RECEPTION AND INVESTIGATION OF FLEXURAL AND MEMBRANE EQUATIONS OF PLATES CONSISTING OF BINARY MIXTURE

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Abstract

A version of linear theory for a body composed of two isotropic homogeneous is studied. Two-dimensional flexural and membrane equations are received. Existence and uniqueness of weak solution of the main mixed boundary value problem is proved. it is shown that the particular flexures of two components of the mixture are equal.

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The theory of mixtures of elastic materials was originated in 1960. Main mechanical properties of new model of elastic medium with complicated internal structure were first formulated in the works of C. Truesdell and R. Toupin [1]. Later this theory was generalized and developed in many directions. Binary and multicomponent models of different type mixtures were created and studied by means of various mathematical methods. There are also being intensively developed the plane theories corresponding to the above-mentioned three-dimensional models.

In this paper, a version of linear theory for a body composed of two isotropic materials suggested by A.E. Green ([2],[3],[4]) is studied. For this types of plates, by means of Ciarlet-Destuynder method ([6],[7]), two-dimensional flexural and membrane equations are obtained. Existence and uniqueness of weak solution of main mixed boundary value problems are proved. It is worth mentioning that, in this case, particular flexures of two components of the mixture are equal.

We assume that an origin and orthonormal basis $\{\mathbf{e}_i\}$ have been chosen in the three-dimensional Euclidean space, which will therefore be identified with the space $\mathbb{R}^3$.

Let ω be a domain in the plane spanned by the vectors $\mathbf{e}_\alpha$ (under domain we mean a bounded connected set with Lipschitz boundary), and let $\varepsilon > 0$

be a dimensionless parameter that may be as small as we please. For each $\varepsilon > 0$, let

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[, \ \ \Gamma_+^\varepsilon := \omega \times \{\varepsilon\}, \ \ \Gamma_-^\varepsilon := \omega \times \{-\varepsilon\},$$

$$\gamma_0 \subset \partial\omega, \ \ length \gamma_0 > 0, \ \ \gamma_1 := \partial\omega - \gamma_0.$$

Let (x_1, x_2) and $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) = (x_i^\varepsilon)$ denote the generic points in the sets $\overline{\omega}$ and $\overline{\Omega^\varepsilon}$,

$$\partial_\alpha = \partial_\alpha^\varepsilon := \frac{\partial}{\partial x_\alpha} \quad and \quad \partial_3^\varepsilon := \frac{\partial}{\partial x_3^\varepsilon}.$$

From now on, we assume that the Latin indices take their values in the set $\{1, 2, 3\}$, Greek indices take their values in the set $\{1, 2\}$, and the repeated index means summation.

We assume that, for each $\varepsilon > 0$, the set $\overline{\Omega^\varepsilon}$ is occupied by an elastic body which contains a mixture of two isotropic, homogeneus, elastic materials.

Statical equilibrium system of equations for a two-component mixture has the form

$$\begin{cases} -\partial_j^\varepsilon \sigma_{ij}^{'\,\varepsilon} + \pi_i^\varepsilon = \rho_1^\varepsilon F_i^{'\varepsilon}, \\ \\ -\partial_j^\varepsilon \sigma_{ij}^{''\,\varepsilon} - \pi_i^\varepsilon = \rho_2^\varepsilon F_i^{''\varepsilon} \quad in \ \ \Omega^\varepsilon, \end{cases} \tag{1}$$

where

$$\sigma_{ij}^{'\,\varepsilon} = \left(-\alpha_2^\varepsilon + \lambda_1^\varepsilon \partial_p^\varepsilon u_p^{'\,\varepsilon} + \lambda_3^\varepsilon \partial_q^\varepsilon u_q^{''\varepsilon}\right)\delta_{ij} + 2\mu_1^\varepsilon e_{ij}^{'\,\varepsilon} + 2\mu_3^\varepsilon e_{ij}^{''\,\varepsilon} + 2\lambda_5^\varepsilon h_{ij}^\varepsilon,$$

$$\sigma_{ij}^{''\,\varepsilon} = \left(\alpha_2^\varepsilon + \lambda_4^\varepsilon \partial_p^\varepsilon u_p^{'\,\varepsilon} + \lambda_2^\varepsilon \partial_q^\varepsilon u_q^{''\varepsilon}\right)\delta_{ij} + 2\mu_3^\varepsilon e_{ij}^{'\,\varepsilon} + 2\mu_2^\varepsilon e_{ij}^{''\,\varepsilon} - 2\lambda_5^\varepsilon h_{ij}^\varepsilon, \tag{2}$$

is Hooke's law, $(\sigma_{ij}^{'\,\varepsilon})$ and $(\sigma_{ij}^{''\,\varepsilon})$ are stress tensors, $(e_{ij}^{'\,\varepsilon})$ and $(e_{ij}^{''\,\varepsilon})$ are strain tensors; ρ_1^ε and ρ_2^ε are densities of components of applied body forces; $?_i^\varepsilon$ are interaction forces between mixture components and

$$\pi_i^\varepsilon := \partial_i^\varepsilon \pi^\varepsilon,$$

where

$$\pi^\varepsilon = \frac{\alpha_2^\varepsilon \rho_2^\varepsilon}{\rho^\varepsilon} \partial_p^\varepsilon u_p^{'\,\varepsilon} + \frac{\alpha_2^\varepsilon \rho_1^\varepsilon}{\rho^\varepsilon} \partial_q^\varepsilon u_q^{''\varepsilon}, \quad \rho^\varepsilon = \rho_1^\varepsilon + \rho_2^\varepsilon;$$

$\mathbf{u}^{'\varepsilon} = (u_i^{'\varepsilon})$, $\mathbf{u}^{''\varepsilon} = (u_i^{''\varepsilon})$ are displacement vector fields; α_2^ε, λ_1^ε, λ_3^ε, λ_4^ε, λ_5^ε, μ_1^ε, μ_2^ε, μ_3^ε are elasticity modulus, furthermore

$$\alpha_2^\varepsilon = \lambda_3^\varepsilon - \lambda_4^\varepsilon;$$

$$e_{ij}^{'\,\varepsilon} = \frac{1}{2}\left(\partial_i^\varepsilon u_j^{'\,\varepsilon} + \partial_j^\varepsilon u_i^{'\varepsilon}\right), \quad e_{ij}^{''\,\varepsilon} = \frac{1}{2}\left(\partial_i^\varepsilon u_j^{''\varepsilon} + \partial_j^\varepsilon u_i^{''\varepsilon}\right); \tag{3}$$

δ_{ij} is Kroneker delta, (h^{ε}_{ij}) is the so called rotation tensor

$$h^{\varepsilon}_{ij}=\frac{1}{2}\left(\partial^{\varepsilon}_{i}{u'_j}^{\varepsilon}-\partial^{\varepsilon}_{j}{u'_i}^{\varepsilon}+\partial^{\varepsilon}_{j}{u''_i}^{\varepsilon}-\partial^{\varepsilon}_{i}{u''_j}^{\varepsilon}\right). \tag{4}$$

For simplicity, we introduce the following notations

$${P'_{ij}}^{\varepsilon}:={\sigma'_{ij}}^{\varepsilon}-\delta_{ij}({}^{\varepsilon}-\alpha^{\varepsilon}_{2}),\ \ {P''_{ij}}^{\varepsilon}:={\sigma''_{ij}}^{\varepsilon}+\delta_{ij}({}^{\varepsilon}-\alpha^{\varepsilon}_{2}), \tag{5}$$

$$\begin{aligned} P^{\varepsilon}_{ij}\ &:=({P'_{ij}}^{\varepsilon},{P''_{ij}}^{\varepsilon})^T,\ \ u^{\varepsilon}_{j}:=({u'_j}^{\varepsilon},{u''_j}^{\varepsilon})^T,\\ e^{\varepsilon}_{ij}\ &:=({e'_{ij}}^{\varepsilon},{e''_{ij}}^{\varepsilon})^T,\ \ \hbar^{\varepsilon}_{ij}:=({h'_{ij}}^{\varepsilon},{h''_{ji}}^{\varepsilon})^T. \end{aligned} \tag{6}$$

For every vector function $(f',f'')^T$

$$\partial_j(f',f'')^T=(\partial_j f',\partial_j f'')^T. \tag{7}$$

By means of the notations (5), (6), relations (1)-(4) take the form

$$-\partial^{\varepsilon}_{j}P^{\varepsilon}_{ij}=F^{\varepsilon}_{i}\quad in\ \ \Omega^{\varepsilon},$$

$$P^{\varepsilon}_{ij}=\Lambda^{\varepsilon}e^{\varepsilon}_{pp}\delta_{ij}+2M^{\varepsilon}e^{\varepsilon}_{ij}+2\lambda_5\hbar^{\varepsilon}_{ij},$$

where

$$e^{\varepsilon}_{ij}=\frac{1}{2}(\partial^{\varepsilon}_{i}u^{\varepsilon}_{j}+\partial^{\varepsilon}_{j}u^{\varepsilon}_{i}),$$

$$\hbar^{\varepsilon}_{ij}=\frac{1}{2}S(\partial^{\varepsilon}_{i}u^{\varepsilon}_{j}-\partial^{\varepsilon}_{j}u^{\varepsilon}_{i}),\ \ S=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},$$

$$\Lambda^{\varepsilon}:=\begin{pmatrix}\lambda^{\varepsilon}_{1}-\dfrac{\alpha^{\varepsilon}_{2}\rho^{\varepsilon}_{2}}{\rho^{\varepsilon}} & \lambda^{\varepsilon}_{3}-\dfrac{\alpha^{\varepsilon}_{2}\rho^{\varepsilon}_{1}}{\rho^{\varepsilon}}\\ \lambda^{\varepsilon}_{4}+\dfrac{\alpha^{\varepsilon}_{2}\rho^{\varepsilon}_{2}}{\rho^{\varepsilon}} & \lambda^{\varepsilon}_{2}+\dfrac{\alpha^{\varepsilon}_{2}\rho^{\varepsilon}_{1}}{\rho^{\varepsilon}}\end{pmatrix},\quad M^{\varepsilon}:=\begin{pmatrix}\mu^{\varepsilon}_{1} & \mu^{\varepsilon}_{3}\\ \mu^{\varepsilon}_{3} & \mu^{\varepsilon}_{2}\end{pmatrix},$$

$$F^{\varepsilon}_{i}:=(\rho^{\varepsilon}_{1}{F'_i}^{\varepsilon},\rho^{\varepsilon}_{2}{F''_i}^{\varepsilon})^T,\ \ \lambda^{\varepsilon}_{3}-\frac{\alpha^{\varepsilon}_{2}\rho^{\varepsilon}_{1}}{\rho^{\varepsilon}}=\lambda^{\varepsilon}_{4}+\frac{\alpha^{\varepsilon}_{2}\rho^{\varepsilon}_{2}}{\rho^{\varepsilon}},$$

$$(\Lambda^{\varepsilon})^T=\Lambda^{\varepsilon},\quad (M^{\varepsilon})^T=M^{\varepsilon}.$$

Consider the following problem

$$\begin{cases} -\partial^{\varepsilon}_{j}P^{\varepsilon}_{ij}=F^{\varepsilon}_{i} & in\ \ \Omega^{\varepsilon},\\ u^{\varepsilon}_{i}=0 & on\ \ \gamma_0\times[-\varepsilon,\varepsilon],\\ P^{\varepsilon}_{ij}n^{\varepsilon}_{j}=\begin{cases} g^{\varepsilon}_{i} & on\ \ \Gamma^{\varepsilon}_{+}\cup\Gamma^{\varepsilon}_{-},\ \ g^{\varepsilon}_{i}=({g'_i}^{\varepsilon},{g''_i}^{\varepsilon})^T,\\ 0 & on\ \ \gamma_1\times[-\varepsilon,\varepsilon], \end{cases} \end{cases} \tag{8}$$

where (n_j^ε) is the unit outer normal vector along the boundary of the set Ω^ε.

Lemma 1. Problem (8), at least formally (1) is equivalent to the following variational problem $P(\Omega^\varepsilon)$: $F_i^\varepsilon \in (L^2(\Omega^\varepsilon))^2$, $g_i^\varepsilon \in (L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon))^2$

$$\begin{aligned}\mathbf{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon) &:= \Big\{\mathbf{v}^\varepsilon = (v_1^{'\varepsilon}, v_2^{'\varepsilon}, v_3^{'\varepsilon}, v_1^{''\varepsilon}, v_2^{''\varepsilon}, v_3^{''\varepsilon}) \in (H^1(\Omega^\varepsilon))^6,\\ &\mathbf{v}^\varepsilon = 0 \ \ on \ \ \gamma_0 \times [-\varepsilon, \varepsilon]\Big\},\end{aligned}$$

$$\int\limits_{\Omega^\varepsilon} \Big\{ \left(\Lambda^\varepsilon e_{pp}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T e_{qq}^\varepsilon(\mathbf{v}^\varepsilon) + 2\left(M^\varepsilon e_{ij}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) - 2\lambda_5^\varepsilon h_{ij}^\varepsilon(\mathbf{u}^\varepsilon) h_{ij}^\varepsilon(\mathbf{v}^\varepsilon)\Big\} dx^\varepsilon$$

$$= \int\limits_{\Omega^\varepsilon} (F_i^\varepsilon)^T v_i^\varepsilon dx^\varepsilon + \int\limits_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} (g_i^\varepsilon)^T v_i^\varepsilon d\Gamma^\varepsilon \qquad for \ all \ \ \mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon). \tag{9}$$

Proof. Multiplying both sides of $(1')$ on the matrix $v_i^\varepsilon = (v_i^{'\varepsilon}, v_i^{''\varepsilon})^T$ and integrating them on Ω^ε, one obtain

$$-\int\limits_{\Omega^\varepsilon} \partial_j^\varepsilon (P_{ij}^\varepsilon)^T v_i^\varepsilon dx^\varepsilon = \int\limits_{\Omega^\varepsilon} (F_{ij}^\varepsilon)^T v_i^\varepsilon dx^\varepsilon.$$

After using the Green formula and taking into account relation $(2')$ and the boundary conditions, the following formula takes place

$$\int\limits_{\Omega^\varepsilon} \Big\{ \left(\Lambda^\varepsilon e_{pp}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T e_{qq}^\varepsilon(\mathbf{v}^\varepsilon) + 2\left(M^\varepsilon e_{ij}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) + 2\lambda_5^\varepsilon (\hbar_{ij}^\varepsilon(\mathbf{u}^\varepsilon))^T \partial_j^\varepsilon v_i^\varepsilon\Big\} dx^\varepsilon$$

$$= \int\limits_{\Omega^\varepsilon} (F_i^\varepsilon)^T v_i^\varepsilon dx^\varepsilon + \int\limits_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} (g_i^\varepsilon)^T v_i^\varepsilon d\Gamma^\varepsilon \quad for \ all \quad \mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon). \tag{10}$$

According to (4), $h_{ij}^\varepsilon = -h_{ji}^\varepsilon$, so that

$$\begin{aligned}\left(\hbar_{ij}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T \partial_j^\varepsilon v_i^\varepsilon &= h_{ij}^\varepsilon \left(\partial_j^\varepsilon v_i^{'\varepsilon} - \partial_j^\varepsilon v_i^{''\varepsilon}\right)\\ &= \frac{1}{2} h_{ij}^\varepsilon \left(\partial_j^\varepsilon v_i^{'\varepsilon} - \partial_i^\varepsilon v_j^{'\varepsilon} - \partial_j^\varepsilon v_i^{''\varepsilon} + \partial_i^\varepsilon v_j^{''\varepsilon}\right)\\ &= -\frac{1}{2} h_{ij}^\varepsilon(\mathbf{u}^\varepsilon) \left(\partial_i^\varepsilon v_j^{'\varepsilon} - \partial_j^\varepsilon v_i^{'\varepsilon} + \partial_j^\varepsilon v_i^{''\varepsilon} - \partial_i^\varepsilon v_j^{''\varepsilon}\right)\\ &= -h_{ij}^\varepsilon(\mathbf{u}^\varepsilon) h_{ij}^\varepsilon(\mathbf{v}^\varepsilon),\\ \left(\hbar_{ij}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T \partial_j^\varepsilon v_i^\varepsilon &= -h_{ij}^\varepsilon(\mathbf{u}^\varepsilon) h_{ij}^\varepsilon(\mathbf{v}^\varepsilon).\end{aligned}$$

By this, from (10), one obtains relation (9).

Now, if $\mathbf{u}^\varepsilon$ is a sufficiently smooth solution of problem $P(\Omega^\varepsilon)$, then, analogously as above, we obtain that u^ε is a solution of problem (8). ∎

Lemma 2. Let

$$W^\varepsilon(e^\varepsilon_{ij}, h^\varepsilon_{ij}) := (\Lambda^\varepsilon e^\varepsilon_{pp})^T e^\varepsilon_{qq} + 2(M^\varepsilon e^\varepsilon_{ij})^T e^\varepsilon_{ij} - 2\lambda_5 h^\varepsilon_{ij} h^\varepsilon_{ij}.$$

Then the estimate

$$W^\varepsilon(e^\varepsilon_{ij}, h^\varepsilon_{ij}) \geq \delta^\varepsilon \left((e^\varepsilon_{ij})^T e^\varepsilon_{ij} + h^\varepsilon_{ij} h^\varepsilon_{ij} \right), \quad \delta^\varepsilon > 0, \tag{11}$$

takes place if and only if

$$\lambda^\varepsilon_5 < 0, \;\; \mu^\varepsilon_1 > 0, \;\; \lambda^\varepsilon_1 - \frac{\alpha^\varepsilon_2 \rho^\varepsilon_2}{\rho^\varepsilon} + \frac{2}{3}\mu^\varepsilon_1 > 0, \;\; det M^\varepsilon > 0, \quad det\left(\Lambda^\varepsilon + \frac{2}{3} M^\varepsilon\right) > 0. \tag{12}$$

One can find the proof in [5].

Lemma 3. Let

$$\begin{aligned}\mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) \;\; : \;\; = & \int\limits_{\Omega^\varepsilon} \Big\{ \left(\Lambda^\varepsilon e^\varepsilon_{pp}(\mathbf{u}^\varepsilon)\right)^T e^\varepsilon_{qq}(\mathbf{v}^\varepsilon) + 2\left(M^\varepsilon e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)\right)^T e^\varepsilon_{ij}(\mathbf{u}^\varepsilon) \\ & - 2\lambda_5 h^\varepsilon_{ij}(\mathbf{u}^\varepsilon) h^\varepsilon_{ij}(\mathbf{v}^\varepsilon) \Big\}\, dx^\varepsilon\end{aligned}$$

and

$$\|\mathbf{v}^\varepsilon\|_{1,\Omega^\varepsilon} := \left\{ \int\limits_{\Omega^\varepsilon} \left[(v^\varepsilon_j)^T v^\varepsilon_j + (\partial^\varepsilon_i v^\varepsilon_j)^T \partial^\varepsilon_i v^\varepsilon_j \right] dx^\varepsilon \right\}^{\frac{1}{2}}. \tag{13}$$

Then: a) The bilinear form $\mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)$ for every ε is bounded, i.e. $\exists k^\varepsilon_1 > 0$ such that

$$|\mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)| \leq k^\varepsilon_1 \|\mathbf{u}^\varepsilon\|_{1,\Omega^\varepsilon} \|\mathbf{v}^\varepsilon\|_{1,\Omega^\varepsilon}; \tag{14}$$

b) If condition (12) is fulfilled then the form $\mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)$ is $\mathbf{V}(\Omega^\varepsilon)$-elliptic for every ε, i.e., $\exists k^\varepsilon_2 > 0$ such that

$$\mathbf{B}^\varepsilon(\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon) \geq k^\varepsilon_2 \|\mathbf{v}^\varepsilon\|^2_{1,\Omega^\varepsilon} \tag{15}$$

for every $\mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$.

Proof. a) If one uses Cauchy-Schwar'z inequality, for each $\varepsilon > 0$, there exists $d^\varepsilon > 0$ such that

$$\mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) \leq d^\varepsilon \sum_{i,j=1}^{3} \left[\left(\int\limits_{\Omega^\varepsilon} \left(\partial^\varepsilon_i u'^{\varepsilon}_j\right)^2 dx^\varepsilon \right)^{\frac{1}{2}} + \left(\int\limits_{\Omega^\varepsilon} \left(\partial^\varepsilon_i u''^{\varepsilon}_j\right)^2 dx^\varepsilon \right)^{\frac{1}{2}} \right]$$

$$\times \sum_{p,q=1}^{3} \left[\left(\int\limits_{\Omega^\varepsilon} \left(\partial_p^\varepsilon {v'_q}^\varepsilon \right)^2 dx^\varepsilon \right)^{\frac{1}{2}} + \left(\int\limits_{\Omega^\varepsilon} \left(\partial_p^\varepsilon {v''_q}^\varepsilon \right)^2 dx^\varepsilon \right)^{\frac{1}{2}} \right]$$

$$\leq k_1^\varepsilon \|\mathbf{u}^\varepsilon\|_{1,\Omega^\varepsilon} \|\mathbf{v}^\varepsilon\|_{1,\Omega^\varepsilon}.$$

b) Let condition (12) be fulfilled. By Korn's inequality with boundary conditions ([6]), for each $\varepsilon > 0$ there exists $\delta^\varepsilon > 0$, $c^\varepsilon > 0$ such that

$$\frac{\delta^\varepsilon}{c^\varepsilon} \|\mathbf{v}^\varepsilon\|_{1,\Omega^\varepsilon}^2 \leq \delta^\varepsilon \int\limits_{\Omega^\varepsilon} (e_{ij}^\varepsilon(\mathbf{v}^\varepsilon))^T e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) dx^\varepsilon \leq \delta^\varepsilon \int\limits_{\Omega^\varepsilon} \left[(e_{ij}^\varepsilon(\mathbf{v}^\varepsilon))^T e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) \right.$$

$$\left. + h_{ij}^\varepsilon(\mathbf{v}^\varepsilon) h_{ij}^\varepsilon(\mathbf{v}^\varepsilon) \right] dx^\varepsilon \leq \mathbf{B}^\varepsilon(\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon).$$

Hence, we obtain (15), where $k_2^\varepsilon = \frac{\delta^\varepsilon}{c^\varepsilon}$. ∎

Theorem 1. Let condition (12) be fulfilled. Assume furthermore that $F_i^\varepsilon \in (L^2(\Omega^\varepsilon))^2$ and $g_i^\varepsilon \in \left(L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)\right)^2$. Then the variational problem $P(\Omega^\varepsilon)$ has one and only one solution.

This solution can also be characterized as the unique solution of the minimization problem. Find $\mathbf{u}^\varepsilon$ such that

$$\mathbf{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon) \ \ and \ \ J^\varepsilon(\mathbf{u}^\varepsilon) = \inf_{\mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)} J^\varepsilon(\mathbf{v}^\varepsilon), \ \ where$$

$$J^\varepsilon(\mathbf{v}^\varepsilon) := \frac{1}{2} \mathbf{B}^\varepsilon(\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{L}^\varepsilon(\mathbf{v}^\varepsilon),$$

$$\mathbf{L}^\varepsilon(\mathbf{v}^\varepsilon) := \int\limits_{\Omega^\varepsilon} (F_i^\varepsilon)^T v_i^\varepsilon dx^\varepsilon + \int\limits_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} (g_i^\varepsilon)^T v_i^\varepsilon d\Gamma^\varepsilon.$$

Proof. For each $\varepsilon > 0$ the linear functional $L^\varepsilon : V(\Omega^\varepsilon) \to R$ is bounded. Thus, by Lemma 3 and Lax-Milgram lemma [6], we obtain that Theorem 1 holds true. ∎

Remark. Lemma 3 and Theorem 1 hold also when $\lambda_5^\varepsilon = 0$.

Our aim is to study the behavior of the displacement fields $\mathbf{u}^\varepsilon$ as $\varepsilon \to 0$. Since these fields are defined on the sets $\overline{\Omega^\varepsilon}$, which themselves vary with ε, it is natural that our first task is transformation of the problem $P(\Omega^\varepsilon)$ into problems posed over a set independent from ε.

Accordingly, we let

$$\Omega := \omega \times]-1, 1[, \ \ \Gamma_+ := \omega \times \{1\}, \ \ \Gamma_- := \omega \times \{-1\}.$$

Let $x = (x_1, x_2, x_3)$ denote a generic point on the set $\overline{\Omega}$ and let

$$\partial_i := \frac{\partial}{\partial x_i}.$$

With each point $x \in \overline{\Omega}$, we associate the point $x^\varepsilon \in \overline{\Omega^\varepsilon}$ through the bijection

$$\pi^\varepsilon : x = (x_1, x_2, x_3) \in \overline{\Omega} \to x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega^\varepsilon}.$$

Note that

$$\partial_\alpha^\varepsilon = \partial_\alpha \ \ and \ \ \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3. \tag{16}$$

With the displacement field $\mathbf{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$, we associate the scaled displacement field $\mathbf{u}(\varepsilon) : \overline{\Omega} \to \mathbb{R}^3$ defined by the scalings

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x) \ \ and \ \ u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x) \ \ for\ all \ \ x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon. \tag{17}$$

We likewise associate with any vector field $\mathbf{v}^\varepsilon = v_i^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$ the scaled vector field $\mathbf{v} = (v_i) : \Omega \to \mathbb{R}^3$ defined by the scalings:

$$v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x) \ \ and \ \ v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x) \ \ for\ all \ \ x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon.$$

We make the following assumptions on the data, we require that the constants $\rho_\alpha,\ \lambda_1, ..., \lambda_5,\ \mu_i,\ \mu_2,\ \mu_3$ do not depended on ε, i.e.

$$\rho_\alpha^\varepsilon = \rho, \ \ \lambda_1^\varepsilon = \lambda_1, ..., \lambda_5^\varepsilon = \lambda_5, \ \mu_1^\varepsilon = \mu_1, \ \mu_2^\varepsilon = \mu_2, \ \mu_3^\varepsilon = \mu_3,$$

furthermore we require that the applied body force density and the applied surface force bensity be of the following form

$$F_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 F_\alpha(x) \ \ and \ \ F_3^\varepsilon(x^\varepsilon) = \varepsilon^3 F_3(x) \ \ for\ all \ \ x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon,$$

$$g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^3 g_\alpha(x) \ \ and \ \ g_3^\varepsilon(x^\varepsilon) = \varepsilon^4 g_3(x) \ \ for\ all \ \ x^\varepsilon = \pi^\varepsilon x \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon.$$

Introduce the following denotations

$$\Lambda := \begin{pmatrix} \lambda_1 - \dfrac{\alpha_2\rho_2}{\rho} & \lambda_3 - \dfrac{\alpha_2\rho_1}{\rho} \\ \lambda_4 + \dfrac{\alpha_2\rho_2}{\rho} & \lambda_2 + \dfrac{\alpha_2\rho_1}{\rho} \end{pmatrix}, \quad M := \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix}.$$

Using the scaling of the displacement and assumptions on the data, we now reformulate the variational problem $P(\Omega^\varepsilon)$ in the following equivalent form.

Lemma 4. The scaled displacement $\mathbf{u}(\varepsilon)$ solves the following variational problem, called the scaled three-dimensional problem $P(\varepsilon,\Omega)$

$$\begin{aligned}
\mathbf{u}(\varepsilon) \;\in\;& \mathbf{V}(\Omega) := \Big\{\mathbf{v} = (v_1^{'}, v_2^{'}, v_3{'}, v_1^{''}, v_2{''}, v_3^{''}) \in (H^1(\Omega))^6, \\
& \mathbf{v} = 0 \;\; on \;\; \gamma_0 \times [-1,1]\big\}, \\
& \int\limits_\Omega \big\{(\Lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)))^T e_{\tau\tau}(\mathbf{v}) + 2(M e_{\alpha\beta}(\mathbf{u}(\varepsilon)))^T e_{\alpha\beta}(\mathbf{v}) \\
& -2\lambda_5 h_{\alpha\beta}(\mathbf{u}(\varepsilon)) h_{\alpha\beta}(\mathbf{v}) dx\} \\
& +\frac{1}{\varepsilon^2}\int\limits_\Omega \big\{(\Lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)))^T e_{33}(\mathbf{v}) + (\Lambda e_{33}(\mathbf{u}(\varepsilon)))^T e_{\tau\tau}(\mathbf{v}) \\
& +4(M e_{\alpha 3}(\mathbf{u}(\varepsilon)))^T e_{\alpha 3}(\mathbf{v}) - 4\lambda_5 h_{\alpha 3}(\mathbf{u}(\varepsilon)) h_{\alpha 3}(\mathbf{v})\big\}\, dx \\
& +\frac{1}{\varepsilon^4}\int\limits_\Omega \left[(\Lambda + 2M) e_{33}(\mathbf{u}(\varepsilon))\right]^T e_{33}(\mathbf{v}) dx \\
=\;& \mathbf{L}(\mathbf{v}) \qquad (18)
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, where

$$e_{ij} := \frac{1}{2}(\partial_i v_j + \partial_j v_i), \;\; h_{ij} := \frac{1}{2}(\partial_i v_j^{'} - \partial_j v_i^{'} + \partial_j v_i^{''} - \partial_i v_j^{''}),$$

$$\mathbf{L}(\mathbf{v}) = \int\limits_\Omega (F_i)^T v_i dx + \int\limits_{\Gamma_+ \cup \Gamma_-} (g_i)^T v_i d\Gamma.$$

Proof. The relations (16), (17) and (18) altogether yield

$$\begin{aligned}
& \left(\Lambda^\varepsilon e_{pp}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T e_{qq}^\varepsilon(\mathbf{v}^\varepsilon) + 2\left(M^\varepsilon e_{ij}^\varepsilon(\mathbf{u}^\varepsilon)\right)^T e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) - 2\lambda_5^\varepsilon h_{ij}^\varepsilon(\mathbf{u}^\varepsilon) h_{ij}^\varepsilon(\mathbf{v}^\varepsilon) \\
=\;& \varepsilon^4 \Big\{(\Lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)))^T e_{\tau\tau}(\mathbf{v}) + \frac{1}{\varepsilon^2}(\Lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)))^T e_{33}(\mathbf{v}) \\
& +\frac{1}{\varepsilon^2}(\Lambda e_{33}(\mathbf{u}(\varepsilon)))^T e_{\tau\tau} + \frac{1}{\varepsilon^4}(\Lambda e_{33}(\mathbf{u}(\varepsilon)))^T e_{33}(\mathbf{v})\Big\} \\
& +\varepsilon^4 \big\{(M e_{\alpha\beta}(\mathbf{u}(\varepsilon)))^T e_{\alpha\beta}(\mathbf{v}) - 2\lambda_5 h_{\alpha\beta}(\mathbf{u}(\varepsilon)) h_{\alpha\beta}(\mathbf{v}) \\
& +\frac{2}{\varepsilon^2}(M e_{\alpha 3}(\mathbf{u}(\varepsilon)))^T e_{\alpha 3}(\mathbf{v}) - \frac{4\lambda_5}{\varepsilon^2} h_{\alpha 3}(\mathbf{u}(\varepsilon)) h_{\alpha 3}(\mathbf{v}) \\
& +\frac{1}{\varepsilon^4}(M e_{33}(\mathbf{u}(\varepsilon)))^T e_{33}(\mathbf{v})\Big\},
\end{aligned}$$

$$\left(F_i^\varepsilon\right)^T v_i^\varepsilon = \varepsilon^4 (F_i)^T v_i, \quad \left(g_i^\varepsilon\right)^T v_i^\varepsilon = \varepsilon^5 (g_i)^T v_i.$$

Furthermore

$$\begin{aligned}\int\limits_{\Omega^\varepsilon} \theta(x^\varepsilon)dx^\varepsilon &= \varepsilon \int\limits_{\Omega} \theta(\pi^\varepsilon x)dx,\\ \int\limits_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} \theta(x^\varepsilon)d\Gamma^\varepsilon &= \int\limits_{\Gamma_+ \cup \Gamma_-} \theta(\pi^\varepsilon x)d\Gamma.\end{aligned}$$

Besides,

$$\mathbf{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon) \Leftrightarrow \mathbf{u}(\varepsilon) \in \mathbf{V}(\Omega).$$

For proof it suffices to combine these relations. ∎

From Theorem 1 and Lemma 4 it follows that if the conditions

$$\lambda_5 < 0,\ \ \mu_1 > 0,\ \ \lambda_1 - \frac{\alpha_2\rho_2}{\rho} + \frac{2}{3}\mu_1 > 0,\ \ detM > 0,\ \ det\left(\Lambda + \frac{2}{3}M\right) > 0$$

are fulfilled, for each $\varepsilon > 0$, the problem $P(\varepsilon, \Omega)$ has a unique solution $\mathbf{u}(\varepsilon)$.
Now we prove an auxiliary lemma which we will use below.

Lemma 5. Let

$$A = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix} \quad and \quad B = \begin{pmatrix} b_1 & b_3 \\ b_3 & b_2 \end{pmatrix}$$

be a symmetric 2×2 matrices such that $detA > 0$, $detB > 0$, $a_1 > 0$, $b_1 > 0$. Then for every $k > 0$, $det(A + kB) > 0$.
Proof.

$$det(A + kB) = a_1a_2 - a_3^2 + k^2(b_1b_2 - b_3^2) + k(a_2b_1 + a_1b_2 - 2a_3b_3).$$

As $a_1,\ a_2,\ b_1,\ b_2 > 0$, so

$$a_2b_1 + a_1b_2 \geq 2\sqrt{a_1a_2b_1b_2} > 2a_3b_3.$$

Therefore

$$det(A + kB) > 0.$$

∎

From condition $(12')$ and Lemma 5, it follows that

$$det(\Lambda + 2M) > 0,$$

as $\Lambda + \frac{2}{3}M$ and M are symmetric matrices.

Introduce the denotation

$$\Lambda^* = \begin{pmatrix} \lambda_1^* & \lambda_3^* \\ \lambda_3^* & \lambda_2^* \end{pmatrix} := \Lambda - \Lambda(\Lambda + 2M)^{-1}\Lambda.$$

The symmetry of this matrix follows from the symmetry of matrices Λ and $\Lambda + 2M$.

Together with condition $(12')$ we will require the fulfillment of the following conditions

$$\lambda_1^* + \mu_1 > 0, \;\; det(\Lambda^* + M) > 0. \tag{19}$$

Theorem 2. Let conditions $(12')$ and (19) be fulfilled. For each $\varepsilon > 0$, let $\mathbf{u}(\varepsilon)$ denote the solution of problem $P(\varepsilon, \Omega)$. Then:

a) As $\varepsilon \to 0$, then family $(\mathbf{u}(\varepsilon))_\varepsilon > 0$ converges weakly in the space

$$\mathbf{V}(\Omega) := \left\{ \mathbf{v} \in (H^1(\Omega))^6; \;\; \mathbf{v} = 0 \;\; on \;\; \gamma_0 \times [-1, 1] \right\}.$$

b) Let $\mathbf{u} = \lim\limits_{\varepsilon \to 0} \mathbf{u}(\varepsilon)$. Then $\mathbf{u}$ satisfies and is the unique solution of the following problem $P_{KL}(\Omega)$:

$$\mathbf{u} \in \mathbf{V}_{KL}(\Omega) := \{ \mathbf{v} \in \mathbf{V}(\Omega); \;\; e_{i3}(\mathbf{v}) = 0, \;\; h_{\alpha 3}(\mathbf{v}) = 0 \;\; in \;\; \Omega \},$$

$$\int\limits_\Omega \left\{ (\Lambda^* e_{\sigma\sigma}(\mathbf{u}))^T e_{\tau\tau}(\mathbf{v}) + 2\, (M e_{\alpha\beta}(\mathbf{u}))^T e_{\alpha\beta}(\mathbf{v}) - 2\lambda_5 h_{\alpha\beta}(\mathbf{u}) h_{\alpha\beta}(\mathbf{v}) \right\} dx$$
$$= \mathbf{L}(\mathbf{v}), \qquad for \quad all \quad \mathbf{v} \in \mathbf{V}_{KL}(\Omega),$$

c) The space $\mathbf{V}_{KL}(\Omega)$ is equivalently defined as

$$\mathbf{V}_{KL}(\Omega) = \Big\{ \mathbf{v} = (v_1', \, v_2', \, v_3', \, v_1'', \, v_2'', \, v_3''), v_3' = v_3'' = \eta_3^*, \; v_\alpha' = \eta_\alpha' - x_3 \partial_\alpha \eta_3^*,$$
$$v_\alpha'' = \eta_\alpha'' - x_3 \partial_\alpha \eta_3^*, \quad with \quad \eta_\alpha', \, \eta_\alpha'' \in H^1(\omega), \; \eta_3^* \in H^2(\omega),$$
$$\eta_\alpha' = \eta_\alpha'' = \eta_3^* = \partial_\nu \eta_3^* = 0 \quad on \quad \gamma_0 \Big\},$$

where ∂_ν denotes the outer normal derivative operator along γ. In particular, there exist functions $\zeta_\alpha^T = (\zeta_\alpha'', \zeta_\alpha'') \in (H^1(\omega))^2$ and $\zeta_3^T = (\zeta_3^*, \zeta_3^*) \in (H^2(\omega))^2$ satisfying $\zeta_\alpha' = \zeta_\alpha'' = \zeta_3^* = \partial_\nu \zeta_3^* = 0$ on γ_0 such that

$$u_\alpha' = \zeta_\alpha' - x_3 \partial_\alpha \zeta_3^*, \;\; u_\alpha'' = \zeta_\alpha'' - x_3 \partial_\alpha \zeta_3^*, \quad and \quad u_3' = u_3'' = \zeta_3^* \quad in \quad \Omega.$$

Let the functions $e_{\alpha\beta}(\boldsymbol{\zeta_H})$, $h_{\alpha\beta}(\boldsymbol{\zeta_H})$, $g_i^{\pm}$ and $q_\alpha \in (L^2(\omega))^2$ be defined by

$$e_{\alpha\beta}(\boldsymbol{\zeta_H}) := \frac{1}{2}(\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha), \;\; h_{\alpha\beta}(\boldsymbol{\zeta_H}) := \frac{1}{2}(\partial_\alpha \zeta_\beta' - \partial_\beta \zeta_\alpha' + \partial_\beta \zeta_\alpha'' - \partial_\alpha \zeta_\beta''),$$

$$g_i^{\pm} := g_i(\cdot, \pm 1),\ p_i := \int_{-1}^{1} F_i dx_3,\ q_\alpha := \int_{-1}^{1} F_\alpha dx_3 + g_\alpha^{+} - g_\alpha^{-}.$$

Then vector field $\boldsymbol{\zeta} = (\zeta_1^{'},\ \zeta_2^{'},\ \zeta_1^{''},\ \zeta_2^{''},\ \zeta_3^{*})$ is obtained by solving two independent variational problems:

1. The function ζ_3^* satisfies the scaled two-dimensional flexural equations:

$$\zeta_3^* \in V_3(\omega) := \left\{\eta_3^* \in H^2(\omega),\ \eta_3^* = \partial_\nu \eta_3^* = 0 \quad on \quad \gamma_0\right\},$$

$$\int_\omega \left\{\frac{2}{3}\widehat{\lambda}\Delta\zeta_3^*\Delta\eta_3^* + \frac{4}{3}\widehat{\mu}\partial_{\alpha\beta}\zeta_3^*\partial_{\alpha\beta}\eta_3^*\right\} d\omega = \int_\omega (p_3^{'} + p_3^{''})\eta_3^* d\omega$$

$$- \int_\omega (q_\alpha^{'} + q_\alpha^{''})\partial_\alpha \eta_3^* d\omega \quad for\ all \quad \eta_3^* \in V_3(\omega),$$

where

$$\widehat{\lambda} = \lambda_1^* + 2\lambda_3^* + \lambda_2^*, \quad \widehat{\mu} = \mu_1 + 2\mu_3 + \mu_2.$$

2. The vector field $\boldsymbol{\zeta}_H = (\zeta_1^{'},\ \zeta_2^{'},\ \zeta_1^{''},\ \zeta_2^{''})$ satisfies the scaled two-dimensional membrane equations

$$\boldsymbol{\zeta}_H \in \mathbf{V}_H(\omega) := \left\{\boldsymbol{\eta}_H = (\eta_1^{'},\ \eta_2^{'},\ \eta_1^{''},\ \eta_2^{''}) \in (H_1(\omega))^4,\ \ \boldsymbol{\eta}_H = 0\ on\ \ \gamma_0\right\},$$

$$\begin{aligned} & \int_\omega \left\{2(\Lambda^* e_{\sigma\sigma}(\boldsymbol{\zeta}_H))^T e_{\tau\tau}(\boldsymbol{\eta}_H) + 4(M e_{\alpha\beta}(\boldsymbol{\eta}_H))^T e_{\alpha\beta}(\boldsymbol{\eta}_H)\right. \\ & \left. -4h_{\alpha\beta}(\boldsymbol{\zeta}_H)h_{\alpha\beta}(\boldsymbol{\eta}_H)\right\} d\omega \\ = & \int_\omega (p_\alpha)^T \eta_\alpha d\omega \quad for\ all \quad \boldsymbol{\eta}_H \in \mathbf{V}_H(\omega), \end{aligned}$$

Proof. (i) Introduce the following notations

$$\boldsymbol{\kappa}(\varepsilon) := (\kappa_{ij}(\varepsilon)),\ \ \overline{\mathbf{h}}(\varepsilon) := (\overline{h}_{ij}(\varepsilon)),$$

$$\kappa_{\alpha\beta}(\varepsilon) := e_{\alpha\beta}(\mathbf{u}(\varepsilon)),\ \ \kappa_{\alpha 3}(\varepsilon) := \frac{1}{\varepsilon}e_{\alpha 3}(\mathbf{u}(\varepsilon)),\ \ \kappa_{33}(\varepsilon) := \frac{1}{\varepsilon^2}e_{33}(\mathbf{u}(\varepsilon)),$$

$$\overline{h}_{\alpha\beta}(\varepsilon) := h_{\alpha\beta}(\mathbf{u}(\varepsilon)),\ \ \overline{h}_{\alpha 3}(\varepsilon) := \frac{1}{\varepsilon}h_{\alpha 3}(\mathbf{u}(\varepsilon)). \qquad (20)$$

Let us show that the norms

$$||\mathbf{u}(\varepsilon)||_{1,\Omega}=\left\{\sum_i\left(\|u_i'(\varepsilon)\|^2_{1,\Omega}+\|u_i''(\varepsilon)\|^2_{1,\Omega}\right)\right\}^{\frac12},$$

$$|\boldsymbol{\kappa}(\varepsilon)|_{0,\Omega}=\left\{\sum_{i,j}\left(|\kappa_{ij}'(\varepsilon)|^2_{0,\Omega}+|\kappa_{ij}''(\varepsilon)|^2_{0,\Omega}\right)\right\}^{\frac12},$$

$$|\overline{\mathbf{h}}(\varepsilon)|_{0,\Omega}=\left\{\sum_{i,j}|\overline{h}_{ij}|^2_{0,\Omega}\right\}^{\frac12}$$

are bounded independently of ε, hence there exists a subsequence, still indexed by ε for denotational convenience, and there exists $\mathbf{u}\in(H^1(\Omega))^6$, $\boldsymbol{\kappa}\in\left((L^2(\Omega))^{2\times1}\right)^{3\times3}_S$, $\overline{\mathbf{h}}\in\left(L^2(\Omega)\right)^{3\times3}_{AS}$ (indices S and AS denote symmetric and antisymmetric matrices, respectively) such that

$$\mathbf{u}(\varepsilon)\rightharpoonup\mathbf{u}\ \ in\ \ (H^1(\Omega))^6\ \ as\ \ \varepsilon\to0, and\ \ \mathbf{u}=0\ \ on\ \ \gamma_0\times[-1,1],$$

$$\boldsymbol{\kappa}(\varepsilon)\rightharpoonup\boldsymbol{\kappa}\ \ in\ \ \left((L^2(\Omega))^{2\times1}\right)^{3\times3}_S\ \ as\ \ \varepsilon\to0,$$

$$\overline{\mathbf{h}}(\varepsilon)\rightharpoonup\overline{\mathbf{h}}\ \ in\ \ \left(L^2(\Omega)\right)^{3\times3}_{AS}\ \ as\ \ \varepsilon\to0.$$

Strong and weak convergences are respectively denoted by $\to$ and $\rightharpoonup$.

Let $\mathbf{v}=\mathbf{u}(\varepsilon)$ in the variational equations of problem $P(\varepsilon,\Omega)$ (see Lemma 4). They then take a pemarkably simple form if the are expressed in above notations, viz.

$$\int_\Omega\left\{(\Lambda\kappa_{pp}(\varepsilon))^T\kappa_{qq}(\varepsilon)+2(M\kappa_{ij}(\varepsilon))^T\kappa_{ij}(\varepsilon)-2\lambda_5\overline{h}_{ij}(\varepsilon)\overline{h}_{ij}(\varepsilon)\right\}dx=\mathbf{L}(\mathbf{u}(\varepsilon)).$$

As conditions $(12')$ are fulfilled, there exists $\delta>0$ such that (see Lemma 2)

$$\int_\Omega\left\{(\Lambda\kappa_{pp}(\varepsilon))^T\kappa_{qq}(\varepsilon)+2(M\kappa_{ij}(\varepsilon))^T\kappa_{ij}(\varepsilon)-2\lambda_5\overline{h}_{ij}(\varepsilon)\overline{h}_{ij}(\varepsilon)\right\}dx$$

$$\geq\delta\left(|\boldsymbol{\kappa}(\varepsilon)|^2_{0,\Omega}+|\overline{\mathbf{h}}(\varepsilon)|^2_{0,\Omega}\right).$$

We may also assume without loss of generality that $\varepsilon\le1$, hence we infer from Korn's inequality with boundary conditions that

$$\delta c\|\mathbf{u}(\varepsilon)\|^2_{1,\Omega}\le\delta|e(\mathbf{u}(\varepsilon))|^2_{0,\Omega}\le\left(|\boldsymbol{\kappa}(\varepsilon)|^2_{0,\Omega}+|\overline{\mathbf{h}}(\varepsilon)|^2_{0,\Omega}\right)$$

$$\leq \int_\Omega \left\{ (\Lambda \kappa_{pp}(\varepsilon))^T \kappa_{qq}(\varepsilon) + 2(M\kappa_{ij}(\varepsilon))^T \kappa_{ij}(\varepsilon) - 2\lambda_5 \overline{h}_{ij}(\varepsilon) \overline{h}_{ij}(\varepsilon) \right\} dx$$

$$\leq \|\mathbf{L}\|_{L((H^1(\Omega))^6,R)} \|\mathbf{u}(\varepsilon)\|_{1,\Omega}.$$

This inequalities imply that the norms $\|\mathbf{u}(\varepsilon)\|_{1,\Omega}$, $|\boldsymbol{\kappa}(\varepsilon)|_{0,\Omega}$ and $|\overline{\mathbf{h}}(\varepsilon)|_{0,\Omega}$ are bounded independently of ε.

(ii) The weak limit $\mathbf{u} \in (H^1(\Omega))^6$ of this subsequence $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ belongs to the subspace

$$\begin{aligned} \mathbf{V}_{KL}(\Omega) \ = \ & \{ \mathbf{v} \in (H^1(\Omega))^6; \ e_{i3}(\mathbf{v}) = 0, \ h_{\alpha 3}(\mathbf{v}) = 0 \ \ in \ \ \Omega, \\ & \mathbf{v} = 0 \ \ on \ \ \gamma_0 \times [-1,1] \} \end{aligned}$$

of the space $\mathbf{V}(\Omega)$.

Since the sequences $(\boldsymbol{\kappa}(\varepsilon))_{\varepsilon>0}$ and $(\overline{\mathbf{h}}(\varepsilon))_{\varepsilon>0}$ are bounded by (i), there exists a constant c independent of ε such that

$$|e_{\alpha 3}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leq c\varepsilon, \ \ |e_{33}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leq c\varepsilon^2, \ \ |h_{\alpha 3}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leq c\varepsilon$$

by definitions. Hence $e_{i3}(\mathbf{u}(\varepsilon)) \to 0 \ \ in \ \ (L^2(\Omega))^2$ and $h_{\alpha 3}(\mathbf{u}(\varepsilon)) \to 0 \ \ in \ \ L^2(\Omega)$ and thus $e_{i3}(\mathbf{u}(\varepsilon)) \rightharpoonup 0 \ \ in \ \ (L^2(\Omega))^2$ and $h_{\alpha 3}(\mathbf{u}(\varepsilon)) \rightharpoonup 0 \ \ in \ \ L^2(\Omega)$.

But $\mathbf{u}(\varepsilon) \rightharpoonup \mathbf{u} \ \ in \ \ (H^1(\Omega))^6$ implies $e_{i3}(\mathbf{u}(\varepsilon)) \rightharpoonup 0 \ \ in \ \ (L^2(\Omega))^2$ and $h_{\alpha 3}(\mathbf{u}(\varepsilon)) \rightharpoonup 0 \ \ in \ \ L^2(\Omega)$. Hence

$$e_{i3}(\mathbf{u}) = (e'_{i3}(\mathbf{u}), \ e''_{i3}(\mathbf{u}))^T = (0,0)^T, \ \ h_{\alpha 3}(\mathbf{u}) = 0.$$

(iii) Let $\omega \in L^2(\Omega)$ be a function such that

$$\int_\Omega \omega \partial_3 v dx = 0 \ \ for \ all \ \ v \in C^\infty(\overline{\Omega}),$$

that satisfy $\mathbf{v} = 0$ on $\gamma_0 \times [-1,1]$. Then $\omega = 0$ [7].

(iv) The components of the weak limit $\kappa_{ij} \in (L^2(\Omega))^2$ of the subsequence $(\boldsymbol{\kappa}(\varepsilon))_{\varepsilon>0}$ and the component weak limit $\overline{h}_{ij} \in L^2(\Omega)$ of the subsequence $(\overline{\mathbf{h}}(\varepsilon))_{\varepsilon>0}$ satisfy

$$\kappa_{\alpha\beta} = e_{\alpha\beta}(\mathbf{u}), \ \ \overline{h}_{\alpha\beta}(\mathbf{u}) = h_{\alpha\beta}(\mathbf{u}), \ \ \kappa_{33} = -(\Lambda + 2M)\Lambda^{-1} e_{\sigma\sigma}(\mathbf{u}).$$

Since $\kappa_{\alpha\beta}(\varepsilon) = e_{\alpha\beta}(\mathbf{u}(\varepsilon))$ and $\mathbf{u}(\varepsilon) \rightharpoonup \mathbf{u}$ in $(H^1(\Omega))^6$, if follows that $\kappa_{\alpha\beta} \rightharpoonup \kappa_{\alpha\beta} = e_{\alpha\beta}(\mathbf{u})$ in $(L^2(\Omega))^2$ and $\overline{h}_{\alpha\beta}(\varepsilon) \rightharpoonup h_{\alpha\beta} = h_{\alpha\beta}(\mathbf{u})$.

The variational equations found in problem $P(\varepsilon,\Omega)$ can be written as

$$\int_\Omega \left\{ (\Lambda \kappa_{pp}(\varepsilon))^T \delta_{\alpha\beta} + 2(M\kappa_{\alpha\beta}(\varepsilon))^T - 2\lambda_5 (\overline{h}_{\alpha\beta}(\varepsilon))^T \right\} \partial_\alpha v_\beta dx$$

$$+\frac{1}{\varepsilon}\int\limits_{\Omega}\left\{\left(2(M\kappa_{\alpha 3}(\varepsilon))^T+2\lambda_5(\overline{\hbar}_{\alpha 3}(\varepsilon))^T\right)\partial_3 v_\alpha\right.$$

$$\left.+\left(2(M\kappa_{\alpha 3}(\varepsilon))^T-2\lambda_5(\overline{\hbar}_{\alpha 3}(\varepsilon))^T\right)\partial_\alpha v_3\right\}dx$$

$$+\frac{1}{\varepsilon^2}\int\limits_{\Omega}\left\{(\Lambda\kappa_{\sigma\sigma}(\varepsilon))^T+[(\Lambda+2M)\kappa_{33}(\varepsilon)]^T\right\}\partial_3 v_3 dx=\mathbf{L}(\mathbf{v}),$$

where

$$\overline{\hbar}_{\alpha j}=(\overline{h}_{\alpha j},\ \overline{h}_{j\alpha}).$$

Letting $v_3=0$ in these equations and multiplyng by ε, we obtain:

$$\int\limits_{\Omega}\left\{2(M\kappa_{\alpha 3}(\varepsilon))^T+2\lambda_5(\overline{\hbar}_{\alpha 3}(\varepsilon))^T\right\}\partial_3 v_\alpha dx=-\varepsilon\int\limits_{\Omega}\{\Lambda\kappa_{pp}(\varepsilon)\delta_{\alpha\beta}$$

$$+2M\kappa_{\alpha\beta}(\varepsilon)-2\lambda_5\overline{\hbar}_{\alpha\beta}(\varepsilon)\big\}^T\partial_\alpha v_\beta dx+\varepsilon L(v)$$

for all $\mathbf{v}\in\mathbf{V}(\Omega)$ such that $v_3=0$. For each such v_i the left-hand side converges to

$$\int\limits_{\Omega}\left\{2M\kappa_{\alpha 3}+2\lambda_5\overline{\hbar}_{\alpha 3}\right\}^T\partial_3 v_\alpha dx$$

as $\varepsilon\to 0$ by the definition of weak convergence, and the right-hand side convergence is bounded.

Hence

$$\int\limits_{\Omega}\left\{2M\kappa_{\alpha 3}+2\lambda_5\overline{\hbar}_{\alpha 3}\right\}^T\partial_3 v_\alpha dx=0\ \ for\ all\ \ v_\alpha\in(H^1(\Omega))^2,\ \ v_3=0$$

that vanish on $\gamma_0\times[-1,1]$ and thus

$$M\kappa_{\alpha 3}+\lambda_5\overline{\hbar}_{\alpha 3}=0 \tag{21}$$

by (iii).

Letting $v_\alpha=0$ in the variational equations and multiplying by ε^2, we likewise obtain

$$\int\limits_{\Omega}\{\Lambda\kappa_{\sigma\sigma}(\varepsilon)+(\Lambda+2M)\kappa_{33}(\varepsilon)\}^T\partial_3 v_3 dx$$

$$=-\varepsilon\int\limits_{\Omega}\left\{2M\kappa_{\alpha 3}(\varepsilon)-2\lambda_5\overline{\hbar}_{\alpha 3}(\varepsilon)\right\}^T\partial_\alpha v_3 dx+\varepsilon^2\mathbf{L}(\mathbf{v})=0,$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$ such that $v_\alpha = 0$. Hence, passing to the limit as $\varepsilon \to 0$, gives

$$\int_\Omega \{\Lambda\kappa_{\sigma\sigma} + (\Lambda + 2M)\kappa_{33}\}^T \partial_3 v_3 dx = 0$$

for all $v_3 \in (H^1(\Omega))^2$ that vanish on $\gamma_0 \times [-1, 1]$ and thus

$$\kappa_{33} = -(\Lambda + 2M)^{-1}\Lambda e_{\sigma\sigma}(\mathbf{u})$$

by (iii). Since $\kappa_{\sigma\sigma} = e_{\sigma\sigma}(\mathbf{u})$, the assertion is established.

(v) The weak limit $\mathbf{u} \in \mathbf{V}_{KL}(\Omega)$ satisfies the variational problem $P_{KL}(\Omega)$ described in the statement of the theorem and it is a unique solution.

Restrict the functions $\mathbf{v} = (v_i) \in \mathbf{V}(\Omega)$ appearing in the variational equations of problem $P(\varepsilon, \Omega)$ to lie in the subspace $\mathbf{V}_{KL}(\Omega)$ defined in (ii). Since $e_{i3}(\mathbf{v}) = 0$ and $h_{\alpha 3}(v) = 0$ in this case, these equations reduce to

$$\int_\Omega \left\{[\Lambda\kappa_{pp}(\varepsilon)\delta_{\alpha\beta} + 2M\kappa_{\alpha\beta}(\varepsilon)]^T e_{\alpha\beta}(\mathbf{v}) - 2\lambda_5 \overline{h}_{\alpha\beta}(\varepsilon) h_{\alpha\beta}(\mathbf{v})\right\} dx = \mathbf{L}(\mathbf{v})$$

for all $\mathbf{v} \in \mathbf{V}_{KL}(\Omega)$.

If $\varepsilon \to 0$, we obtain

$$\int_\Omega \left\{[\Lambda^* e_{\sigma\sigma}(\mathbf{u})\delta_{\alpha\beta} + 2M e_{\alpha\beta}(\mathbf{u})]^T e_{\alpha\beta}(\mathbf{v}) - 2\lambda_5 h_{\alpha\beta}(\mathbf{u}) h_{\alpha\beta}(\mathbf{v})\right\} dx = \mathbf{L}(\mathbf{v})$$

for all $\mathbf{v} \in \mathbf{V}_{KL}(\Omega)$.

The bilinear form associated with problem $P_{KL}(\Omega)$ is thus $\mathbf{V}_{KL}(\Omega)$-elliptic; hence problem $P_{KL}(\Omega)$ has one and only one solution, as a consequence of the Lax-Milgram lemma.

(vi) Two definitions of the space $\mathbf{V}_{KL}(\Omega)$ coincide, i.e.

$$\{\mathbf{v} \in \mathbf{V}(\Omega);\ e_{i3} = 0,\ h_{\alpha 3}(\mathbf{v}) = 0\ in\ \Omega\} = \Big\{\mathbf{v} = (v_1^{'},\ v_2^{'},\ v_3^{'},\ v_1^{''},\ v_2^{''},\ v_3^{''}),$$

$$v_3^{'} = v_3^{''} = \eta_3^*,\ v_\alpha^{'} = \eta_\alpha^{'} - x_3\partial_\alpha\eta_3^*,\ v_\alpha^{''} = \eta_\alpha^{''} - x_3\partial_\alpha\eta_3^*,$$

$$\eta_\alpha^{'}, \eta_\alpha^{''} \in H^1(\omega), \eta_3^* \in H^2(\omega), \eta_\alpha^{'} = \eta_\alpha^{''} = \eta_3^* = \partial_\nu\eta_3^* = 0\ on\ \gamma_0\Big\}.$$

The relations $e_{33} = \partial_3 v_3 = 0\ in\ \Omega\ and\ v_3 = 0\ on\ \gamma_0 \times [-1, 1]$ imply that there exists a function $\eta_3 \in (H^1(\omega))^2$ such that $\eta_3 = 0$ on γ_0 and

$$v_3(x^{'}, x_3) = \eta_3(x^{'})\ for\ almost\ all\ (x^{'}, x_3) \in \Omega = \omega \times]-1, 1[.$$

The equation $\partial_\alpha v_3 + \partial_3 v_\alpha = 0$ imply that $\partial_{33} v = -\partial_\alpha(\partial_3 v_3) = 0$ in Ω. Hence there exist functions $\eta_\alpha, \eta^1_\alpha \in (H^1(\omega))^2$ such that $\eta_\alpha = \eta^1_\alpha = 0$ on γ_0 and

$$v_\alpha(x', x_3) = \eta_\alpha(x') + x_3\eta^1_\alpha(x') \; for \; almost \; all \; (x', x_3) \in \Omega.$$

Since $0 = \partial_\alpha v_3 + \partial_3 v_\alpha = \partial_\alpha \eta_3 + \eta^1_\alpha$, we conclude that $\eta_3 \in (H^2(\omega))^2$.

Since $\partial_\alpha \eta_3 = -\eta^1_\alpha \in (H^1(\omega))^2$, and that $\partial_\nu \eta_3 = 0$ on γ_0 since $\partial_\alpha \eta_3 = -\eta^1_\alpha = 0$ on γ_0.

Since $h_{\alpha 3} = 0$ we conclude that $\partial_\alpha v'_3 - \partial_3 v'_\alpha + \partial_3 v''_\alpha - \partial_\alpha v''_3 = \partial_\alpha \eta'_3 + \partial_\alpha \eta'_3 - \partial_\alpha \eta''_3 - \partial_\alpha \eta''_3 = 0, \partial_\alpha(\eta' - \eta''_3) = 0$ on $\eta'_3 = \eta''_3 = 0$ on γ_0. Hence $\eta'_3 = \eta''_3 \equiv \eta^*_3$. These relations imply that

$$v'_\alpha = \eta'_\alpha - x_3\partial_\alpha \eta^*_3, \; v''_\alpha = \eta''_\alpha - x_3\partial_\alpha \eta^*_3, \; \eta'_\alpha, \; \eta''_\alpha \in H^1(\omega), \; \eta^*_3 \in H^2(\omega)$$

and

$$\eta'_\alpha = \eta''_\alpha = \eta^*_3 = \partial_\nu \eta^*_3 = 0 \; on \; \gamma_0.$$

Hence the two definitions are equivalent.

(viii) The functions ζ_3 and $\boldsymbol{\zeta}_H = (\zeta'_1, \; \zeta'_2, \; \zeta''_1, \; \zeta''_2)$ satisfy the variational prtoblems announced in the statement of the theorem.

Replacing the components u_i and v_i of the functions $\mathbf{u}, \mathbf{v} \in \mathbf{V}_{KL}(\Omega)$ by

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3, \; u_3 = \zeta_3, \; and \; v_\alpha = \eta_\alpha - x_\alpha\partial_\alpha\eta_3, \; v_3 = \eta_3,$$

where

$$\zeta_3 := (\zeta^*_3, \zeta^*_3)^T, \; \eta_3 := (\eta^*_3, \eta^*_3)^T,$$

we obtain

$$e_{\alpha\beta}(\mathbf{u}) = e_{\alpha\beta}(\boldsymbol{\zeta}_H) - x_3\partial_{\alpha\beta}\zeta_3, \; e_{\alpha\beta}(\mathbf{v}) = e_{\alpha\beta}(\boldsymbol{\eta}_H) - x_3\partial_{\alpha\beta}\eta_3,$$

$$e_{\sigma\sigma}(\mathbf{u}) = e_{\sigma\sigma}(\boldsymbol{\zeta}_H) - x_3\Delta\zeta_3, \; e_{\tau\tau}(\mathbf{v}) = e_{\tau\tau}(\boldsymbol{\eta}_H) - x_3\Delta\zeta_3,$$

$$\begin{aligned} h_{\alpha\beta}(\mathbf{u}) &= \frac{1}{2}\left(\partial_\alpha\zeta'_\beta - \partial_\beta\zeta'_\alpha + \partial_\beta\zeta''_\alpha - \partial_\alpha\zeta''_\beta\right), \; h_{\alpha\beta}(\mathbf{v}) \\ &= \frac{1}{2}\left(\partial_\alpha\eta'_\beta - \partial_\beta\eta'_\alpha + \partial_\beta\eta''_\alpha - \partial_\alpha\eta''_\beta\right) \end{aligned}$$

and we find the desired variational equations simply by noticing that $\int\limits_{-1}^{1} dx_3 = 2, \; \int\limits_{-1}^{1} x_3 dx_3 = 0, \; \int\limits_{-1}^{1} x_3^2 dx_3 = \frac{2}{3}$. ∎

The existence of the limit $\mathbf{u}$ found in Theorem 2 provides de facto an existence theory for the limit scaled two-dimensional problem, hence for both the scaled flexural and membrane equations. The uniqueness of their solution likewise follows from the uniqueness of the limit u established in the same theorem.

But below we give a "direct" proof of existence and uniqueness for each variational problem. We also write the two-dimensional boundary value problems that are, at least formally, equivalent to these variational problem.

Theorem 3. (a) Assume that $p_3^{'}+p_3^{''},\ q_\alpha^{'}+q_\alpha^{''}\in L^2(\omega)$. The scaled flexural equations of a linearly elastic plate, viz., find ζ_3^* such that

$$\zeta_3^*\in V_3(\omega):=\left\{\eta_3^*\in H^2(\omega),\ \eta_3^*=\partial_\nu\eta_3^*=0\ on\ \gamma_0\right\},$$

$$\int\limits_\omega\left\{\frac{2}{3}\widehat{\lambda}\Delta\zeta_3^*\Delta\eta_3^*+\frac{4}{3}\widehat{\mu}\partial_{\alpha\beta}\zeta_3^*\partial_{\alpha\beta}\eta_3^*\right\}d\omega=\int\limits_\omega(p_3^{'}+p_3^{''})\eta_3^*d\omega$$

$$-\int\limits_\omega(q_\alpha^{'}+q_\alpha^{''})\partial_\alpha\eta_3^*d\omega\ \ for\ all\ \ \eta_3^*\in V_3(\omega),$$

where

$$\widehat{\mu}>0\ \ and\ \widehat{\lambda}+\frac{2}{3}\widehat{\mu}>0, \tag{22}$$

has one and only one solution. If $\gamma_0=\gamma$, the variational equations may also be written as

$$\frac{2}{3}(\widehat{\lambda}+2\widehat{\mu})\int\limits_\omega\Delta\zeta_3^*\Delta\eta_3^*d\omega=\int\limits_\omega(p_3^{'}+p_3^{''})\eta_3^*d\omega-\int\limits_\omega(q_\alpha^{'}+q_\alpha^{''})\partial_\alpha\eta_3^*d\omega.$$

(b) Assume that the boundary γ of ω, the functions $p_3^{'}+p_3^{''},\ q_\alpha^{'}+q_\alpha^{''}$ and the solution ζ_3 are smooth enough. Then ζ_3 is also a solution of the following boundary value problem

$$-\partial_{\alpha\beta}m_{\alpha\beta}=p_3^{'}+p_3^{''}+\partial_\alpha(q_\alpha^{'}+q_\alpha^{''})\ \ in\ \ \omega,$$

$$\zeta_3^*=\partial_\nu\zeta_3^*=0\ \ on\ \ \gamma_0,$$

$$m_{\alpha\beta}\nu_\alpha\nu_\beta=0\ \ on\ \ \gamma_1,$$

$$(\partial_\alpha m_{\alpha\beta})\nu_\beta+\partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta)=-(q_\alpha^{'}+q_\alpha^{''})\nu_\alpha\ \ on\ \ \gamma_1,$$

where $\gamma_1=\gamma-\gamma_0$, (ν_α) is the unit outer normal vector along γ, $\tau_1:=-\nu_2,\ \tau_2:=\nu_1$

$$m_{\alpha\beta}:=-\left\{\frac{2}{3}\widehat{\lambda}\Delta\zeta_3^*\delta_{\alpha\beta}+\frac{4}{3}\widehat{\mu}\partial_{\alpha\beta}\zeta_3^*\right\}=-\frac{1}{3}a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3^*,$$

$$a_{\alpha\beta\sigma\tau} = \frac{2}{3}\widehat{\lambda}\delta_{\alpha\beta}\delta_{\sigma\tau} + 2\widehat{\mu}(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta\beta\sigma).$$

The partial differential equation satisfied by ζ_3^* may be also written as a biharmonic equation

$$\frac{2}{3}(\widehat{\lambda} + 2\widehat{\mu})\Delta^2\zeta_3^* = p_3^{'} + p_3^{''} + \partial_\alpha(q_\alpha^{'} + q_\alpha^{''}) \;\; in \;\; \omega,$$

where $\Delta^2 = \Delta\Delta = \partial_{\alpha\alpha}\partial_{\beta\beta}$ denotes the biharmonic operator.

Proof. (i) Let ω be a domain in $\mathbb{R}^2$, and let γ_0 be a measurable subset of γ with length $\gamma_0 > 0$. Then there exists a constant $c > 0$ such that

$$c^{-1}\|\eta^*\|_{2,\omega} \leq |\eta^*|_{2,\omega}$$

for all $\eta^* \in V_3(\omega)$.

To see this, we first notice that the semi-norm $|\cdot|_{2,\omega}$ is a norm in the space $V_3(\omega)$. For $|\eta^*|_{2,\omega} = 0$ implies that

$$\eta^*(x_1, x_2) = a_0 + a_1x_2 + a_2x_2,$$

by a classical result from the distribution theory, the boundary conditions $\eta^* = \partial_\nu\eta^* = 0$ on γ_0 then imply that $a_1 = a_2 = a_0 = 0$, since length $\gamma_0 > 0$ by assumption.

If the announced inequality is false, there exists a sequence (η^{*k}) of functions $\eta^{*k} \in V_3(\omega),\ k = 0, 1, ...,$ such that

$$\|\eta^{*k}\|_{2,\omega} = 1 \;\; for\ all \;\; k, \;\; and \;\; \lim_{k\to\infty} |\eta^{*k}|_{2,\omega} = 0.$$

By the Rellich-Kondrasov theorem [7], there exists a subsequence (η^{*l}) that converges in $H^1(\omega)$. Since each subsequence $(\partial_{\alpha\beta}\eta^{*l})$ converges in $L^2(\omega)$ (to 0), the subsequence is a Cauchy sequence in $H^2(\omega)$; hence it converges to some element $\eta^* \in V_3(\omega)$. From $|\eta^*|_{2,\omega} = \lim_{l\to\infty} |\eta^{*l}|_{2,\omega} = 0$, we infer that $\eta^* = 0$ since we have just showed that $|\cdot|_{2,\omega}$ is a norm on $V_3(\omega)$, but this contradicts $\|\eta^{*l}\|_{2,\omega}$ for all l.

Consequently, the bilinear form in the flexural equations in $V_3(\omega)$ is elliptic, since

$$\int_\omega \frac{2}{3}\left\{\widehat{\lambda}\Delta\eta^*\Delta\eta^* + 2\widehat{\mu}\partial_{\alpha\beta}\eta^*\partial_{\alpha\beta}\eta^*\right\}d\omega = \int_\omega \frac{2}{3}\left\{\left(\widehat{\lambda} + \frac{2}{3}\widehat{\mu}\right)\Delta\eta^*\Delta\eta^*\right.$$

$$\left.-\frac{2}{3}\widehat{\mu}\partial_{\alpha\alpha}\eta^*\partial_{\beta\beta}\eta^* + 2\widehat{\mu}\partial_{\alpha\beta}\eta^*\partial_{\alpha\beta}\eta^*\right\}d\omega = \int_\omega \frac{2}{3}\left\{\left(\widehat{\lambda} + \frac{2}{3}\widehat{\mu}\right)\Delta\eta^*\Delta\eta^*\right.$$

$$+\frac{2}{3}\widehat{\mu}(\partial_{11}\eta^* - \partial_{22}\eta^*)^2 + \frac{2}{3}\widehat{\mu}(\partial_{11}\eta^{*2} + \partial_{22}\eta^{*2} + 6(\partial_{12}\eta^{*2})\Big\} d\omega$$

$$\geq \frac{4\widehat{\mu}}{3}|\eta^*|^2_{2,\omega} \geq \frac{4\widehat{\mu}}{3}c^{-2}|\eta^*|^2_{2,\omega} \ \ for\ all\ \ \eta^* \in V_3(\omega)$$

by condition (22) (it follows from conditions $(12')$ and (19)). The bilinear and linear forms in the flexural equations being continuous with respect to $\|\cdot\|_{2,\omega}$, the existence and uniqueness of a solution follow from the Lax-Milgram lemma.

If $\gamma_0 = \gamma$, the space $V_3(\omega)$ coincides with $H_0^2(\omega)$. Since, by Green's formula.

$$\int\limits_\omega \partial_{\alpha\beta}\varphi\partial_{\alpha\beta}\psi d\omega = -\int\limits_\omega \partial_{\alpha\beta}\varphi\partial_{\alpha\beta}\psi d\omega = \int\limits_\omega \partial_{\alpha\alpha}\varphi\partial_{\beta\beta}\psi d\omega$$

for all $\varphi,\ \psi \in D(\omega)$ and $D(\omega)$ is dense in $H_0^2(\omega)$, these relations remain valid for all $\varphi,\ \psi \in H_0^2(\omega)$. Hence the last assertion in part (a) is established.

(ii) In view of finding the boundary value problem solved by ζ_3^*, we first note that the left-hand side of the variational equations may also be written as

$$\int\limits_\omega \left\{\frac{2}{3}\widehat{\lambda}\Delta\zeta\eta^*\Delta\eta^* + \frac{4\widehat{\mu}}{3}\partial_{\alpha\beta}\zeta^*\partial_{\alpha\beta}\eta^*\right\} d\omega = -\int\limits_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta^* d\omega,$$

where $m_{\alpha\beta}$ is defined in the theorem. Two applications of the Green formula then give

$$\begin{aligned} -\int\limits_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta^* d\omega &= -\int\limits_\omega (\partial_{\alpha\beta}m_{\alpha\beta})\eta^* d\omega + \int\limits_\gamma (\partial_\alpha m_{\alpha\beta})\nu_\beta\eta^* d\gamma \\ &\quad -\int\limits_\gamma m_{\alpha\beta}\nu_\alpha\partial_\beta\eta^* d\gamma. \end{aligned}$$

Since $\partial_\beta\eta^* = \nu_\beta\partial_\nu\eta^* + \tau_\beta\partial_\tau\eta^*$, we may write

$$-\int\limits_\gamma m_{\alpha\beta}\nu_\alpha\nu_\beta d\gamma = \int\limits_\gamma m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta^* d\gamma + \int\limits_\gamma m_{\alpha\beta}\nu_\alpha\tau_\beta\partial_\tau\eta^* d\gamma.$$

Observing that

$$\int\limits_\gamma \varphi\partial_\tau\eta^* d\gamma = -\int\limits_\gamma (\partial_\tau\varphi)\eta^* d\gamma, \ \ since \ \ \int\limits_\gamma \partial_\tau(\varphi\eta^*)d\gamma = 0.$$

So

$$-\int\limits_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta^* d\omega = -\int\limits_\omega (\partial_{\alpha\beta}m_{\alpha\beta})\eta^* d\omega + \int\limits_\gamma \{(\partial_\alpha m_{\alpha\beta})\nu_\beta$$

$$+\,\partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta)\}\,\eta^* d\gamma - \int\limits_\gamma m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta^* d\gamma$$

is valid for all $m_{\alpha\beta} \in H^2(\omega)$ and $\eta^* \in H^2(\omega)$.

For all $q_\alpha^{'} + q_\alpha^{''} \in H^1(\omega)$ and $\eta^* \in H^1(\omega)$ vanishing on γ_0

$$-\int\limits_\omega (q_\alpha^{'} + q_\alpha^{''})\partial_\alpha\eta^* d\omega = \int\limits_\omega \partial_\alpha(q_\alpha^{'} + q_\alpha^{''})\eta^* d\omega - \int\limits_{\gamma_1}(q_\alpha^{'} + q_\alpha^{''})\nu_\alpha\eta^* d\gamma.$$

Hence part (b) is proved. ∎

Theorem 4. Let the following conditions be fulfilled

$$\lambda_5 < 0,\ \mu_1 > 0,\ \lambda_1^* + \mu_1 > 0,\ det M > 0,\ det\,(\Lambda^* + M) > 0. \qquad (23)$$

Then: (a) Assume that $p_\alpha \in (L^2(\omega))^2$. The scaled membrane equations, viz., find $\boldsymbol{\zeta}_H$ such that

$$\boldsymbol{\zeta}_H \in \mathbf{V}_H(\omega) := \left\{\boldsymbol{\eta}_H = (\eta_1^{'},\ \eta_2^{'},\ \eta_1^{''},\ \eta_2^{''}) \in (H_1(\omega))^4,\ \boldsymbol{\eta}_H = 0\ on\ \gamma_0\right\},$$

$$\begin{aligned} &\int\limits_\omega \left\{2(\Lambda^* e_{\sigma\sigma}(\boldsymbol{\zeta}_H))^T e_{\tau\tau}(\boldsymbol{\eta}_H) + 4(Me_{\alpha\beta}(\boldsymbol{\zeta}_H))^T e_{\alpha\beta}(\boldsymbol{\eta}_H)\right. \\ &-4\lambda_5 h_{\alpha\beta}(\boldsymbol{\zeta}_H)h_{\alpha\beta}(\boldsymbol{\eta}_H)\}\,d\omega \\ = &\int\limits_\omega (p_\alpha)^T\eta_\alpha d\omega\ \ for\ all\ \ (\boldsymbol{\eta}_H) \in V_H(\omega), \end{aligned}$$

where

$$e_{\alpha\beta}(\boldsymbol{\zeta}_H) := \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha),\ \ h_{\alpha\beta}(\boldsymbol{\zeta}_H) = \frac{1}{2}(\partial_\alpha\zeta_\beta^{'} - \partial_\beta\zeta_\alpha^{'} + \partial_\beta\zeta_\alpha^{''} - \partial_\alpha\zeta_\beta^{''}),$$

have one and only one solution.

(b) A smooth enough solution $\boldsymbol{\zeta}_H$ of these equations is also a solution of the following boundary value problem:

$$-\partial_\beta n_{\alpha\beta} = p_\alpha\ \ in\ \ \omega,$$

$$\boldsymbol{\zeta}_H = 0\ \ on\ \ \gamma_0,$$

$$n_{\alpha\beta}\nu_\beta = 0\ \ on\ \ \gamma_1,$$

where

$n_{\alpha\beta} := 2\Lambda^* e_{\sigma\sigma}(\boldsymbol{\zeta}_H)\delta_{\alpha\beta} + 4Me_{\alpha\beta}(\zeta_H) + 4\lambda_5\hbar_{\alpha\beta}(\boldsymbol{\zeta}_H)$,
$\hbar_{\alpha\beta}(\boldsymbol{\zeta}_H) = (h_{\alpha\beta}(\boldsymbol{\zeta}_H),\ h_{\beta\alpha}(\boldsymbol{\zeta}_H))^T$.

Proof. Let ω be a domain in $\mathbb{R}^2$, and let γ_0 be a measurable subset of $\gamma = \partial\omega$ with length $\gamma_0 > 0$. Then there exists a constant $c > 0$ such that

$$c^{-1}\|\boldsymbol{\eta}_H\|_{1,\omega} \leq |\mathbf{e}(\boldsymbol{\eta}_H)|_{0,\omega} \leq \left\{\sum_{\alpha,\beta}(|e'_{\alpha\beta}|^2_{0,\omega} + |e''_{\alpha\beta}|^2_{0,\omega})\right\}^{\frac{1}{2}}$$

for all $\boldsymbol{\eta}_H \in \mathbf{V}_H(\omega)$.

To prove this two-dimensional Korn inequality, first we notice that the semi-norm $|\mathbf{e}(\boldsymbol{\eta}_H)|_{0,\omega}$ is a norm on the space $V_H(\omega)$. For $|\mathbf{e}(\boldsymbol{\eta}_H)|_{0,\omega} = 0$ implies that

$$\partial_{\alpha\beta}\eta_\alpha = \partial_\alpha e_{\beta\sigma}(\boldsymbol{\eta}_H) + \partial_\beta e_{\alpha\sigma}(\boldsymbol{\eta}_H) - \partial_\sigma e_{\alpha\beta}(\boldsymbol{\eta}_H) = 0 \ \ in\ \ D'(\omega),$$

hence $\eta_1(x_1,x_2) = a_1 - bx_2$ and $\eta_1(x_1,x_2) = a_2 + bx_1$, where $a_\alpha = (a'_\alpha,\ a''_\alpha)^T,\ \ b = (b',\ b'')^T$. These relations, together with the boundary conditions $\eta_\alpha = 0$ on γ_0, show that $\boldsymbol{\eta}_H = 0$.

If the announced inequality is false, there exists a sequence $(\boldsymbol{\eta}^k_H)$ of functions $\boldsymbol{\eta}^k_H \in \mathbf{V}_H(\omega),\ \ k = 0,1,...$, such that

$$\|\boldsymbol{\eta}^k_H\|_{1,\omega} = 1, \quad for\ all\ \ k\ \ and\ \ \lim_{k\to 0}|\mathbf{e}(\boldsymbol{\eta}^k_H)|_{0,\omega} = 0.$$

By the Rellich-Kondrasov theorem, there exists a subsequence $(\boldsymbol{\eta}^l_H)$ that converges in $(L^2(\omega))^4$. Since the subsequence $(\mathbf{e}(\boldsymbol{\eta}^l_H))$ converges in $(L^2(\omega)^{2\times 1})^{2\times 2}_s$ (to 0), the subsequence $(\boldsymbol{\eta}^l_H)$ is a Cauchy sequence with respect to the norm

$$\boldsymbol{\eta}_H \to \left\{|\boldsymbol{\eta}_H|^2_{0,\omega} + |\mathbf{e}(\boldsymbol{\eta}_H)|^2_{0,\omega}\right\}^{\frac{1}{2}}.$$

By the two-dimensional Korn inequality without boundary conditions, this norm is equivalent to the norm $\|\cdot\|_{1,\omega}$ over the space $\mathbf{V}_H(\omega)$. Hence the $(\boldsymbol{\eta}^l_H)$ is also Cauchy sequence with respect to $\|\cdot\|_{1,\omega}$ and this converges to the some element $\boldsymbol{\eta}_H \in \mathbf{V}_H(\omega)$. From $|\mathbf{e}(\boldsymbol{\eta}_H)|_{0,\omega} = \underset{l\to\infty}{lim}|\mathbf{e}(\boldsymbol{\eta}^l_H)|_{0,\omega} = 0$, we infer that $\boldsymbol{\eta}_H = 0$, since $|e(\cdot)|_{0,\omega}$ is a norm on $V_H(\omega)$; but this contradicts $\|\boldsymbol{\eta}^l_H\|_{1,\omega} = 1$ for all l.

By conditions (27) and Lemma 2, there exists a constant $\delta^* > 0$ such that

$$\begin{aligned}
&\int_\omega \left\{2(\Lambda^* e_{\sigma\sigma}(\boldsymbol{\eta}_H))^T e_{\tau\tau}(\boldsymbol{\eta}_H) + 4(Me_{\alpha\beta}(\boldsymbol{\eta}_H))^T e_{\alpha\beta}(\boldsymbol{\eta}_H)\right.\\
&\left.-4h_{\alpha\beta}(\boldsymbol{\eta}_H)h_{\alpha\beta}(\boldsymbol{\eta}_H)d\omega\right\}\\
\geq\ & \delta^*(|\mathbf{e}(\boldsymbol{\eta}_H)|^2_{0,\omega} + |\mathbf{h}(\boldsymbol{\eta}_H)|^2_{0,\omega})\\
\geq\ & \delta^* c^{-2}\|\boldsymbol{\eta}_H\|^2_{1,\omega}\ \ for\ all\ \ \ \boldsymbol{\eta}_H \in \mathbf{V}_H(\omega).
\end{aligned}$$

The existence and uniqueness of a solution then follow from Lax-Milgram lemma. Hence part (a) is proved.

(ii) In view of finding the boundary value problem solved by $\boldsymbol{\zeta}_H$, we first note that the left-hand side of the variational equations may also be written as

$$\begin{aligned}&\int_\omega \{2(\Lambda^* e_{\sigma\sigma}(\boldsymbol{\eta}_H))^T e_{\tau\tau}(\boldsymbol{\eta}_H) + 4(Me_{\alpha\beta}(\boldsymbol{\eta}_H))^T e_{\alpha\beta}(\boldsymbol{\eta}_H)\\ &-4\lambda_5 h_{\alpha\beta}(\boldsymbol{\eta}_H) h_{\alpha\beta}(\boldsymbol{\eta}_H)\}\, d\omega\\ =\ &\int_\omega n_{\alpha\beta}^T \partial_\beta \eta_\alpha d\omega,\end{aligned}$$

where $n_{\alpha\beta}$ is defined in the theorem. The Green formula

$$\int_\omega (n_{\alpha\beta})^T \partial_\beta \eta_\alpha d\omega = -\int_\omega (\partial_\beta n_{\alpha\beta})^T \eta_\alpha d\omega + \int_{\gamma_1} (n_{\alpha\beta}\nu_\beta)^T \eta_\alpha d\gamma,$$

valid for all $n_{\alpha\beta} \in (H^1(\omega))^2$ and $\boldsymbol{\eta}_H \in \mathbf{V}_H(\omega)$, then yields the partial differential equations and boundary conditions on γ_1 that are satisfied by $\boldsymbol{\zeta}_H$. Hence part (b) is proved. ∎

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