ON A HIERARCHICAL MODEL OF ELASTIC RODS WITH VARIABLE CROSS-SECTIONS

G. Avalishvili, M. Avalishvili

Institute of Applied Mathematics,
Tbilisi State University
0143 University Str. 2, Tbilisi, Georgia

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Abstract

The present paper is devoted to the construction and investigation of one-dimensional hierarchical model of elastic rod with variable non-rectangular cross-sections. The three-dimensional static boundary value problem is reduced to a sequence of one-dimensional ones and the existence and uniqueness of their solutions in suitable spaces are proved. Under the conditions of solvability of the original problem the convergence of the sequence of vector functions of three variables restored from the solutions of the constructed one-dimensional problems is proved and the rate of convergence is estimated.

Key words and phrases: Linearly elastic rods, hierarchical modeling, a priori error estimates.

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In recent years heirarchical modeling is widely used while constructing and investigating various lower-dimensional models in the theory of elasticity and mathematical physics [1-8]. One of the methods of constructing the hierarchical two-dimensional models for linearly elastic prismatic shells was suggested by I. Vekua in [1]. This method is based on approximation of the components of the displacement vector function by partial sums of the orthogonal Fourier-Legendre series with respect to the variable of plate thickness. The classical Kirchhoff-Love and Reissner-Mindlin models can be incorporated into the hierarchy obtained by I. Vekua so that it may be viewed as an extension of the mentioned plate models. It must be pointed out that in [1] initial boundary value problems were considered only in the classical spaces of regular functions and the convergence of the sequence of approximate solutions to the exact solution of the original three-dimensional problem was not investigated. For static boundary value problem the existence and uniqueness of solution of the reduced two-dimensional problem obtained by I. Vekua [1] in Sobolev spaces first were investigated by D. Gordeziani in [9]. The rate of approximation of the exact solution of the original problem by vector functions of three variables restored from the solutions of the reduced problems in C^k spaces was estimated in [10]. Later on, the hierarchical models constructed by I. Vekua's method and its generalizations were studied in [11-15].

In the present paper we construct and investigate a hierarchy of one-dimensional problems for static boundary value problem for nonhomogeneous anisotropic linearly elastic rod. Hierarchical models of elastic rods with variable rectangular cross-sections were obtained in [13-15] and the existence and uniqueness of solutions of the reduced one-dimensional problems and the relation of the models to the original problem were investigated in [14, 15]. In this work we consider boundary value problem for linearly elastic rod with variable cross-sections, which, in general, aren't rectangular, and generalizing I. Vekua's approach we construct a hierarchy of one-dimensional models. For the obtained boundary value problems we investigate the existence and uniqueness of their solutions in suitable functional spaces. In addition, we prove that the sequence of vector functions constructed by means of the solutions of the reduced one-dimensional problems converges to the solution of the three-dimensional problem and under certain conditions we obtain a priori error estimate of the rate of convergence.

Throughout this paper we assume that the subscripts and superscripts i, j, p, q take their values in the set $\{1, 2, 3\}$ and the partial derivative $\partial/\partial x_i$ with respect to the variable x_i we denote by ∂_i. For each real $s \geq 0$, $H^s(\Omega)$ and $H^s(\check{\Gamma})$ denote the usual Sobolev spaces of real-valued functions based on $H^0(\Omega) = L^2(\Omega)$ and $H^0(\check{\Gamma}) = L^2(\check{\Gamma})$, respectively, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded Lipschitz domain (i.e., connected open set with a Lipschitz-continuous boundary, the set Ω being locally on one side of its boundary) and $\check{\Gamma}$ is an element of a Lipschitz dissection of the boundary $\Gamma = \partial\Omega$ [16]. $H_0^s(\Omega)$ denotes the closure of the set $\mathfrak{D}(\Omega)$ of infinitely differentiable functions with compact support in Ω in the space $H^s(\Omega)$. The corresponding spaces of vector-valued functions we denote by $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^3$, $\mathbf{H}_0^s(\Omega) = [H_0^s(\Omega)]^3$, $\mathbf{H}^s(\check{\Gamma}) = [H^s(\check{\Gamma})]^3$, $s \geq 0$.

Let us consider an elastic body with initial configuration $\overline{\Omega}$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain with boundary $\Gamma = \partial\Omega$. The body Ω consists of nonhomogeneous and anisotropic linearly elastic material for which the stress tensor (σ_{ij}) linearly depends on the strain tensor $(e_{pq}(\boldsymbol{u}))$, $\sigma_{ij} = \sum\limits_{p,q=1}^{3} a_{ijpq} e_{pq}(\boldsymbol{u})$, $i, j = \overline{1, 3}$, where $e_{pq}(\boldsymbol{u}) = 1/2(\partial_p u_q + \partial_q u_p)$, $\boldsymbol{u} = (u_i)$ is the displacement vector function, a_{ijpq} are the elastic coefficients depending on $x = (x_1, x_2, x_3) \in \Omega$ [17].

We assume that the part Γ_0 of the boundary Γ of Ω is clamped, i.e., $\boldsymbol{u} = \mathbf{0}$ on Γ_0, and a surface force is acting on the remaining part $\Gamma_1 = \Gamma \backslash \overline{\Gamma_0}$ of the boundary, where $\partial\Omega = \Gamma_0 \cup \Gamma_{01} \cup \Gamma_1$ is a Lipschitz dissection of $\partial\Omega$ [16]. The applied body force density we denote by $\boldsymbol{f} = (f_i)$ and the density of the surface force which act on Γ_1 we denote by $\boldsymbol{g} = (g_i)$, $g_i = \sum\limits_{j=1}^{3} \sigma_{ij}\nu_j$, where $\boldsymbol{\nu} = (\nu_j)$ is the outward normal to Γ_1. The variational formulation of the corresponding static three-dimensional problem of linearized elasticity is of the following form: Find a vector function $\boldsymbol{u} = (u_i) \in V(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \boldsymbol{tr}_\Gamma(\boldsymbol{v}) = \mathbf{0} \text{ on } \Gamma_0\}$ such that for all $\boldsymbol{v} \in V(\Omega)$,

$$\sum_{i,j,p,q=1}^{3} \int_\Omega a_{ijpq}(x) e_{pq}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{g}, \boldsymbol{tr}_{\Gamma_1}(\boldsymbol{v}) \rangle_{\Gamma_1}, \qquad (1)$$

where $\boldsymbol{tr}_\Gamma$ denotes the trace operator from $\mathbf{H}^1(\Omega)$ into $\mathbf{H}^{1/2}(\Gamma)$ and $\boldsymbol{tr}_{\Gamma_1}(\boldsymbol{v})$ is the restriction of $\boldsymbol{tr}_\Gamma(\boldsymbol{v})$ on Γ_1 for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{f} = (f_i) \in \widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{g} = (g_i) \in \mathbf{H}^{-1/2}(\Gamma_1)$, $\widetilde{\mathbf{H}}^{-1}(\Omega)$ and $\mathbf{H}^{-1/2}(\Gamma_1)$ are the dual spaces of $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma_1)$ [16], respectively, and $\langle .,. \rangle_\Omega$, $\langle .,. \rangle_{\Gamma_1}$ denote the duality relations between the corresponding spaces. The bilinear form in the left-hand side of the equation (1) we denote by $A(\boldsymbol{u}, \boldsymbol{v})$ and the linear form in the right-hand side by $L(\boldsymbol{v})$.

The three-dimensional problem (1) possesses a unique solution, when $a_{ijpq} \in L^\infty(\Omega)$, $i,j,p,q = \overline{1,3}$, and the elasticity tensor (a_{ijpq}) satisfies the following ellipticity and symmetry conditions for almost all $x \in \Omega$ and for all $\varepsilon_{ij} \in \mathbb{R}$, $\varepsilon_{ij} = \varepsilon_{ji}$,

$$\sum_{i,j,p,q=1}^{3} a_{ijpq}(x) \varepsilon_{ij} \varepsilon_{pq} \geq \alpha \sum_{i,j=1}^{3} \varepsilon_{ij}\varepsilon_{ij},\ a_{ijpq}(x) = a_{jipq}(x) = a_{pqij}(x), \qquad (2)$$

where $\alpha = const > 0$, $i,j,p,q = \overline{1,3}$. In addition, the solution $\boldsymbol{u}$ of the problem (1) is also a unique solution of the following minimization problem: Find $\boldsymbol{u} \in V(\Omega)$ such that

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in V(\Omega)} J(\boldsymbol{v}), \quad J(\boldsymbol{v}) = \frac{1}{2} A(\boldsymbol{v}, \boldsymbol{v}) - L(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega).$$

We consider the particular case of the three-dimensional problem (1), when Ω is a rod with variable cross-sections

$$\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ h_1^-(x_2, x_3) < x_1 < h_1^+(x_2, x_3),$$
$$h_2^-(x_3) < x_2 < h_2^+(x_3),\ x_3 \in I\},$$

where $I=(h_3^-,h_3^+)$, $h_3^+>h_3^-$, $h_3^\pm\in\mathbb{R}$, $h_1^\pm\in C^0(\overline{\omega})\cap C^1(\omega)$, $\omega=\{(x_2,x_3)\in\mathbb{R}^2;\ h_2^-(x_3)<x_2<h_2^+(x_3),\ x_3\in I\}$, $h_1^-(x_2,x_3)<h_1^+(x_2,x_3)$, for all $(x_2,x_3)\in\omega$ and $x_3=h_3^+$, $h_2^\pm\in C^0(\overline{I})\cap C^1(I)$, $h_2^-(x_3)<h_2^+(x_3)$, for all $h_3^-<x_3\le h_3^+$. We assume that the clamped part of the body Ω coincides with the upper face $\Gamma_0=\{x\in\partial\Omega;x_3=h_3^+\}$ of the rod. Note that the cross-sections of the rod with planes which are parallel to the plane $x_3=0$ depend on x_3 and, in general, may be non-rectangular; the thickness or width of the rod may vanish on the lower part of the rod for $x_3=h_3^-$.

To construct one-dimensional hierarchical model of the problem (1) let us consider the subspaces $V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)\subset V(\Omega)$, $\mathbf{N}^\alpha=(N_1^\alpha,N_2^\alpha,N_3^\alpha)\in[\mathbb{N}\cup\{0\}]^3$, $\alpha=1,2$, of vector functions the i-th components of which are polynomials of degree N_i^1 with respect to the variable x_1 and of degree N_i^2 with respect to the variable x_2, i.e.,

$$\begin{aligned}&V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)=\{\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i})\in\mathbf{H}^1(\Omega);v_{\mathbf{N}^1\mathbf{N}^2i}=\Phi_{\mathbf{N}^1\mathbf{N}^2i}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=\\&=\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\overset{[r_i^1r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2i}\,x_1^{r_i^1}x_2^{r_i^2},\boldsymbol{tr}_\Gamma(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})=\mathbf{0}\ \text{ on }\Gamma_0,\vec{v}_{\mathbf{N}^1\mathbf{N}^2}=(\overset{[r_i^1r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2i}),\\&\overset{[r_i^1r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in H^1_{loc}((h_3^-;h_3^+]),\ 0\le r_i^\alpha\le N_i^\alpha,\ \alpha=1,2,\ i=1,2,3\},\end{aligned}$$

where $H^1_{loc}((h_3^-;h_3^+])$ is the space of functions which belong to $H^1(\widetilde{h}_3^-;\widetilde{h}_3^+)$, $h_3^-<\widetilde{h}_3^-<\widetilde{h}_3^+\le h_3^+$. On the subspace $V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ from the original three-dimensional problem (1) we obtain the following variational problem: Find the unknown vector function $\boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2}=(w_{\mathbf{N}^1\mathbf{N}^2i})\in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $w_{\mathbf{N}^1\mathbf{N}^2i}=\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\overset{[r_i^1r_i^2]}{w}_{\mathbf{N}^1\mathbf{N}^2i}\,x_1^{r_i^1}x_2^{r_i^2}$, $i=\overline{1,3}$, which satisfies the equation

$$A(\boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2},\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})=L(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}),\qquad\forall\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}\in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega).\tag{3}$$

Note that the elements of the space $H^1_{loc}((h_3^-;h_3^+])$ are continuous on the segment $(h_3^-;h_3^+]$ and $\|.\|_{\mathbf{H}^1(\Omega)}$ defines the norm $\|.\|_*$ of vector functions $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}=(\overset{[r_i^1r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2i})$ in the space $[H^1_{loc}((h_3^-;h_3^+])]^{N^{1,2}_{1,2,3}}$, $N^{1,2}_{1,2,3}=\sum_{i=1}^3(N_i^1+1)(N_i^2+1)$, such that $\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*=\|\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{\mathbf{H}^1(\Omega)}$, where $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i})$ corresponds to $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}$, $v_{\mathbf{N}^1\mathbf{N}^2i}=\Phi_{\mathbf{N}^1\mathbf{N}^2i}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})$, $i=\overline{1,3}$. Indeed, if the norm $\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*$ equals to zero, then the corresponding $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}\equiv\mathbf{0}$ and hence $\partial_1^{N_i^1}\partial_2^{N_i^2}v_{\mathbf{N}^1\mathbf{N}^2i}\equiv\overset{[N_i^1N_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2i}\equiv0$, $i=\overline{1,3}$. Similarly we obtain that $\partial_1^{r_i^1}\partial_2^{r_i^2}v_{\mathbf{N}^1\mathbf{N}^2i}\equiv\overset{[r_i^1r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2i}\equiv0$, for $r_i^1+r_i^2=N_i^1+N_i^2-1$, then for $r_i^1+r_i^2=N_i^1+N_i^2-2$ and so on for $r_i^1+r_i^2=N_i^1+N_i^2-k$, $k=3,4,...,N_i^1+N_i^2$, $i=1,2,3$.

Since in the problem (3) the unknown functions are $\overset{[r_i^1 r_i^2]}{w}_{\mathbf{N}^1\mathbf{N}^2 i}$ the problem (3) is equivalent to the following one: Find a vector function $\vec{w}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I) = \{\vec{v}_{\mathbf{N}^1\mathbf{N}^2} = (\overset{[r_i^1 r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2 i}) \in [H^1_{loc}((h_3^-; h_3^+])]^{N^{1,2}_{1,2,3}}; \|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_* < \infty$, $\overset{[r_i^1 r_i^2]}{v}_{\mathbf{N}^1\mathbf{N}^2 i} = 0$ for $x_3 = h_3^+$, $r_i^\alpha = \overline{0, N_i^\alpha}$, $\alpha = 1, 2, i = \overline{1, 3}\}$ such that

$$A_{\mathbf{N}^1\mathbf{N}^2}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}), \qquad \forall \vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I), \tag{4}$$

where $A_{\mathbf{N}^1\mathbf{N}^2}(\vec{u}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2})$ and $L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})$ denote the restrictions $A(\boldsymbol{u}_{\mathbf{N}^1\mathbf{N}^2}, \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})$ and $L(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})$ of the forms $A(.,.)$ and $L(.)$ on the subspace $V_{\mathbf{N}^1\mathbf{N}^2}(\Omega) \subset V(\Omega)$ which are considered as the forms with respect to the vector functions $\vec{u}_{\mathbf{N}^1\mathbf{N}^2}$ and $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}$ of one variable corresponding to $\boldsymbol{u}_{\mathbf{N}^1\mathbf{N}^2}$ and $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}$, respectively.

So, we have constructed the hierarchy of one-dimensional models of linearly elastic rod with variable, in general, non-rectangular cross-sections. In the following theorem we prove the well-posedness of the corresponding boundary value problem (4).

Theorem 1. *Suppose the elasticity tensor (a_{ijpq}) satisfies the conditions (2) and $a_{ijpq} \in L^\infty(\Omega)$, i, j, p, $q = \overline{1, 3}$. If $\boldsymbol{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega)$ and $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_1)$, then the reduced one-dimensional problem (4) possesses a unique solution $\vec{w}_{\mathbf{N}^1\mathbf{N}^2}$, which is also a solution of the following minimization problem: Find $\vec{w}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$ such that*

$$J_{\mathbf{N}^1\mathbf{N}^2}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2}) = \inf_{\vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)} J_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}),$$
$$J_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = \frac{1}{2} A_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) - L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}).$$

Proof. We first prove that the space $\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$ is complete. Let $\{\vec{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2}\}_{l=1}^\infty$ be a Cauchy sequence in $\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, i.e.,

$$\|\vec{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2} - \vec{v}^{(m)}_{\mathbf{N}^1\mathbf{N}^2}\|_* \to 0 \qquad \text{as } l, m \to \infty.$$

Then it follows from the definition of the norm $\|.\|_*$ that $\{\boldsymbol{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2}\}_{l=1}^\infty$ is a Cauchy sequence in the space $\mathbf{H}^1(\Omega)$, where $\boldsymbol{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2} = (v^{(l)}_{\mathbf{N}^1\mathbf{N}^2 i})$, $v^{(l)}_{\mathbf{N}^1\mathbf{N}^2 i} = \Phi_{\mathbf{N}^1\mathbf{N}^2 i}(\vec{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2})$, $i = \overline{1, 3}$. Hence, there exists a vector function $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2} = (v_{\mathbf{N}^1\mathbf{N}^2 i}) \in \mathbf{H}^1(\Omega)$ such that $\boldsymbol{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2} \to \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}$ in $\mathbf{H}^1(\Omega)$ as $l \to \infty$.

Note that for any subdomain Ω^* of Ω the sequence of restrictions $\{\boldsymbol{v}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2}\}_{l=1}^\infty$ of $\{\boldsymbol{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2}\}_{l=1}^\infty$ on Ω^* converges to the restriction $\boldsymbol{v}^*_{\mathbf{N}^1\mathbf{N}^2}$ of $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}$ on Ω^* in the space $\mathbf{H}^1(\Omega^*)$ as $l \to \infty$. Let us consider the subdomain Ω^* which is of the same geometric form as Ω, i.e.,

$$\Omega^* = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ h_1^{*,-}(x_2, x_3) < x_1 < h_1^{*,+}(x_2, x_3),$$
$$h_2^{*,-}(x_3) < x_2 < h_2^{*,+}(x_3),\ x_3 \in I^*\},$$

where $I^* = (h_3^{*,-}, h_3^+)$, $h_3^+ > h_3^{*,-}$, $h_3^{*,-} \in \mathbb{R}$, $h_1^{*,\pm} \in C^1(\overline{\omega^*})$, $\omega^* = \{(x_2, x_3) \in \mathbb{R}^2;\ h_2^{*,-}(x_3) < x_2 < h_2^{*,+}(x_3),\ x_3 \in I^*\}$, $h_2^{*,\pm} \in C^1(\overline{I^*})$, $h_1^{*,+} > h_1^{*,-}$ on $\overline{\omega^*}$ and $h_2^{*,+} > h_2^{*,-}$ on $\overline{I^*}$. The restriction $v^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i}$ of $v^{(l)}_{\mathbf{N}^1\mathbf{N}^2 i}$ on Ω^* belongs to $H^1(\Omega^*)$ and can be represented as follows

$$v^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} = \sum_{r_i^1=0}^{N_i^1} \frac{1}{h_1^*}\left(r_i^1 + \frac{1}{2}\right) \overset{r_i^1}{v}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} P_{r_i^1}(z_1^*),\ \overset{r_i^1}{v}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} \in H^1(\omega^*), i = \overline{1,3},$$

where $z_1^* = (x_1 - \overline{h}_1^*)/h_1^*$, $\overline{h}_1^* = (h_1^{*,+} + h_1^{*,-})/2$, $h_1^* = (h_1^{*,+} - h_1^{*,-})/2$ and P_r denotes the Legendre polynomial of order $r \in \mathbb{N} \cup \{0\}$. Since $v^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} \to v^*_{\mathbf{N}^1\mathbf{N}^2 i}$ in $H^1(\Omega^*)$ as $l \to \infty$, we have that for all $r \in \mathbb{N} \cup \{0\}$,

$$\overset{r}{v}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} = \int_{h_1^{*,-}}^{h_1^{*,+}} v^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} P_r(z_1^*) dx_1 \to$$

$$\to \overset{r}{v}{}^*_{\mathbf{N}^1\mathbf{N}^2 i} = \int_{h_1^{*,-}}^{h_1^{*,+}} v^*_{\mathbf{N}^1\mathbf{N}^2 i} P_r(z_1^*) dx_1 \quad \text{in } H^1(\omega^*) \text{ as } l \to \infty, i = \overline{1,3}, \tag{5}$$

and therefore $\overset{r_i^1}{v}{}^*_{\mathbf{N}^1\mathbf{N}^2 i} \equiv 0$, for $r_i^1 > N_i^1$, $i = \overline{1,3}$. Thus the vector function $v^*_{\mathbf{N}^1\mathbf{N}^2 i}$ is of the following form

$$v^*_{\mathbf{N}^1\mathbf{N}^2 i} = \sum_{r_i^1=0}^{N_i^1} \frac{1}{h_1^*}\left(r_i^1 + \frac{1}{2}\right) \overset{r_i^1}{v}{}^*_{\mathbf{N}^1\mathbf{N}^2 i} P_{r_i^1}(z_1^*),\ \overset{r_i^1}{v}{}^*_{\mathbf{N}^1\mathbf{N}^2 i} \in H^1(\omega^*), i = \overline{1,3}.$$

From (5) it follows that the coefficient $\overset{r_i^1}{b}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i}$ of $x_1^{r_i^1}$ in the expression of $v^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i}$ tends to the corresponding coefficient $\overset{r_i^1}{b}{}^*_{\mathbf{N}^1\mathbf{N}^2 i}$ of $x_1^{r_i^1}$ in $v^*_{\mathbf{N}^1\mathbf{N}^2 i}$, $i = 1, 2, 3$. Consequently, since $\overset{r_i^1}{b}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i}$, $r_i^1 = \overline{0, N_i^1}$, $i = \overline{1,3}$, are polynomials with respect to the variable x_2, we have

$$\overset{r_i^1}{b}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} = \sum_{r_i^2=0}^{N_i^2} \overset{[r_i^1 r_i^2]}{v}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} x_2^{r_i^2} = \sum_{r_i^2=0}^{N_i^2} \frac{1}{h_2^*}\left(r_i^2 + \frac{1}{2}\right) \overset{r_i^1 r_i^2}{b}{}^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i} P_{r_i^2}(z_2^*) \to$$

$$\to \overset{r_i^1}{b}{}^*_{\mathbf{N}^1\mathbf{N}^2 i} = \sum_{r_i^2=0}^{N_i^2} \frac{1}{h_2^*}\left(r_i^2 + \frac{1}{2}\right) \overset{r_i^1 r_i^2}{b}{}^*_{\mathbf{N}^1\mathbf{N}^2 i} P_{r_i^2}(z_2^*) \text{ in } H^1(\omega^*) \text{ as } l \to \infty,$$

where $z_2^* = \frac{x_2 - \overline{h}_2^*}{h_2^*}$, $\overline{h}_2^* = \frac{h_2^{*,+} + h_2^{*,-}}{2}$, $h_2^* = \frac{h_2^{*,+} - h_2^{*,-}}{2}$, $\overset{r_i^1 r_i^2}{b^*_{\mathbf{N}^1\mathbf{N}^2 i}} \in H^1(I^*)$, and $\overset{r_i^1 r_i^2}{b^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i}} \to \overset{r_i^1 r_i^2}{b^*_{\mathbf{N}^1\mathbf{N}^2 i}}$ in $H^1(I^*)$ as $l \to \infty$, $r_i^\alpha = \overline{0, N_i^\alpha}$, $\alpha = 1, 2$, $i = \overline{1,3}$. Hence $\overset{r_i^1 r_i^2}{b^*_{\mathbf{N}^1\mathbf{N}^2 i}}(x_3) = 0$ for $x_3 = h_3^+$, because $\overset{r_i^1 r_i^2}{b^{*(l)}_{\mathbf{N}^1\mathbf{N}^2 i}}(x_3) = 0$ for $x_3 = h_3^+$, $r_i^\alpha = \overline{0, N_i^\alpha}$, $\alpha = 1, 2$, $i = \overline{1,3}$. So, the components $v^*_{\mathbf{N}^1\mathbf{N}^2 i}$ of the vector function $\boldsymbol{v}^*_{\mathbf{N}^1\mathbf{N}^2}$ are polynomials with respect to the variables x_1 and x_2 with coefficients from the space $H^1(I^*)$ which equal to zero for $x_3 = h_3^+$.

Since $v_{\mathbf{N}^1\mathbf{N}^2 i}$ equals to $v^*_{\mathbf{N}^1\mathbf{N}^2 i}$ on Ω^* and for arbitrary $\varepsilon > 0$ there exists subdomain Ω^* such that the measure of the set $\Omega \backslash \Omega^*$ is less than ε, we have that $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2} \in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ and the corresponding vector function $\vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$ is a limit of the sequence $\{\vec{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2}\}_{l=1}^\infty$,

$$\|\vec{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2} - \vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_* = \|\boldsymbol{v}^{(l)}_{\mathbf{N}^1\mathbf{N}^2} - \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{\mathbf{H}^1(\Omega)} \to 0 \quad \text{as } l \to \infty.$$

Thus, $\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$ is a Hilbert space with respect to the scalar product defined by the norm $\|.\|_*$.

The conditions of the theorem imply that the bilinear form $A(.,.)$ is $V(\Omega)$-elliptic [17], i.e. $A(\boldsymbol{v}, \boldsymbol{v}) \geq c_A \|\boldsymbol{v}\|^2_{V(\Omega)}$, for all $\boldsymbol{v} \in V(\Omega)$. Since $V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ is a subspace of $V(\Omega)$ the form $A(.,.)$ is symmetric and $V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$-elliptic and, hence, the bilinear form $A_{\mathbf{N}^1\mathbf{N}^2}(.,.)$ is symmetric and $\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$-elliptic,

$$A_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = A(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}, \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}) \geq c_A \|\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}\|^2_{\mathbf{H}^1(\Omega)} = c_A \|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|^2_*,$$

for all $\vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2} = (v_{\mathbf{N}^1\mathbf{N}^2 i})$, $v_{\mathbf{N}^1\mathbf{N}^2 i} = \Phi_{\mathbf{N}^1\mathbf{N}^2 i}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})$, $i = \overline{1,3}$. From $\boldsymbol{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_1)$ it follows that the linear form $L(.)$ is continuous and hence the form $L_{\mathbf{N}^1\mathbf{N}^2}(.)$ is continuous too,

$$L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = L(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}) \leq c_L \|\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{\mathbf{H}^1(\Omega)} = c_L \|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*.$$

So, the assumptions of the Lax-Milgram lemma in its symmetric version are satisfied and, consequently, the problem (4) has one and only one solution, which may be equivalently characterized as the solution of the minimization problem stated in the theorem. $\square$

One of the ways to justify the use of the constructed one-dimensional models consists in estimating the difference between the exact solution of the original three-dimensional problem and the vector function $\boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2} \in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ which corresponds to the solution $\vec{w}_{\mathbf{N}^1\mathbf{N}^2}$ of the reduced problem (4). The next theorem gives the results on the relation of the obtained hierarchy and the three-dimensional problem, but before we formulate it let us introduce the following anisotropic weighted Sobolev spaces when h_1^+ and h_1^- are independent of the variable x_2,

$$\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)=\{\boldsymbol{v}\in\mathbf{H}^1(\Omega);\ \partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}\in\mathbf{H}^2(\Omega),\ (h^{\pm}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}\in\mathbf{L}^2(\Omega),\\ 0\le\widetilde{k}\le s-2,\ (h^{\pm}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}\in\mathbf{L}^2(\Omega),\ 1\le k\le s,\ \alpha=1,2\},$$

where $s\ge 2$, $s\in\mathbb{N}$, $(h^{\pm}_{\alpha})'$ denote derivatives of the functions $h^{\pm}_{\alpha}$, $\alpha=1,2$, and this space is equipped with the norm $\|.\|_{\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)}$ defined as follows

$$\|\boldsymbol{v}\|^2_{\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)}=\sum_{\alpha=1}^{2}\left(\sum_{\widetilde{k}=0}^{s-2}\left(\|\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)}+\|(h^{+}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)}+\right.\right.\\ \left.\left.+\|(h^{-}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)}\right)+\sum_{k=1}^{s}\left(\|(h^{+}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)}+\|(h^{-}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)}\right)\right).$$

Note that $\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)$ is a Hilbert space. Indeed, if $\{\boldsymbol{v}_n\}_{n\ge1}$ is a Cauchy sequence in $\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)$, then $\{\boldsymbol{v}_n\}_{n\ge1}$ is a Cauchy sequence in the space $\mathbf{H}^2(\Omega)$ and, consequently, $\boldsymbol{v}_n\to\boldsymbol{v}$ in $\mathbf{H}^2(\Omega)$ as $n\to\infty$. Therefore $\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}_n\to\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}$ in $\mathbf{H}^2(\Omega)$ as $n\to\infty$, $\widetilde{k}=\overline{0,s-2}$, $\alpha=1,2$. Since $h^{\pm}_1$, $h^{\pm}_2\in C^1(I)$ we have that $h^{\pm}_1,h^{\pm}_2\in C^1(\overline{I_1})$, where I_1 is any subinterval of I, $\overline{I_1}\subset I$, and hence

$$\begin{array}{l}(h^{\pm}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}_n\to(h^{\pm}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v},\\ (h^{\pm}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}_n\to(h^{\pm}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v},\end{array}\quad \text{in }\ \mathbf{L}^2(\Omega_1)\ \text{ as } n\to\infty,\alpha=1,2,\tag{6}$$

where $k=\overline{1,s}$, $\widetilde{k}=\overline{0,s-2}$, Ω_1 is any subdomain of Ω, $\overline{\Omega_1}\subset\Omega$. The convergence of the sequences $\{(h^{\pm}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}_n\}_{n\ge1}$ and $\{(h^{\pm}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}_n\}_{n\ge1}$ in $\mathbf{L}^2(\Omega)$ and (6) together imply that $(h^{\pm}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}_n\to(h^{\pm}_{\alpha})'\partial^{k}_{\alpha}\boldsymbol{v}$ and $(h^{\pm}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}_n\to(h^{\pm}_{\alpha})'\partial_1\partial_2\partial^{\widetilde{k}}_{\alpha}\boldsymbol{v}$ in $\mathbf{L}^2(\Omega)$ as $n\to\infty$, $\alpha=1,2$, and so the space $\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)$ is complete.

Theorem 2. *Assume that the components of the elasticity tensor $a_{ijpq}\in L^{\infty}(\Omega)$ and the conditions (2) are satisfied (i, j, p, $q=\overline{1,3}$). If $\boldsymbol{f}\in\widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{g}\in\mathbf{H}^{-1/2}(\Gamma_1)$, then the vector function $\boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2}=(w_{\mathbf{N}^1\mathbf{N}^2 i})$, $w_{\mathbf{N}^1\mathbf{N}^2 i}=\Phi_{\mathbf{N}^1\mathbf{N}^2 i}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2})$, $i=\overline{1,3}$, which corresponds to the solution $\vec{w}_{\mathbf{N}^1\mathbf{N}^2}\in\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$ of the one-dimensional problem (4) tends to the solution $\boldsymbol{u}$ of the three-dimensional problem (1) in the space $\mathbf{H}^1(\Omega)$ as $\min\{N^1,N^2\}\to\infty$, $N^{\alpha}=\min\limits_{1\le i\le 3}\{N^{\alpha}_i\}$, $\alpha=1,2$. If, in addition, $h^{\pm}_1\in C^0(\overline{I})\cap C^1(I)$ and $\boldsymbol{u}\in\mathbf{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)$, $s\in\mathbb{N}$, $s\ge 2$, then*

$$\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2}\|^2_{\mathbf{H}^1(\Omega)}\le\left(\frac{1}{(N^1)^{2\widehat{s}}}+\frac{1}{(N^2)^{2\widehat{s}}}\right)o(\Omega,\Gamma_0,h^{\pm}_1,h^{\pm}_2,\mathbf{N}^1,\mathbf{N}^2),\tag{7}$$

where $\widehat{s} = \min\{(s-1)/2, 1\}(s-1)$ *and* $o(\Omega, \Gamma_0, h_1^{\pm}, h_2^{\pm}, \mathbf{N}^1, \mathbf{N}^2) \to 0$ *as* $\min\{N^1, N^2\} \to \infty$.

Proof. The conditions of the theorem ensure the existence and uniqueness of solutions of the problems (1) and (4), which also minimize the corresponding energy functionals $J(.)$ and $J_{\mathbf{N}^1\mathbf{N}^2}(.)$ on the spaces $V(\Omega)$ and $\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, respectively. Therefore

$$\frac{1}{2}A_{\mathbf{N}^1\mathbf{N}^2}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2}, \vec{w}_{\mathbf{N}^1\mathbf{N}^2}) - L_{\mathbf{N}^1\mathbf{N}^2}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2}) \leq$$
$$\leq \frac{1}{2}A_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) - L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}), \ \ \forall \vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I). \qquad (8)$$

By the definition of the forms $A_{\mathbf{N}^1\mathbf{N}^2}(.,.)$ and $L_{\mathbf{N}^1\mathbf{N}^2}(.)$ we have that $A_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = A(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}, \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})$ and $L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = L(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})$, for all $\vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$,where $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2} = (v_{\mathbf{N}^1\mathbf{N}^2 i})$, $v_{\mathbf{N}^1\mathbf{N}^2 i} = \Phi_{\mathbf{N}^1\mathbf{N}^2 i}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})$, $i = \overline{1,3}$. Since $A(\boldsymbol{u}, \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}) = L(\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2})$, for all $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2} \in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, from (8) we obtain that for all $\boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2} \in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$,

$$A(\boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2}, \boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2}) \leq A(\boldsymbol{u} - \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}, \boldsymbol{u} - \boldsymbol{v}_{\mathbf{N}^1\mathbf{N}^2}). \qquad (9)$$

Applying the latter inequality we prove the convergence result of the theorem. By trace theorems for Sobolev spaces [16] for any $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{tr}_{\Gamma}(\boldsymbol{v}) = \mathbf{0}$ on Γ_0 there exists a continuation $\boldsymbol{v}^1 \in \mathbf{H}^1_0(\Omega_1)$ of $\boldsymbol{v}$, where Ω_1 is a Lipschitz domain such that $\Omega_1 \supset \Omega$, $\partial\Omega_1 \cap \partial\Omega = \Gamma_0$. Let us consider the subdomain Ω^{**} of Ω_1, $\Omega \subset \Omega^{**} \subset \Omega_1$, of the following form

$$\Omega^{**} = \{x \in \mathbb{R}^3;\ h_1^{**,-}(x_2, x_3) < x_1 < h_1^{**,+}(x_2, x_3),$$
$$h_2^{**,-}(x_3) < x_2 < h_2^{**,+}(x_3),\ x_3 \in I\},$$

where $h_1^{**,\pm} \in C^{\infty}(\overline{\omega^{**}})$, $\omega^{**} = \{(x_2, x_3) \in \mathbb{R}^2;\ h_2^{**,-}(x_3) < x_2 < h_2^{**,+}(x_3)$, $x_3 \in I\}$, $h_2^{**,\pm} \in C^{\infty}(\overline{I})$, $h_1^{**,-} \leq h_1^- \leq h_1^+ \leq h_1^{**,+}$ on $\overline{\omega}$, $h_2^{**,-} \leq h_2^- \leq h_2^+ \leq h_2^{**,+}$ on $\overline{I}$, $h_1^{**,-} < h_1^{**,+}$ on $\overline{\omega^{**}}$, $h_2^{**,-} < h_2^{**,+}$ on $\overline{I}$. Since the set $[\mathfrak{D}(\Omega_1)]^3$ is dense in $\mathbf{H}^1_0(\Omega_1)$ we have, that the set of infinitely differentiable vector functions $[C^{\infty}_{\Gamma_0^{**}}(\Omega^{**})]^3$ which vanish on $\Gamma_0^{**} = \{x \in \Gamma^{**} = \partial\Omega^{**}; x_3 = h_3^+\} \supset \Gamma_0$ is dense in the space of vector functions $\boldsymbol{v}^{**} \in \mathbf{H}^1(\Omega^{**})$, $\boldsymbol{tr}_{\Gamma^{**}}(\boldsymbol{v}^{**}) = \mathbf{0}$ on Γ_0^{**}. For any vector function $\boldsymbol{v}^{**} \in [C^{\infty}_{\Gamma_0^{**}}(\Omega^{**})]^3$ the sequence of vector functions $\boldsymbol{v}^{**}_{\mathbf{N}^1} = (v^{**}_{\mathbf{N}^1 i})$,

$$v^{**}_{\mathbf{N}^1 i} = \sum_{r_i^1=0}^{N_i^1} \frac{1}{h_1^{**}}\left(r_i^1 + \frac{1}{2}\right) \overset{r_i^1}{v}{}^{**}_{\mathbf{N}^1 i} P_{r_i^1}(z_1^{**}),$$

where $z_1^{**} = (x_1 - \overline{h}_1^{**})/h_1^{**}$,$\overline{h}_1^{**} = (h_1^{**,+} + h_1^{**,-})/2$,$h_1^{**} = (h_1^{**,+} - h_1^{**,-})/2$,

$\overset{r_i^1}{v}{}^{**}_{\mathbf{N}^1 i} = \int\limits_{h_1^{**,-}}^{h_1^{**,+}} v^{**}_{\mathbf{N}^1 i} P_{r_i^1}(z_1^{**}) dx_1$, $i = 1,2,3$, belongs to $[C^\infty_{\Gamma_0^{**}}(\Omega^{**})]^3$ and converges to $\boldsymbol{v}^{**}$ in the space $\mathbf{H}^1(\Omega^{**})$ as $\min\limits_{1 \le i \le 3}\{N_i^1\} \to \infty$. The functions $v^{**}_{\mathbf{N}^1 i}$ are polynomials with respect to the variable x_1 and the coefficient of $x_1^{r_i^1}$, which we denote by $\overset{r_i^1}{b}_{\mathbf{N}^1 i}$, belongs to $C^\infty(\overline{\omega^{**}})$, $\overset{r_i^1}{b}_{\mathbf{N}^1 i} = 0$ for $x_3 = h_3^+$, $r_i^1 = \overline{0, N_i^1}$, $i = \overline{1,3}$. As in the case of three-dimensional domain we obtain that for each function $\overset{r_i^1}{b}_{\mathbf{N}^1 i}$ the sequence of functions

$$\overset{r_i^1}{b}_{\mathbf{N}^1 \mathbf{N}^2 i} = \sum_{r_i^2=0}^{N_i^2} \frac{1}{h_2^*}\left(r_i^2 + \frac{1}{2}\right) \overset{r_i^1 r_i^2}{b}_{\mathbf{N}^1 \mathbf{N}^2 i} \, P_{r_i^2}(z_2^{**}),$$

where $z_2^{**} = (x_2 - \overline{h}_2^{**})/h_2^{**}$, $\overline{h}_2^{**} = (h_2^{**,+} + h_2^{**,-})/2$, $h_2^{**} = (h_2^{**,+} - h_2^{**,-})/2$, $\overset{r_i^1 r_i^2}{b}_{\mathbf{N}^1 \mathbf{N}^2 i} = \int\limits_{h_2^{**,-}}^{h_2^{**,+}} \overset{r_i^1}{b}_{\mathbf{N}^1 \mathbf{N}^2 i} \, P_{r_i^2}(z_2^{**}) dx_2$, is a subset of $C^\infty(\overline{\omega^{**}})$, $\overset{r_i^1 r_i^2}{b}_{\mathbf{N}^1 \mathbf{N}^2 i} = 0$ for $x_3 = h_3^+$, and converges to $\overset{r_i^1}{b}_{\mathbf{N}^1 i}$ in the space $H^1(\omega^{**})$ as $N_i^2 \to \infty$, $r_i^1 = \overline{0, N_i^1}$, $i = \overline{1,3}$. Since $\overset{r_i^1}{b}_{\mathbf{N}^1 \mathbf{N}^2 i}$ is a polynomial with respect to the variable x_2, there exists a sequence of polynomials with respect to the variables x_1 and x_2 with coefficients from $H^1(I)$, vanishing for $x_3 = h_3^+$, which converges to vector function $\boldsymbol{v}^{**} \in [C^\infty_{\Gamma_0^{**}}(\Omega^{**})]^3$ in the space $\mathbf{H}^1(\Omega^{**})$ as $\min\limits_{1 \le i \le 3}\{N_i^1, N_i^2\} \to \infty$. The restrictions of these polynomials on Ω are elements of the subspaces $V_{\mathbf{N}^1 \mathbf{N}^2}(\Omega)$ and, consequently, the union $\bigcup\limits_{\mathbf{N}^1, \mathbf{N}^2 \ge \mathbf{0}} V_{\mathbf{N}^1 \mathbf{N}^2}(\Omega)$ of the subspaces $V_{\mathbf{N}^1 \mathbf{N}^2}(\Omega)$ for all $N_i^1, N_i^2 \in \mathbb{N} \cup \{0\}$, $i = \overline{1,3}$, is dense in $V(\Omega)$. So it follows from $V(\Omega)$-ellipticity of the bilinear form $A(.,.)$ and the inequality (9) that the sequence of vector functions $\{\boldsymbol{w}_{\mathbf{N}^1 \mathbf{N}^2}\}$ converges to the solution $\boldsymbol{u}$ of the original problem in the space $\mathbf{H}^1(\Omega)$ as $\min\limits_{1 \le i \le 3}\{N_i^1, N_i^2\} \to \infty$.

Let us now suppose that the functions $h_1^\pm$ are independent of x_2, $h_1^\pm \in C^0(\overline{I}) \cap C^1(I)$ and the solution $\boldsymbol{u}$ satisfies the following condition $\boldsymbol{u} \in \mathbf{H}^{s,s,2}_{h_{1,2}^\pm}(\Omega)$, $s \in \mathbb{N}$, $s \ge 2$. We assume that $s \ge 3$, because in the case of $s = 2$ the estimate of the theorem is proved in [15]. In order to obtain the estimate (7) we consider the Fourier-Legendre expansions of the components u_i of $\boldsymbol{u}$ with respect to the variables x_1, x_2, and then construct the modified partial sums $\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}$ of the Fourier-Legendre series which has the following form

$$\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}=\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\frac{1}{h_1h_2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right)\overset{r_i^1r_i^2}{u_i}\,P_{r_i^1}(z_1)P_{r_i^2}(z_2)+$$
$$+\sum_{\alpha=1}^{2}\sum_{r_i^\alpha=0}^{N_i^\alpha-2}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\frac{1}{2h_\alpha}\left(r_i^\alpha+\frac{1}{2}\right)\overset{r_i^1r_i^2}{\partial_{3-\alpha}u_i}\,P_{r_i^\alpha}(z_\alpha)P_{r_i^{3-\alpha}-1}(z_{3-\alpha})+$$
$$+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\frac{1}{4}\Big(\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_2u_i}+\overset{r_i^1-1,r_i^2+1}{\partial_1\partial_2u_i}-\overset{r_i^1+1,r_i^2+1}{\partial_1\partial_2u_i}\Big)P_{r_i^1}(z_1)P_{r_i^2}(z_2),$$

where $z_\alpha=(x_\alpha-\overline{h}_\alpha)/h_\alpha$, $\overline{h}_\alpha=(h_\alpha^++h_\alpha^-)/2$, $h_\alpha=(h_\alpha^+-h_\alpha^-)/2$, $\alpha=1,2$, and $\overset{r^1r^2}{v}=\int\limits_{h_1^-}^{h_1^+}\int\limits_{h_2^-}^{h_2^+}vP_{r^1}(z_1)P_{r^2}(z_2)dx_1dx_2$, for all $v\in L^2(\Omega)$, $r^1,r^2\in\mathbb{N}\cup\{0\}$.

Let us prove that the constructed vector function $\widetilde{\boldsymbol{u}}_{\mathbf{N}^1\mathbf{N}^2}=(\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i})$ belongs to the subspace $V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$. The Legendre polynomials and their derivatives satisfy the following recurrence relations

$$\begin{aligned}P_r(t)&=\frac{1}{2r+1}(P'_{r+1}(t)-P'_{r-1}(t)), && r\ge 1,\\ tP'_r(t)&=P'_{r+1}(t)-(r+1)P_r(t), && r\ge 0,\end{aligned}\tag{10}$$

from which it follows that for all $r^1,r^2\in\mathbb{N}$ and $i=1,2,3$,

$$\overset{r^1r^2}{u_i}=\frac{h_1}{2r^1+1}\Big(\overset{r^1-1,r^2}{\partial_1u_i}-\overset{r^1+1,r^2}{\partial_1u_i}\Big)=\frac{h_2}{2r^2+1}\Big(\overset{r^1,r^2-1}{\partial_2u_i}-\overset{r^1,r^2+1}{\partial_2u_i}\Big),\tag{11}$$

$$h_1h_2\partial_3\left(\frac{1}{h_1h_2}\overset{r^1r^2}{u_i}\right)=\overset{r^1r^2}{\partial_3u_i}+\overline{h}_1{}'\,\overset{r^1r^2}{\partial_1u_i}+h_1{}'\left(\frac{1}{h_1}r^1\,\overset{r^1r^2}{u_i}+\overset{r^1+1,r^2}{\partial_1u_i}\right)+$$
$$+\overline{h}_2{}'\,\overset{r^1r^2}{\partial_2u_i}+h_2{}'\left(\frac{1}{h_2}r^2\,\overset{r^1r^2}{u_i}+\overset{r^1,r^2+1}{\partial_2u_i}\right).$$

Applying the latter formulas we obtain expressions for derivatives of the functions $\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}$ $(i=\overline{1,3})$

$$\frac{\partial\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}}{\partial x_\alpha}=\sum_{r_i^\alpha=0}^{N_i^\alpha-1}\sum_{r_i^{3-\alpha}=0}^{N_i^{3-\alpha}}\frac{1}{h_1h_2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right)\overset{r_i^1r_i^2}{\partial_\alpha u_i}\,P_{r_i^1}(z_1)P_{r_i^2}(z_2)+$$
$$+\sum_{r_i^\alpha=0}^{N_i^\alpha-1}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\frac{1}{2h_\alpha}\left(r_i^\alpha+\frac{1}{2}\right)\overset{r_i^1r_i^2}{\partial_1\partial_2u_i}\,P_{r_i^\alpha}(z_\alpha)P_{r_i^{3-\alpha}-1}(z_{3-\alpha}),\alpha=1,2,$$

$$\frac{\partial \widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}}{\partial x_3} = \sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2} \frac{1}{h_1 h_2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right) \overset{r_i^1 r_i^2}{\partial_3 u_i}\, P_{r_i^1}(z_1)P_{r_i^2}(z_2)+$$
$$+\sum_{\alpha=1}^{2}\sum_{r_i^{3-\alpha}=0}^{N_i^{3-\alpha}}\sum_{r_i^\alpha=N_i^\alpha}^{N_i^\alpha} \frac{\overline{h}_\alpha{}'}{h_1 h_2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right) \overset{r_i^1 r_i^2}{\partial_\alpha u_i}\, P_{r_i^\alpha}(z_\alpha)P_{r_i^{3-\alpha}}(z_{3-\alpha})+$$
$$+\sum_{\alpha=1}^{2}\sum_{r_i^{3-\alpha}=0}^{N_i^{3-\alpha}-2}\sum_{r_i^\alpha=N_i^\alpha}^{N_i^\alpha+1} \frac{1}{2h_{3-\alpha}}\left(r_i^{3-\alpha}+\frac{1}{2}\right)\left(\overset{r_i^1 r_i^2}{\partial_3\partial_\alpha u_i} + \overline{h}_\alpha{}'\, \overset{r_i^1 r_i^2}{\partial_\alpha\partial_\alpha u_i}\right)P_{r_i^\alpha-1}(z_\alpha)\times$$
$$\times P_{r_i^{3-\alpha}}(z_{3-\alpha}) + \sum_{\alpha=1}^{2}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\sum_{r_i^\alpha=1}^{1} \frac{h_\alpha{}'}{4h_\alpha}\, \overset{r_i^1 r_i^2}{\partial_1\partial_2 u_i}\, P_{r_i^{3-\alpha}-1}(z_{3-\alpha})+$$
$$+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\frac{1}{4}\left(\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_2\partial_3 u_i} + \overline{h}_1{}'\, \overset{r_i^1+1,r_i^2-1}{\partial_1\partial_1\partial_2 u_i} - h_1{}'\left(\frac{1}{h_1}\overset{r_i^1-1,r_i^2+1}{\partial_1\partial_2 u_i} -\right.\right.$$
$$\left.\left. - \overset{r_i^1,r_i^2+1}{\partial_1\partial_1\partial_2 u_i}\right) + \overline{h}_2{}'\, \overset{r_i^1-1,r_i^2+1}{\partial_1\partial_2\partial_2 u_i} - h_2{}'\left(\frac{1}{h_2}\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_2 u_i} - \overset{r_i^1+1,r_i^2}{\partial_1\partial_2\partial_2 u_i}\right)\right)P_{r_i^1}(z_1)\times$$
$$\times P_{r_i^2}(z_2) + \sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\frac{2r_i^1+1}{4h_1}\, \overset{r_i^1,r_i^2+1}{\partial_2\partial_3 u_i}\, P_{r_i^1}(z_1)P_{r_i^2}(z_2)+$$
$$+\sum_{\alpha=1}^{2}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\sum_{r_i^\alpha=N_i^\alpha}^{N_i^\alpha}\frac{\overline{h}_\alpha{}'(2r_i^\alpha+1)}{4h_\alpha}\, \overset{r_i^1 r_i^2}{\partial_1\partial_2 u_i}\, P_{r_i^\alpha}(z_\alpha)P_{r_i^{3-\alpha}-1}(z_{3-\alpha})+$$
$$+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\frac{h_1{}'(2r_i^2+1)}{2h_2}\left(\frac{r_i^1+1}{h_1}\overset{r_i^1+1,r_i^2}{\partial_1 u_i} + \frac{1}{2}\overset{r_i^1+2,r_i^2}{\partial_1\partial_1 u_i}\right)P_{r_i^1}(z_1)P_{r_i^2}(z_2)+$$
$$+\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\frac{h_2{}'(2r_i^1+1)}{2h_1}\left(\frac{r_i^2+1}{h_2}\overset{r_i^1,r_i^2+1}{\partial_2 u_i} + \frac{1}{2}\overset{r_i^1,r_i^2+2}{\partial_2\partial_2 u_i}\right)P_{r_i^1}(z_1)P_{r_i^2}(z_2).$$

Hence, from the conditions $u_i, \partial_j u_i, \partial_j\partial_\beta u_i, \partial_1\partial_2\partial_3 u_i, \partial_\alpha\partial_\alpha\partial_\beta u_i \in L^2(\Omega)$ and $h_\alpha^{\pm}\partial_\alpha u_i$, $(h_\alpha^{\pm})'\partial_\alpha\partial_\beta u_i$, $(h_\alpha^{\pm})'\partial_1\partial_2\partial_\alpha u_i \in L^2(\Omega)$, $\alpha,\beta=1,2$, $i,j=\overline{1,3}$, it follows that $\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i} \in H^1(\Omega)$, $i=\overline{1,3}$. Since $\boldsymbol{u}\in V(\Omega)$ we have that $\boldsymbol{tr}_\Gamma(\boldsymbol{u}) = \mathbf{0}$ on Γ_0. So, $\boldsymbol{tr}_\Gamma(\widetilde{\boldsymbol{u}}_{\mathbf{N}^1\mathbf{N}^2}) = \mathbf{0}$ on Γ_0 and $\widetilde{\boldsymbol{u}}_{\mathbf{N}^1\mathbf{N}^2} \in V_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$.

Applying the orthogonality of the Legendre polynomials, the expressions for $\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}$, $\partial_j(\widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i})$ $(i,j=\overline{1,3})$ and the Parseval relation for the remainder term $\boldsymbol{\varepsilon}_{\mathbf{N}^1\mathbf{N}^2} = (\varepsilon_{\mathbf{N}^1\mathbf{N}^2 i})$, $\varepsilon_{\mathbf{N}^1\mathbf{N}^2 i} = u_i - \widetilde{u}_{\mathbf{N}^1\mathbf{N}^2 i}$, $i=1,2,3$, we obtain

$$\|\varepsilon_{\mathbf{N}^1\mathbf{N}^2 i}\|^2_{L^2(\Omega)} = \sum_{r_i^1=N_i^1+1}^{\infty} \vee \sum_{r_i^2=N_i^2+1}^{\infty} \int_I \prod_{\alpha=1}^{2}\frac{1}{h_\alpha}\left(r_i^\alpha+\frac{1}{2}\right)(\overset{r_i^1 r_i^2}{u_i})^2 dx_3+$$

$$+\sum_{\alpha=1}^{2}\sum_{r_i^\alpha=0}^{N_i^\alpha-2}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\int_I \frac{h_{3-\alpha}(2r_i^\alpha+1)}{4h_\alpha(2r_i^{3-\alpha}-1)}(\overset{r_i^1 r_i^2}{\partial_{3-\alpha}u_i})^2dx_3+$$

$$+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\frac{1}{4}\int_I\prod_{\alpha=1}^{2}\frac{h_\alpha}{2r_i^\alpha+1}(\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_2u_i}+\overset{r_i^1-1,r_i^2+1}{\partial_1\partial_2u_i}-\overset{r_i^1+1,r_i^2+1}{\partial_1\partial_2u_i})^2dx_3,$$

$$\left\|\frac{\partial\varepsilon_{\mathbf{N}^1\mathbf{N}^2i}}{\partial x_\alpha}\right\|^2_{L^2(\Omega)}=\sum_{r_i^\alpha=N_i^\alpha}^{\infty}\vee\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}+1}^{\infty}\int_I\prod_{\beta=1}^{2}\frac{1}{h_\beta}\left(r_i^\beta+\frac{1}{2}\right)(\overset{r_i^1 r_i^2}{\partial_\alpha u_i})^2dx_3+$$

$$+\sum_{r_i^\alpha=0}^{N_i^\alpha-1}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\int_I\frac{h_{3-\alpha}(2r_i^\alpha+1)}{4h_\alpha(2r_i^{3-\alpha}-1)}(\overset{r_i^1 r_i^2}{\partial_1\partial_2u_i})^2dx_3,\qquad\alpha=1,2,$$

$$\left\|\frac{\partial\varepsilon_{\mathbf{N}^1\mathbf{N}^2i}}{\partial x_3}\right\|^2_{L^2(\Omega)}\leq 9\Bigg(\sum_{r_i^1=N_i^1+1}^{\infty}\vee\sum_{r_i^2=N_i^2+1}^{\infty}\int_I\prod_{\beta=1}^{2}\frac{1}{h_\beta}\left(r_i^\beta+\frac{1}{2}\right)(\overset{r_i^1 r_i^2}{\partial_3u_i})^2dx_3+$$

$$+\sum_{\alpha=1}^{2}\sum_{r_i^\alpha=N_i^\alpha}^{N_i^\alpha}\sum_{r_i^{3-\alpha}=0}^{N_i^{3-\alpha}}\int_I\frac{(\overline{h}_\alpha{}')^2}{h_1h_2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right)(\overset{r_i^1 r_i^2}{\partial_\alpha u_i})^2dx_3+$$

$$+\sum_{\alpha=1}^{2}\sum_{r_i^\alpha=N_i^\alpha}^{N_i^\alpha+1}\sum_{r_i^{3-\alpha}=0}^{N_i^{3-\alpha}-2}\int_I\frac{h_\alpha(2r_i^{3-\alpha}+1)}{4h_{3-\alpha}(2r_i^\alpha-1)}(\overset{r_i^1 r_i^2}{\partial_3\partial_\alpha u_i}+\overline{h}_\alpha{}'\,\overset{r_i^1 r_i^2}{\partial_\alpha\partial_\alpha u_i})^2dx_3+$$

$$+\sum_{\alpha=1}^{2}\sum_{r_i^\alpha=1}^{1}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\int_I\frac{(h_\alpha{}')^2h_{3-\alpha}}{4h_\alpha(2r_i^{3-\alpha}-1)}(\overset{r_i^1 r_i^2}{\partial_1\partial_2u_i})^2dx_3+$$

$$+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\int_I\frac{h_1h_2}{4(2r_i^1+1)(2r_i^2+1)}(\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_2\partial_3u_i}+\overline{h}_1{}'\,\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_1\partial_2u_i}-$$

$$-h_1{}'\left(\frac{1}{h_1}\overset{r_i^1-1,r_i^2+1}{\partial_1\partial_2u_i}-\overset{r_i^1,r_i^2+1}{\partial_1\partial_1\partial_2u_i}\right)+\overline{h}_2{}'\,\overset{r_i^1-1,r_i^2+1}{\partial_1\partial_2\partial_2u_i}-h_2{}'\Bigg(\frac{1}{h_2}\overset{r_i^1+1,r_i^2-1}{\partial_1\partial_2u_i}-$$

$$-\overset{r_i^1+1,r_i^2}{\partial_1\partial_2\partial_2u_i}\Bigg)\Bigg)^2dx_3+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\int_I\frac{h_2(2r_i^1+1)}{4h_1(2r_i^2+1)}(\overset{r_i^1,r_i^2+1}{\partial_2\partial_3u_i})^2dx_3+$$

$$+\sum_{\alpha=1}^{2}\sum_{r_i^\alpha=N_i^\alpha}^{N_i^\alpha}\sum_{r_i^{3-\alpha}=N_i^{3-\alpha}}^{N_i^{3-\alpha}+1}\int_I\frac{(\overline{h}_\alpha{}')^2h_{3-\alpha}}{2h_\alpha(2r_i^{3-\alpha}-1)}\left(r_i^\alpha+\frac{1}{2}\right)(\overset{r_i^1 r_i^2}{\partial_1\partial_2u_i})^2dx_3+$$

$$+\sum_{r_i^1=N_i^1-1}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\int_I\frac{(h_1{}')^2h_1(2r_i^2+1)}{h_2(2r_i^1+1)}\left(\frac{r_i^1+1}{h_1}\overset{r_i^1+1,r_i^2}{\partial_1u_i}+\frac{1}{2}\overset{r_i^1+2,r_i^2}{\partial_1\partial_1u_i}\right)^2dx_3+$$

$$+\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=N_i^2-1}^{N_i^2}\int_I \frac{(h_2')^2h_2(2r_i^1+1)}{h_1(2r_i^2+1)}\left(\frac{r_i^2+1}{h_2}\overset{r_i^1,r_i^2+1}{\partial_2 u_i}+\frac{1}{2}\overset{r_i^1,r_i^2+2}{\partial_2\partial_2 u_i}\right)^2 dx_3\Bigg),$$

where $\sum\limits_{k^1=\widehat{N}^1}^{\infty}\vee\sum\limits_{k^2=\widehat{N}^2}^{\infty}$ denotes the sum with respect to the variables k^1 and k^2 for all pairs $(k^1,k^2)\in\mathbb{N}\times\mathbb{N}$, $k^1\geq\widehat{N}^1$ or $k^2\geq\widehat{N}^2$. Application of the formula (11) to vector function $\boldsymbol{v}=(v_i)\in\mathbf{H}^{n,n,2}_{h^{\pm}_{1,2}}(\Omega)$, $n\in\mathbb{N}$, $n\geq 2$, gives us the following estimates

$$\begin{aligned}
&(\overset{k^1k^2}{v_i})^2\leq\frac{c_1h_1^{2n}}{(k^1)^{2n}}\sum_{\widetilde{k^1}=k^1-n}^{k^1+n}(\overset{\widetilde{k^1}k^2}{\partial_1^n v_i})^2,\\
&(\overset{k^1k^2}{v_i})^2\leq\frac{c_2h_2^{2n}}{(k^2)^{2n}}\sum_{\widetilde{k^2}=k^2-n}^{k^2+n}(\overset{k^1\widetilde{k^2}}{\partial_2^n v_i})^2,
\end{aligned}\qquad \min\{k^1,k^2\}\geq n,\ k^1,k^2\in\mathbb{N}, \tag{12}$$

where c_1, c_2 are positive constants independent of $h_1^{\pm}$, $h_2^{\pm}$ and k^1, k^2. Using (12) for $\boldsymbol{\varepsilon}_{\mathbf{N}^1\mathbf{N}^2}$ we obtain

$$\|\varepsilon_{\mathbf{N}^1\mathbf{N}^2 i}\|^2_{L^2(\Omega)}\leq\left(\frac{1}{(N_i^1)^{2(s-1)}}+\frac{1}{(N_i^2)^{2(s-1)}}\right)o(h_1^{\pm},h_2^{\pm},N_i^1,N_i^2),$$
$$\|\partial_j\varepsilon_{\mathbf{N}^1\mathbf{N}^2 i}\|^2_{L^2(\Omega)}\leq\left(\frac{1}{(N_i^1)^{2(s-1)}}+\frac{1}{(N_i^2)^{2(s-1)}}\right)o(h_1^{\pm},h_2^{\pm},N_i^1,N_i^2),$$

where $o(h_1^{\pm},h_2^{\pm},N_i^1,N_i^2)\to 0$ as $\min\{N_i^1,N_i^2\}\to\infty$, $i,j=1,2,3$. Therefore, the inequality (9) and $V(\Omega)$-ellipticity of the bilinear form $A(.,.)$ together imply

$$\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}^1\mathbf{N}^2}\|^2_{\mathbf{H}^1(\Omega)}\leq\left(\frac{1}{(N^1)^{2(s-1)}}+\frac{1}{(N^2)^{2(s-1)}}\right)o(\Omega,\Gamma_0,h_1^{\pm},h_2^{\pm},\mathbf{N}^1,\mathbf{N}^2),$$

where $N^\alpha=\min\limits_{1\leq i\leq 3}\{N_i^\alpha\}$, $\alpha=1,2$ and $o(\Omega,\Gamma_0,h_1^{\pm},h_2^{\pm},\mathbf{N}^1,\mathbf{N}^2)\to 0$ as $\min\limits_{1\leq\alpha\leq 2}\{N^\alpha\}\to\infty$. □

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