frac{2}{3}\tau\mu\rho+\frac{1}{9}\tau^2\rho^2, \tag{4.10}$$

$$\begin{aligned}|P_2(\tau z)|^2 &= \left|1+\frac{2}{3}\tau\rho(\cos\varphi+i\sin\varphi)\right.\\ &\left.+\frac{1}{6}\tau^2\rho^2(\cos(2\varphi)+i\sin(2\varphi))\right|^2\\ &= 1+\frac{4}{3}\tau\mu\rho+\left(\frac{1}{9}+\frac{2}{3}\mu^2\right)\tau^2\rho^2\\ &+\frac{2}{9}\tau^3\mu\rho^3+\frac{1}{36}\tau^4\rho^4, \qquad (4.11)\end{aligned}$$

where $\mu=\cos\varphi$, $\varphi=\arg(z)$, $|z|=\rho$.

From (4.9) and (4.10) it follows:

$$(1+\tau\mu_1\rho)^2|P_1(\tau z)|^2\leq|P_2(\tau z)|^2, \quad \mu_1=\frac{1}{3}\mu.$$

From here we obtain:

$$|W(\tau,z)|=\frac{|P_1(\tau z)|}{|P_2(\tau z)|}\leq\frac{1}{1+\mu_1\tau\rho}. \qquad (4.12)$$

Let us estimate the absolute value of the function $W_0(\tau,z)-W(\tau,z)$. We obviously have:

$$|W_0(\tau,z)-W(\tau,z)|=\frac{\frac{1}{4}\tau^2\rho^2}{(1+2\tau\mu\rho+\tau^2\rho^2)^{\frac{1}{2}}|P_2(\tau z)|}.$$

From here, taking into account the inequality $|P_2(\tau z)|\geq(1+\tau\mu_1\rho)^2$, it follows:

$$|W_0(\tau,z)-W(\tau,z)|\leq\frac{\tau^2\rho^2}{(1+\mu_1\tau\rho)^3}. \qquad (4.13)$$

For the absolute value of $W_0(\tau,z)$, the following estimate holds:

$$\begin{aligned}|W_0(\tau,z)| &= \frac{1}{|1+\tau\rho(\cos\varphi+i\sin\varphi)|}\\ &= \frac{1}{(1+2\tau\mu\rho+\tau^2\rho^2)^{\frac{1}{2}}}\leq\frac{1}{1+\mu\tau\rho}. \qquad (4.14)\end{aligned}$$

Let us present the operator-function $W_0^k(\tau,A)-W^k(\tau,A)$ by means of the Danford-Taylor integral:

$$W_0^k(\tau,A)-W^k(\tau,A)=\frac{1}{2\pi i}\int_\Gamma\left(W_0^k(\tau,z)-W^k(\tau,z)\right)(zI-A)^{-1}dz,$$

where Γ is the bound of the sector $\left\{|\arg z| \le \varphi, \quad \varphi_0 \le \varphi < \frac{\pi}{2}\right\}$. From here, according to (4.8), we obtain:

$$\begin{aligned} W_0^k(\tau,A) - W^k(\tau,A) &= \frac{1}{2\pi i}\int\limits_{\Gamma} \Big((W_0(\tau,z) - W(\tau,z)) \\ &\quad \times \sum_{i=0}^{k-1} W_0^i(\tau,z)\, W^{k-i-1}(\tau,z)\Big)(zI-A)^{-1}\,dz, \end{aligned}$$

From here, with account of inequalities (4.1),(4.11),(4.12) and (4.13), we obtain the following estimate:

$$\begin{aligned} \left\|W_0^k(\tau,A) - W^k(\tau,A)\right\| &\le c\int\limits_0^{\infty}\Bigg(\frac{\tau^2\rho^2}{(1+\tau\mu_1\rho)^3} \\ &\quad \times \sum_{i=0}^{k-1}\frac{1}{(1+\tau\mu\rho)^i}\frac{1}{(1+\tau\mu_1\rho)^{k-i-1}}\Bigg)\frac{1}{\rho}d\rho \\ &\le ck\tau\int\limits_0^{\infty}\frac{\tau\rho d\rho}{(1+\tau\mu_1\rho)^{k+1}} \\ &= ck\int\limits_0^{\infty}\frac{xd\rho}{(1+x)^{k+1}} = c. \end{aligned}$$

From this inequality and the estimate of Lemma 4.1, according to the triangle inequality, follows the sought estimate ■

Theorem 4.2 *Assume that the operator A satisfies conditions of the Theorem 1.1.*

Then, for any $\tau > 0$ and natural k, the following estimate holds:

$$\left\|W^k(\tau,A)\right\| \le c, \quad c = const > 0, \tag{4.15}$$

where

$$\begin{aligned} W(\tau,A) &= aI + b(I+\lambda\tau A)^{-1} + c(I+\lambda\tau A)^{-2}, \\ \lambda &= \frac{1}{2} + \frac{1}{2\sqrt{3}}, \\ a &= 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}, \\ b &= \frac{3}{\lambda} - \frac{1}{\lambda^2}, \\ c &= \frac{1}{2\lambda^2} - \frac{1}{\lambda}. \end{aligned}$$

Proof. Estimate (4.14) was proven by Alibekov and Sobolevskii (see [1]), for the case when the operator A, instead of condition (4.1), satisfies the following condition:

$$\|zI - A\| \leq \frac{c}{1+|z|}, \quad c = const > 0. \tag{4.16}$$

The above-mentioned authors present the operator $W^k(\tau, A)$ as the sum of the following three addends:

$$\begin{aligned} W^k(\tau, A) &= \left((a+b+c) + (2a+b)\lambda\tau A + a\lambda^2\tau^2 A^2\right) \\ &\quad \times (I+\lambda\tau A)^{-2} W^{k-1}(\tau, A) \\ &= \left(1 + (2a+b)\lambda\tau A + a\lambda^2\tau^2 A^2\right)(I+\lambda\tau A)^{-2} W^{k-1}(\tau, A) \\ &= J_{1,k}(\tau, A) + J_{2,k}(\tau, A) + J_{3,k}(\tau, A), \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} J_{1,k}(\tau, A) &= (I+\lambda\tau A)^{-2} W^{k-1}(\tau, A), \\ J_{2,k}(\tau, A) &= 2a_0\lambda\tau A (I+\lambda\tau A)^{-2} W^{k-1}(\tau, A), \ a_0 = 2a+b, \\ J_{3,k}(\tau, A) &= a\lambda^2\tau^2 A^2 (I+\lambda\tau A)^{-2} W^{k-1}(\tau, A). \end{aligned}$$

It should be noted that the estimates (for any $\tau > 0$ and natural k):

$$\|J_{l,k}(\tau, A)\| \leq c, \quad l = 2, 3, \quad c = const > 0 \tag{4.18}$$

are valid in the case when the operator A satisfies condition (4.1). The above-mentioned authors need rather heavier condition (4.15) to obtain for the operator $J_{1,k}(\tau, A)$ an estimate, analogous to estimate (4.17), since in this case they use fraction powers of the operator A. Below we give the estimate of the operator $J_{1,k}(\tau, A)$ in the case of condition (4.1).

Let us estimate the norm of the operator $J_{1,k}(\tau, A)$. At first we estimate the module of the scalar function $W(\tau, z)$. Obviously we have:

$$W(\tau, z) = \frac{P_3(\tau z)}{P_4(\tau z)},$$

where

$$\begin{aligned} P_3(z) &= 1 + a_0\lambda z + a\lambda^2 z^2, \\ P_4(z) &= (1+\lambda z)^2. \end{aligned}$$

Let us calculate the modules of the polynomials $P_3(\tau z)$ and $P_4(\tau z)$:

$$\begin{aligned} |P_3(\tau z)|^2 &= |1 + a_0\lambda\tau\rho(\cos\varphi + i\sin\varphi) \\ &\quad + a\lambda^2\tau^2\rho^2(\cos(2\varphi) + i\sin(2\varphi))|^2 \\ &= 1 + 2a_0\mu\lambda\tau\rho + 2\left(1 + 2a\mu^2\right)\lambda^2\tau^2\rho^2 \qquad (4.19) \\ &\quad + 2aa_0\mu\lambda^3\tau^3\rho^3 + a^2\lambda^4\tau^4\rho^4, \\ |P_4(\tau z)| &= |1 + \lambda\tau\rho(\cos\varphi + i\sin\varphi)|^2 \\ &= 1 + 2\mu\lambda\tau\rho + \lambda^2\tau^2\rho^2. \qquad (4.20) \end{aligned}$$

From (4.18) and (4.19) it follows:

$$|P_3(\tau z)|^2 \le |P_4(\tau z)|^2.$$

From here follows the estimate:

$$|W(\tau, z)| \le 1. \qquad (4.21)$$

In order to estimate the norm of the operator $J_{1,k}(\tau, A)$, we compare it to the following operator:

$$W_1(\tau, A) = \left((I + a_0\lambda\tau A)(I + \tau A)^{-2}\right)^{k-1}(I + \lambda\tau A)^{-2},$$

Let us present the difference between the operators $J_{1,k}(\tau, A)$ and $W_1(\tau, A)$ in the form:

$$\begin{aligned} J_{1,k}(\tau, A) - W_1(\tau, A) &= (I + \lambda\tau A)^{-2} \\ &\quad \times \left(W^{k-1}(\tau, A) - \left((I + a_0\lambda\tau A)(I + \lambda\tau A)^{-2}\right)^{k-1}\right) \\ &= (I + \lambda\tau A)^{-2}\left(W(\tau, A) - (I + a_0\lambda\tau A)(I + \lambda\tau A)^{-2}\right) \\ &\quad \times \sum_{i=0}^{k-2}\left((I + a_0\lambda\tau A)(I + \lambda\tau A)^{-2}\right)^i W^{k-i-2}(\tau, A) \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{(1 + \lambda\tau z)^2} \\ &\quad \times \left(\frac{1 + a_0\lambda\tau z + a\lambda^2\tau^2 z^2}{(1 + \lambda\tau z)^2} - \frac{1 + a_0\lambda\tau z}{(1 + \lambda\tau z)^2}\right) \\ &\quad \times \sum_{i=0}^{k-2}\left(\frac{1 + a_0\lambda\tau z}{(1 + \lambda\tau z)^2}\right)^i W^{k-i-2}(\tau, z)(zI - A)^{-1}\,dz \\ &= \frac{1}{2\pi i}\sum_{i=0}^{k-2}\int_\Gamma \frac{a\lambda^2\tau^2 z}{(1 + \lambda\tau z)^4}\left(\frac{1 + a_0\lambda\tau z}{(1 + \lambda\tau z)^2}\right)^i \\ &\quad \times W^{k-i-2}(\tau, z)\,z\,(zI - A)^{-1}\,dz. \qquad (4.22) \end{aligned}$$

By simple calculations we obtain:

$$\begin{aligned}\left|\frac{1+a_0\lambda\tau z}{(1+\lambda\tau z)^2}\right| &= \frac{|1+a_0\lambda\tau\rho\,(\cos\varphi+i\sin\varphi)|}{|1+\lambda\tau\rho\,(\cos\varphi+i\sin\varphi)|^2}\\ &\le \frac{1}{(1+2\lambda\tau\mu\rho+\lambda^2\tau^2\rho^2)^{\frac{1}{2}}}\le\frac{1}{1+\lambda\tau\mu\rho}. \qquad (4.23)\end{aligned}$$

From (4.21), with account of inequalities (4.1), (4.20) and (4.22), we obtain:

$$\begin{aligned}\|J_{1,k}(\tau,A)-W_1(\tau,A)\| &\le c\sum_{i=0}^{k-2}\int_0^\infty\frac{\tau^2\rho}{(1+\lambda\tau\mu\rho)^{i+4}}d\rho\\ &= c\sum_{i=0}^{k-2}\int_0^\infty\frac{x}{(1+x)^{i+4}}dx\\ &= c\sum_{i=0}^{k-2}\int_0^\infty\left(\frac{1}{(1+x)^{i+3}}-\frac{1}{(1+x)^{i+4}}\right)dx\\ &= c\sum_{i=0}^{k-2}\left(\frac{1}{i+2}-\frac{1}{i+3}\right)\\ &= \left(\frac{1}{2}-\frac{1}{k+1}\right)c\le c\cdot \qquad (4.24)\end{aligned}$$

In order to obtain the final estimate, we need to estimate the norm of the operator $W_1(\tau,A)$. According to the Lemma 4.1 and the inequality $a_0=2a+b<1$, we have:

$$\begin{aligned}\|W_1(\tau,A)\| &\le \left\|\left((I+a_0\lambda\tau A)(I+\lambda\tau A)^{-2}\right)^k(I+\lambda\tau A)^{-2}\right\|\\ &\le \left\|\left((I+a_0\lambda\tau A)(I+\lambda\tau A)^{-1}\right)^k\right\|\left\|(I+\lambda\tau A)^{-(k+2)}\right\|\\ &\le c\left\|\left(a_0I+(1-a_0)(I+\lambda\tau A)^{-1}\right)^k\right\|\\ &\le c\sum_{i=0}^{k}\binom{i}{k}a_0^i(1-a_0)^{k-i}\left\|(I+\lambda\tau A)^{-(k-i)}\right\|\\ &\le c\sum_{i=0}^{k}\binom{i}{k}a_0^i(1-a_0)^{k-i}=c.\end{aligned}$$

From here and (4.23), due to the triangle inequality, it follows:

$$\|J_{1,k}(\tau,A)\|\le c,\quad c=const>0. \qquad (4.25)$$

From (4.16), with account of inequalities (4.17) and (4.24), we obtain the sought estimate ■

Theorem 4.3 *Assume that the linear, closed, densely defined operators A_1 and A_2 in the Banach space X satisfy the following conditions:*

(a) The sector $S = \left\{z : |\arg z| < \varphi_0,\ z \neq 0,\ 0 < \varphi_0 < \frac{\pi}{3}\right\}$ completely includes spectrums of the operators A_1 and A_2 and for any $z \notin S$ $(z \neq 0)$ the inequality holds:

$$\left\|(zI - A_j)^{-1}\right\| \leq \frac{c}{|z|}, \quad c = const > 0, \quad j = 1, 2;$$

(b) There exists such point $z_0 \notin S$ that the resolvents of the operators A_1 and A_2 are commutative at the point z_0.

Then, for any $\tau > 0$, for the transition operators corresponding to the decomposition schemes defined by formulas (1.4), the following estimate is valid:

$$\left\|V^k(\tau)\right\| \leq c, \quad c = const > 0 \quad (k = 1, 2, ...),$$

where

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left(V_1(\tau) + V_2(\tau)\right), \\ V_1(\tau) &= W(\tau, \alpha A_1)\, W(\tau, A_2)\, W(\tau, \overline{\alpha} A_1), \\ V_2(\tau) &= W(\tau, \alpha A_2)\, W(\tau, A_1)\, W(\tau, \overline{\alpha} A_2). \end{aligned}$$

Proof. It follows from the condition *(b)* of the theorem that the resolvents of the operators A_1 and A_2 are commutative at any points $z_1, z_2 \notin S$, respectively. From here it follows that the operators $W(\tau, A_1)$ and $W(\tau, A_2)$ are commutative. Therefore the equalities are valid:

$$V_1^k(\tau) = W^k(\tau, \alpha A_1)\, W^k(\tau, A_2)\, W^k(\tau, \overline{\alpha} A_1), \tag{4.26}$$

$$V_2^k(\tau) = W^k(\tau, \alpha A_2)\, W^k(\tau, A_1)\, W^k(\tau, \overline{\alpha} A_2). \tag{4.27}$$

It is obvious that if the operators A_1 and A_2 satisfy conditions of the Theorem 4.3, then the operators γA_1 and γA_2 $(\gamma = 1, \alpha, \overline{\alpha})$ will satisfy conditions of the Theorem 4.1. Therefore, from formulas (4.25) and (4.26), due to the Theorem 4.1 (Theorem 4.2), follow the estimates:

$$\left\|V_l^k(\tau)\right\| \leq c, \quad l = 1, 2, \quad c = const > 0. \tag{4.28}$$

From the commutativity of the operators $W(\tau, A_1)$ and $W(\tau, A_2)$ follows the commutativity of the operators $V_1(\tau)$ and $V_2(\tau)$, hence the representation is valid:

$$V^k(\tau) = \left(\frac{1}{2}\left(V_1(\tau) + V_2(\tau)\right)\right)^k = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} V_1^{k-j}(\tau)\, V_2^j(\tau).$$

From here, according to inequalities (4.27), follows the estimate:

$$\left\| V^k(\tau) \right\| \le \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \left\| V_1^{k-j}(\tau) \right\| \left\| V_2^j(\tau) \right\| \le \frac{1}{2^k} c \sum_{j=0}^{k} \binom{k}{j} = c.$$

■

Theorem 4.4 *Assume that A_1 and A_2 are linear, normal, densely defined operators in the Hilbert space H. Assume further that the sector $S = \left\{ z : |\arg z| < \varphi_0,\ z \ne 0,\ 0 < \varphi_0 \le \frac{\pi}{3} \right\}$ completely includes the spectrums of the operators A_1 and A_2.*

Then, for any $\tau > 0$, for the transition operators corresponding to the decomposition schemes defined by formulas (1.4), the following estimate is valid:

$$\| V(\tau) \| \le 1.$$

Proof. Since the operators A_1 and A_2 are normal, their corresponding resolvents also will be normal operators (see T. Kato [35], Ch. 5, §3). From here it follows that $W(\tau, \gamma A_1)$ and $W(\tau, \gamma A_2)$ are also normal operators. Therefore, due to inequalities (4.11) and (4.20), the estimate is valid:

$$\| W(\tau, \gamma A_j) \| \le \sup_{z \in S} | W(\tau, \gamma z) | \le 1.$$

From here follows the estimate to be proven ■

Remark 4.1. *Estimate (1.10) holds when the operators A_1 and A_2 satisfy the conditions of the Theorem 4.3, the operator A satisfies the conditions of the Theorem 4.1, and besides the conditions (c) and (d) of the Theorem 1.1 are valid.*

Remark 4.2. *It is obvious that if the resolvents of the operators A_1 and A_2 are commutative, then for exponential splitting we have an exact coincidence. As regards resolvent splitting, it has an essential value even for the commutative case, as the exact coincidence does not take place and therefore, it is important to construct a stable splitting with the high order precision.*

5. Conclusion

In the case when the operators A_1, A_2 are matrices, it is obvious that the conditions of the Theorem 1.1 are automatically satisfied. The conditions of Theorem 1.1 are also satisfied if A_1, A_2 and A are self-adjoint, positive definite operators. Moreover, the conditions of the Theorem 1.1 are automatically satisfied if the operators A_1, A_2 and A are normal operators.

However, in this case, certain restrictions are imposed on the spectrums of this operators: the spectrum of the operator A have to be included in the right half-plane and the spectrums of the operators A_1 and A_2 have to be included in the sector with angle of 120^0, in order the spectrums of the operators A_1 and A_2 to remain in the right half-plane after turning by $\pm 30^0$ (this is caused by multiplication of the operators A_1 and A_2 on the parameters α and $\overline{\alpha}$).

The third order precision is reached by introducing a complex parameter. For this reason, each equation of the given decomposed system is replaced by a pair of real equations, unlike the lower order precision schemes. To solve the specific problem, (for example) the matrix factorization may be used, where the coefficients are the matrices of the second order, unlike the lower order precision schemes, where the common factorization may be used.

It must be noted that, unlike the high order precision decomposition schemes considered in [13], the sum of absolute values of coefficients of the addends of the transition operator $V(\tau)$ equals to one. Hence the considered scheme is stable for any bounded operators A_1, A_2.

6. Numerical example

There are computed the following test problems:

$$
\begin{aligned}
\frac{\partial u(t,x,y)}{\partial t} - a(x,y)\frac{\partial^2 u(t,x,y)}{\partial x^2} - b(x,y)\frac{\partial^2 u(t,x,y)}{\partial y^2} &= f(t,x,y), \\
(x,y) &\in [0;1]\times[0;1], \\
t &\geq 0, \\
u(0,x,y) &= \varphi(x,y), \\
u(t,x,0) &= u(t,x,1) = 0, \\
u(t,0,y) &= u(t,1,y) = 0.
\end{aligned}
$$

Test 6.1

$$
\begin{aligned}
f(t,x,y) &= 0; \\
\varphi(x,y) &= \sin(\pi x)\sin(\pi y); \\
a(x,y) &= b(x,y) = 1.
\end{aligned}
$$

Solution of the problem is $u(t,x,y) = e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y)$.

This test is interesting by that with increase of t the solution decreases very quickly (converges to machine zero) and for this reason it becomes quite difficult to catch the behavior of the solution. The proposed high precision scheme allows to obtain good accuracy, what is confirmed by the numerical calculations (see Tables 1-2).

Test 6.2

$$\begin{aligned} f(t,x,y) &= e^{2\pi^2 t}\left(\pi^2\left(2+a(x,y)+b(x,y)\right)\sin(m\pi t)+m\pi\cos(m\pi t)\right) \\ &\quad \times\sin(\pi x)\sin(\pi y); \\ \varphi(x,y) &= 0; \\ a(x,y) &= 2+\sin(\pi x)\sin(\pi y) \\ b(x,y) &= 2+0.5\sin(\pi x)\sin(\pi y). \end{aligned}$$

Solution of the problem is $u(t,x,y)=e^{2\pi^2 t}\sin(m\pi t)\sin(\pi x)\sin(\pi y)$.

This test is interesting by that the increase of the parameter m causes the fast alternating-sign oscillation of the solution. In addition, we can regularize the frequency of the oscillation according to time coordinates at the expence of m. As the algorithm provides the high accuracy with respect to time coordinate, it is natural that we take the oscillation with respect to t. Obviously the factor $e^{2\pi^2 t}$, with the increase of t, induces the fast increase of the oscillation amplitude. This fact along with the oscillation makes difficult to catch the behavior of the solution and for this reason it is necessary to use the high precision schemes. It can be well seen on Table 3. Note that in this test the operators A_1 and A_2 are noncommutative.

Table 1: Test 6.1: $(x,y)=(0.5,0.5)$; $\tau=1/64$; $h=1/100$

t	$\widetilde{u}$	u	$\lvert u-\widetilde{u}\rvert$	$\lvert(u-\widetilde{u})/u\rvert$
0.125	8479.62 E-05	8480.53 E-05	9.1 E-06	0.11 E-03
0.250	719.03 E-05	719.19 E-05	1.6 E-06	0.21 E-03
0.375	609.71 E-06	609.91 E-06	2.0 E-07	0.32 E-03
0.500	517.01 E-07	517.23 E-07	2.2 E-08	0.43 E-03
0.625	438.40 E-08	438.64 E-08	2.4 E-09	0.53 E-03
0.750	371.75 E-09	371.99 E-09	2.4 E-10	0.64 E-03
0.875	315.23 E-10	315.46 E-10	2.3 E-11	0.75 E-03
1.000	267.30 E-11	267.53 E-11	2.3 E-12	0.85 E-03

Table 2: Test 6.1: $(x, y) = (0.5, 0.5)$; $\tau = 1/100$; $h = 1/142$

t	$\widetilde{u}$	u	$\|u - \widetilde{u}\|$	$\|(u - \widetilde{u})/u\|$
0.1	13891.28 E-05	13891.11 E-05	1.7 E-06	0.12 E-04
0.2	19296.77 E-06	19296.30 E-06	4.7 E-07	0.24 E-04
0.3	26805.69 E-07	26804.71 E-07	9.8 E-08	0.37 E-04
0.4	3723.65 E-07	3723.47 E-07	1.8 E-08	0.48 E-04
0.5	5172.63 E-08	5172.32 E-08	3.1 E-09	0.60 E-04
0.6	7185.45 E-09	7184.90 E-09	5.5 E-10	0.77 E-04
0.7	9981.51 E-10	9980.66 E-10	8.5 E-11	0.85 E-04
0.8	1386.56 E-10	1386.43 E-10	1.3 E-11	0.94 E-04
0.9	1926.11 E-11	1925.90 E-11	2.1 E-12	0.11 E-03
1.0	2675.61 E-12	2675.29 E-12	3.2 E-13	0.12 E-03

Table 3: Test 6.2: $m = 101$; $(x, y) = (0.5, 0.5)$; $\tau = 1/1000$; $h = 1/100$

t	$\widetilde{u}$	u	$\|u - \widetilde{u}\|$	$\|(u - \widetilde{u})/u\|$
0.395	-788.00 E 00	-788.18 E 00	1.8 E-01	0.23 E-03
0.396	-30.91 E 00	-31.19 E 00	2.8 E-01	0.89 E-02
0.397	759.82 E 00	759.47 E 00	3.5 E-01	0.46 E-03
0.398	1504.72 E 00	1504.33 E 00	3.9 E-01	0.26 E-03
...	...	...	...	...
0.602	8436.00 E 01	8436.47 E 01	4.7 E 00	0.56 E-04
0.603	442.96 E 02	443.07 E 02	1.1 E 01	0.26 E-03
0.604	-19.10 E 02	-18.93 E 02	1.7 E 01	0.90 E-03
0.605	-497.81 E 02	-497.60 E 02	2.1 E 01	0.43 E-02
...	...	...	...	...
0.750	-19008.69 E 02	-19008.90 E 02	2.1 E 01	0.11 E-04

Here u is an exact solution and $\widetilde{u}$ - an approximate solution. In addition we note that at the other nodes the error does not increase.

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