A POSTERIORI ERROR ESTIMATION FOR hp-APPROXIMATION OF THE 3D-BASED FIRST ORDER SHELL MODEL PART 1. THEORETICAL ASPECTS

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Abstract

The paper presents theoretical aspects dealing with a posteriori error estimation for the finite element hp-approximation applied to the 3D-based first order shell model. The assignment of the presented error estimation is its application to hierarchical modelling and adaptive analysis of shell parts of complex structures consisting of thin-walled, thick-walled and solid parts. The main feature of the 3D-based formulation is that it is equipped with 3D degrees of freedom, while its mechanical model corresponds to the classical first order shell theory. The hp-approximation applied to the elaborated 3D-based model allows local p- and h-adaptivities, where h is the element size parameter while p is the transverse approximation order of the element.

The organization of the paper is the following. First we present the model shell problem through definition of the three-dimensional geometry of shell and introduction of the three-dimensional local and variational formulations of the Reissner-Mindlin theory. Next we apply the hp-approximation to the problem. The main body of the paper is devoted to a posteriori approximation error estimation for the elaborated model. Within this subject we present local and global characteristics of the error. The global characteristic is based on the difference of the potential energies corresponding to the exact and approximated solutions. It has been proved in the paper that the proposed error estimator is equivalent to the estimator based on the strain energy defined on the local error. We have also shown the upper bound property of the proposed global error estimate. Moreover, we have introduced the element approximation error indicators, the sum of which gives us the mentioned global estimate. The values of the local indicators can be obtained through solution of the element local problems. The practical method of obtaining these indicators is based on the finite element discretization of the local problems. The paper is completed with the conclusions.

The implementation details corresponding to the elaborated method of a posteriori approximation error estimation of the 3D-based first order shell model will

be presented in the forthcoming paper [1].

Key words and phrases: A posteriori error estimation, Shell model, Finite element method, Hierarchical modelling.

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1. Introduction

The main objective of the paper is to present problems concerning a posteriori error estimation for the first order shell theory based on three-dimensional approach. The proposed approach is characterized by application of the classical mechanical model corresponding to Reissner-Mindlin first order shell theory, while the displacement field is that corresponding to three-dimensional theory of elasticity. In our approach we use three displacements at any point of the three-dimensional body of the shell as a primary unknowns instead of three displacements and two rotations of the mid-shell surface. In order to retain equivalence of the classical and our approach we introduce the proper constraints reflecting deformation of the straight lines perpendicular to the mid-shell surface into straight lines without elongation. This way we form a constrained six-parameter model equivalent to the classical five parameter model.

The main motivation of introduction of three-dimensional approach proposed in [2] for description of the first order shell theory is to obtain the formulation which is compatible with 3D-based hierarchical shell models and three-dimensional theory of elasticity as well. Thanks to that all the mentioned theories of the first order shell, hierarchical shell and three-dimensional elasticity can be applied together for hierarchical modelling and adaptive finite element analysis of complex structures consisting of thin- and thick-walled parts, solid parts and transition zones between them. With the three-dimensional description one can use the consistent approach (based on the same type of dofs and the same shape functions) to the finite element approximation of all mechanical models applied within a complex structure. Also, the consistent error estimation and adaptive procedures based on three-dimensional degrees of freedom can be applied to the each part of the structure, without any necessity to introduce different approaches for different mechanical models. Such an advantage is of great importance from the implementation point of view, as it very much simplifies the finite element formulation assigned for adaptive modelling and solution of the problems in which are utilized more than one mechanical models.

In this paper we concentrate on some theoretical and implementation aspects of a method of a posteriori approximation error estimation, based on the equilibrated residual approach. The theoretical basis for this approach can be found in the initiating work of Ainsworth and Oden [3], which constitutes a generalization of the previous works of Kelly [4] as well as of Bank and Weisser [5]. Application of the equilibrated residual approach to finite element methods can be found again in the works of Ainworth and Oden [6, 7]. In that context they considered elliptic problems of the second order in [8, 9] and together with Wu [10] the specific case of linear elasticity which is close to our interest. In the method proposed by them they start with introduction of the ad hoc error functional leading to the error estimator defined as the strain energy of the difference between the exact and approximated solutions.

In this paper we will pay our attention to the main difference between our and their approaches. Note that in our approach we will start with the functional based on the difference of the potential energies corresponding to the exact and approximated solutions. The advantage of our approach is that it has clearer physical interpretation, easily understood by practitioners. In our method the error estimate can be equivalently expressed as a difference of the strain energies of both solutions. We will prove that both error estimates are equivalent. A consequence of our approach is that we will search for the estimate of the exact solution rather than for the estimated value of the error. The error estimate will be calculated next as a difference of the solutions.

As far as the implementation aspects of the finite element error estimation are concerned, we will focus our attention on the aspects resulting from the constrained character of our 3D-based formulation of the first order shell model. Our formulation will lead to the modified definition of the splitting functions of the equilibrated stress fluxes utilized in solution of the element local problems, which leads us to the searched estimate. Other implementation details will be shown in [1].

2. The model problem

2.1. Three-dimensional geometry of a shell

Classical description. Let us define a three-dimensional shell geometry through introduction of the mid-shell surface $\overset{s}{S} \subset R^2$, called sometimes the reference surface and viewed as an open bounded region of a piecewise smooth boundary $\partial\overset{s}{S}$, and through introduction of the notion of the shell

thickness t. The sufficiently smooth open bounded region representing the volume $V \subset R^3$ of the shell can be viewed as: $V = \overset{s}{S} \times \left(-\frac{t}{2}, \frac{t}{2}\right)$. In order to describe the geometry more precisely let us utilize the specific curvilinear coordinates $\boldsymbol{\eta}$ called normal coordinates consisting of two natural coordinates (η^1, η^2), which are tangent to the mid-shell surface and the third Cartesian coordinate $\eta^3 \equiv x_3'$ normal to the mid-shell surface at any point. This way one can obtain the classical shell geometry description of the form:

$$V = \left\{ \boldsymbol{\eta} : (\eta^1, \eta^2, \eta^3) \subset R^3 | (\eta^1, \eta^2) \in \overset{s}{S}, \eta^3 \equiv x_3' \in \left(-\frac{t}{2}, \frac{t}{2}\right) \right\} \tag{2.1}$$

Local and global Cartesian descriptions. Apart from the above classical description we will also apply the local and global Cartesian systems of coordinates very useful for the finite element approximation introduced in the next sections.

Let us note that at any point (η^1, η^2) of the mid-shell surface the local Cartesian coordinate system $\mathbf{x}' = \mathbf{x}'(\eta^1, \eta^2)$ can be proposed with two first axes tangent and the third axis normal to the mid-shell surface. Hence the alternative local Cartesian description of the shell body is possible:

$$V = \left\{ \mathbf{x}' : (x_1', x_2', x_3') \subset R^3 | (x_1', x_2') \in \overset{s}{S}, x_3' \in \left(-\frac{t}{2}, \frac{t}{2}\right) \right\} \tag{2.2}$$

Also the global representation of the shell geometry can be equivalently introduced with the fixed Cartesian system of coordinates $\mathbf{x}$ typical for problems of three-dimensional elasticity

$$V = \left\{ \mathbf{x} : (x^1, x^2, x^3) \subset R^3 | (x_1', x_2') \in \overset{s}{S}, x_3' \in \left(-\frac{t}{2}, \frac{t}{2}\right), \mathbf{x}' = \boldsymbol{\theta}\mathbf{x} \right\}, \tag{2.3}$$

where the matrix function $\boldsymbol{\theta} = \boldsymbol{\theta}(\eta^1, \eta^2)$ transforms the global Cartesian coordinates to the local ones. Due to three-dimensional approach to formulation of the first order shell theory we will apply the above description later on in the paper.

For the sake of formal completeness of the geometry description let us finally introduce the lateral boundary $\overset{l}{S} = \partial\overset{s}{S} \times \left(-\frac{t}{2}, \frac{t}{2}\right)$, as well as the

top surface and bottom surface boundaries defined as follows: $\overset{t}{S} = \overset{s}{S} \times \left\{ \frac{t}{2} \right\}$, $\overset{b}{S} = \overset{s}{S} \times \left\{ -\frac{t}{2} \right\}$. The closure of V constitutes a total boundary of the shell: $\partial V \equiv \overline{S}$, which can be obtained by the closed versions of the component boundaries $\overline{S} = \overset{t}{\overline{S}} \cup \overset{b}{\overline{S}} \cup \overset{l}{\overline{S}}$.

2.2. The 3D-based Reissner-Mindlin model

Local formulation

Starting relations of three-dimensional elasticity. In order to introduce our 3D-based formulation of the Reissner-Mindlin first order shell theory let us start with application of the classical formulation of three-dimensional elasticity to the three-dimensional shell body defined in the previous subsection. The governing matrix equilibrium, constitutive, and compatibility equations of the theory of elasticity within the volume V of the body can be written as follows

$$\left.\begin{aligned} \boldsymbol{\Gamma}^T \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} &= \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}) \\ \boldsymbol{\varepsilon}(\mathbf{u}) &= \boldsymbol{\Gamma}\mathbf{u} \end{aligned}\right\}, \quad \mathbf{x} \in V, \tag{2.4}$$

where: $\mathbf{x} = (x_1, x_2, x_3)^T$ are the global Cartesian coordinates from the previous subsection, $\mathbf{u} = (u_1, u_2, u_3)^T$ is a sufficiently smooth field of global displacements, the terms $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})^T$ and $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T$ denote the six-component strain and stress vectors, and $\mathbf{D}$ represents the symmetric positive definite matrix of three-dimensional elasticity, satisfying the ellipticity condition of the following form: $\mathbf{y}^T\mathbf{D}\mathbf{y} \geq \alpha \mathbf{y}^T\mathbf{y}, \forall \mathbf{y} \in R^6, \alpha > 0$. The body load vector $\mathbf{f}$ represents sufficiently smooth (Lipschitzian) data, e.g. $\mathbf{f} \in (L^2(V))^3$, while $\boldsymbol{\Gamma}$ is the matrix divergence operator which for the case of global Cartesian description takes the form

$$\mathbf{\Gamma} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \end{bmatrix} \tag{2.5}$$

The above set of relations have to be completed with the displacement and traction boundary conditions on the parts P and Q into which the surface of the shell body can be subdivided according to: $\partial V \equiv S = P \cup Q$, $P \cap Q = \emptyset$. The proper Dirichlet conditions are

$$\mathbf{H}(\boldsymbol{\nu})\, \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{p}, \quad \mathbf{x} \in P \tag{2.6}$$

while the Newmann supporting conditions can be written as

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{x} \in \mathbf{Q}. \tag{2.7}$$

In the first of the above two equations the surface load vector $\mathbf{p}$ represents smooth, Lipschitzian set of data: $\mathbf{p} \in \left(L^2(S)\right)^3$, while the matrix $\mathbf{H}$ is expressed by the components of the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^T$ in the following way

$$\mathbf{H}(\boldsymbol{\nu}) = \begin{bmatrix} \nu_1 & 0 & 0 & \nu_2 & 0 & \nu_3 \\ 0 & \nu_2 & 0 & \nu_1 & \nu_3 & 0 \\ 0 & 0 & \nu_3 & 0 & \nu_2 & \nu_1 \end{bmatrix} \tag{2.8}$$

Equivalent formulation of three-dimensional elasticity. Let us express now the equilibrium equation in the following modified form corresponding to substitution of the third and second relation (4) into the first one

$$\mathbf{\Gamma}^T \mathbf{D}\, \mathbf{\Gamma} \mathbf{u} - \mathbf{f} = \mathbf{0} \tag{2.9}$$

We can change now the description of the equilibrium by choosing the local Cartesian coordinates $\mathbf{x}' = (x_1', x_2', x_3')^T$. The proper equilibrium equation will hold now

$$\mathbf{\Gamma'}^T \mathbf{D}\, \mathbf{\Gamma'} \mathbf{u'} - \mathbf{f} = \mathbf{0} \tag{2.10}$$

where $\mathbf{u'} = \boldsymbol{\theta}\mathbf{u},\ \mathbf{f'} = \boldsymbol{\theta}\mathbf{f}$ are local vectors of displacements and body forces with $\boldsymbol{\theta}$ being the transformation matrix introduced in the previous subsection. The local divergence operator $\mathbf{\Gamma'}$ takes the form (5) in which global derivatives with respect to three components of $\mathbf{x}$ have to be replaced by local derivatives with respect to the components of the local coordinates $\mathbf{x'}$. With this operator the local strains can be defined: $\boldsymbol{\varepsilon'} = \mathbf{\Gamma'}\mathbf{u'}$. Substitution of the definition of $\mathbf{u'}$ into relation (10) and left multiplication of this relation by the transposition of $\boldsymbol{\theta}$ gives

$$\boldsymbol{\theta}^T \mathbf{\Gamma'}^T \mathbf{D}\, \mathbf{\Gamma'} \boldsymbol{\theta} \mathbf{u} - \boldsymbol{\theta}^T \mathbf{f'} = \mathbf{0} \tag{2.11}$$

which after introduction of the simplifying relation: $\mathbf{f} = \boldsymbol{\theta}^T \mathbf{f'}$ reads

$$\mathbf{\Gamma''}^T \mathbf{D}\, \mathbf{\Gamma''} \mathbf{u} - \mathbf{f} = \mathbf{0} \tag{2.12}$$

In the above equilibrium equation we have also applied the modified version $\mathbf{\Gamma''}$ of the local divergence operator $\mathbf{\Gamma'}$, defined in accordance with

$$\mathbf{\Gamma''} = \mathbf{\Gamma'}\boldsymbol{\theta} = \begin{bmatrix} \theta_{11}\partial/\partial x'_1 & \theta_{12}\partial/\partial x'_1 & \theta_{13}\partial/\partial x'_1 \\ \theta_{21}\partial/\partial x'_2 & \theta_{22}\partial/\partial x'_2 & \theta_{23}\partial/\partial x'_2 \\ \theta_{13}\partial/\partial x'_3 & \theta_{23}\partial/\partial x'_3 & \theta_{33}\partial/\partial x'_3 \\ \theta_{11}\partial/\partial x'_2 + \theta_{21}\partial/\partial x'_1 & \theta_{12}\partial/\partial x'_2 + \theta_{22}\partial/\partial x'_1 & \theta_{13}\partial/\partial x'_2 + \theta_{23}\partial/\partial x'_1 \\ \theta_{21}\partial/\partial x'_3 + \theta_{31}\partial/\partial x'_2 & \theta_{22}\partial/\partial x'_3 + \theta_{32}\partial/\partial x'_2 & \theta_{23}\partial/\partial x'_3 + \theta_{33}\partial/\partial x'_2 \\ \theta_{11}\partial/\partial x'_3 + \theta_{31}\partial/\partial x'_1 & \theta_{12}\partial/\partial x'_3 + \theta_{32}\partial/\partial x'_1 & \theta_{13}\partial/\partial x'_3 + \theta_{33}\partial/\partial x'_1 \end{bmatrix} \tag{2.13}$$

It is necessary to notice that the initial and modified forms (9) and (11) of the equilibrium equations of three-dimensional elasticity are totally equivalent. The difference is that the former relation is based on the global strains $\boldsymbol{\varepsilon}$ while in the latter one we utilize local strains $\boldsymbol{\varepsilon'}$. However, the corresponding divergence operators act on the global displacement vector $\mathbf{u}$ in both cases. These features of the latter formulation will be of use while deriving the 3D-based Reissner-Mindlin equations.

3D-based Reissner-Mindlin equations. In order to obtain the equilibrium equations of the Reissner-Mindlin shell model we have to introduce the thin shell assumption of plane stress into the formulation presented in the paragraph above. The mentioned assumption can be expressed in the form

$$\varepsilon'_{33} = \frac{\nu}{1-\nu}(\varepsilon'_{11} + \varepsilon'_{22}) \tag{2.14}$$

where$\boldsymbol{\nu}$ is Poisson's ratio. Relation (14) can be obtained from the third scalar equation of the constitutive matrix equation defined locally: $\boldsymbol{\sigma}'(\mathbf{u}') = \mathbf{D}\varepsilon'(\mathbf{u}')$ after introduction of the condition $\sigma'_{33} = 0$.

Introduction of (14) into the local strain vector expressed through global displacements $\boldsymbol{\varepsilon}'(\mathbf{u}') = \boldsymbol{\Gamma}'\mathbf{u}' = \boldsymbol{\Gamma}''\mathbf{u} = \boldsymbol{\varepsilon}'(\boldsymbol{\theta}\mathbf{u})$ leads to the following modified form $\boldsymbol{\Gamma}'''$ of the divergence matrix operator $\boldsymbol{\Gamma}''$, the terms of which are given by (13):

$$\boldsymbol{\Gamma}''' = \begin{bmatrix} \Gamma'''_{11} & \Gamma'''_{12} & \Gamma'''_{13} \\ \Gamma'''_{21} & \Gamma'''_{22} & \Gamma'''_{23} \\ \Gamma'''_{31} & \Gamma'''_{32} & \Gamma'''_{33} \\ \Gamma'''_{41} & \Gamma'''_{42} & \Gamma'''_{43} \\ \Gamma'''_{51} & \Gamma'''_{52} & \Gamma'''_{53} \\ \Gamma'''_{61} & \Gamma'''_{62} & \Gamma'''_{63} \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma''_{11} & \Gamma''_{12} & \Gamma''_{13} \\ \Gamma''_{21} & \Gamma''_{22} & \Gamma''_{23} \\ \frac{\nu}{1-\nu}(\Gamma''_{11} + \Gamma''_{21}) & \frac{\nu}{1-\nu}(\Gamma''_{12} + \Gamma''_{22}) & \frac{\nu}{1-\nu}(\Gamma''_{13} + \Gamma''_{23}) \\ \Gamma''_{41} & \Gamma''_{42} & \Gamma''_{43} \\ \Gamma''_{51} & \Gamma''_{52} & \Gamma''_{53} \\ \Gamma''_{61} & \Gamma''_{62} & \Gamma''_{63} \end{bmatrix}. \tag{2.15}$$

Thus the final form of the equilibrium equations of the 3D-based first order shell model can now be written in the form

$$\boldsymbol{\Gamma}'''^{T}\mathbf{D}\,\boldsymbol{\Gamma}'''\mathbf{u} - \mathbf{f} = \mathbf{0}, \quad \mathbf{x} \in \mathbf{V}. \tag{2.16}$$

The above equation has to be enriched with the internal constraints corresponding to the kinematic assumptions of the Reissner-Mindlin theory. The first of them is deformation of the straight lines perpendicular to the mid-shell surface into other straight (not necessarily perpendicular) lines

$$u'_k = \overset{s}{u'}_k - \frac{x'_3}{t}(\overset{t}{u'}_k - \overset{b}{u'}_k), \quad u'_k = \theta_{ki} u_i, \quad \mathbf{x} \in V, \tag{2.17}$$

where $k = 1, 2$ represents two local tangent directions, while the local displacements of the middle, top and bottom surfaces are: $u'_k = u'_k(x'_1, x'_2, 0)$, $\overset{t}{u'_k} = u'_k\left(x'_1, x'_2, \frac{t}{2}\right)$, $\overset{b}{u'}_k = u'_k\left(x'_1, x'_2, -\frac{t}{2}\right)$. If necessary, the above constraint equations can be expressed by the global displacements with use of the terms $\theta_{ji}, j, i = 1, 2, 3$ of the transformation matrix $\boldsymbol{\theta}$.

The second assumption is the lack of elongation of the lines perpendicular to mid-shell surface during deformation: $\partial u'_3 / \partial x'_3 = 0$. Utilizing the terms of the transformation matrix the proper constraint equation can be expressed as

$$\theta_{3i} \frac{\partial u_i}{\partial x_j} \theta_{j3} = 0, \quad \mathbf{x} \in V. \tag{2.18}$$

The final form of the governing equation of the $3D$-based Reissner-Mindlin shell theory consists of the relations (16), (17) and (18) holding in the volume V of the three-dimensional shell body as well as of the unchanged traction and kinematic boundary conditions (6) and (7) on parts P and Q of the surface S of the body.

Variational formulation

Let us write now the potential energy Π of the shell body conforming the 3D-based Reissner-Mindlin shell model from the previous sections

$$\Pi(\mathbf{u}) = \frac{1}{2} \int_V \mathbf{u}^T \boldsymbol{\Gamma}^T \mathbf{D} \boldsymbol{\Gamma}''' \mathbf{u} \, dV - \int_V \mathbf{u}^T \mathbf{f} \, dV - \int_P \mathbf{u}^T \mathbf{p} \, dS. \tag{2.19}$$

We can treat this energy as a functional of $\mathbf{u} = \mathbf{u}(\mathbf{x})$ being the kinematically admissible displacements from the proper space $\mathcal{U}$ defined in accordance with

$$\begin{aligned} \mathcal{U}(V) \; = \; \Big\{ & \mathbf{u} \in \left(H^1(V)\right)^3 : \; u'_k = \overset{s}{u'}_k - \frac{x'_3}{t}\left(\overset{t}{u'}_k - \overset{b}{u'}_k\right), \quad \frac{\partial u'_3}{\partial x'_3} = 0, \\ & u'_j = \theta_{ji} u_i, \quad x'_3 = \theta_{3i} x_i \quad \text{in} : V, \quad \mathbf{u} = 0 \quad \text{on} : Q \Big\} \end{aligned} \tag{2.20}$$

where $k = 1, 2$ and $j = 1, 2, 3$ refer to local directions, while $i = 1, 2, 3$ corresponds to global ones.

Minimization of the above functional $\delta\Pi(\mathbf{u}) = 0$ by taking its first variation with respect to the admissible displacements $\mathbf{u}$ leads to the following stationary condition

$$\int_V \delta\mathbf{u}^T \boldsymbol{\Gamma}'''^T \mathbf{D} \boldsymbol{\Gamma}''' \mathbf{u}\, dV - \int_V \delta\mathbf{u}^T \mathbf{f}\, dV - \int_P \mathbf{u}^{\mathbf{T}} \mathbf{p}\, dS = 0 \tag{2.21}$$

After introduction of the simplifying notation reflecting the bilinear and linear character of the virtual strain energy and virtual work of external forces

$$\begin{gathered} B(\mathbf{u},\delta\mathbf{u}) = \textstyle\int_V \delta\mathbf{u}^T \boldsymbol{\Gamma}'''^T \mathbf{D} \boldsymbol{\Gamma}''' \mathbf{u}\, dV \\ L(\delta\mathbf{u}) = \textstyle\int_V \delta\mathbf{u}^T \mathbf{f}\, dV + \int_P \mathbf{u}^{\mathbf{T}} \mathbf{p}\, dS \end{gathered} \tag{2.22}$$

the condition (21) can be written as

$$B(\mathbf{u}, \delta\mathbf{u}) = L(\delta\mathbf{u}). \tag{2.23}$$

Note that finding the solution to equilibrium equation (23) consists in search for $\mathbf{u} \in \mathcal{U}(V)$ conforming (23) for $\forall\, \delta\mathbf{u} \in \mathcal{U}(V)$. It can be proved with the standard methods of linear elasticity that the solution to (23) satisfies the set of (16), (17), (18) as well as of (6) and (7), and conversely. Also the existence and uniqueness of the solution to (23) can be proved with the standard approach presented for example in [10].

2.3. Introduction of the global finite element approximation (misha)

Let us divide our three-dimensional shell body of volume V into a set of E finite elements V_e according to the partitioning $\mathcal{D}$ of the following properties

$$\begin{gathered} \overline{V} = \bigcup_{e=1}^{E} \overline{V}_e, \\ \bigwedge_{e \neq f} V_e \cap V_f = \emptyset, \\ \bigwedge_{e \neq f} \overline{V}_e \cap \overline{V}_f = \partial V_e \cap \partial V_f \equiv \overline{S}_e \cap \overline{S}_f = \overline{S}_{ef}, \end{gathered} \tag{2.24}$$

where $e, f = 1, 2, ..., E(\mathcal{D})$ are the numbers of the elements. In the above relations we obviously have: $\overline{V}_e = V_e + \partial V_e \equiv V_e + \overline{S}_e$.

Let us show now geometry of each finite element described by the approximated coordinates $\overset{e}{\mathbf{x}}{}^{hp}$as an image of the global approximated coordinates $\mathbf{x}^{hp}$ restricted to the element e: $\mathbf{x}^{hp}|_{V_e} \equiv \overset{e}{\mathbf{x}}{}^{hp}$. The local approximated geometry can be obtained from the global exact representation $\mathbf{x}$ with the element-wise projection $\overset{e}{\mathcal{P}}$ of the real geometry of the body onto the local three-dimensional polynomial space $(W(V_e))^3$ in accordance with: $\overset{e}{\mathbf{x}}{}^{hp} = \overset{e}{\mathcal{P}}(\mathbf{x})$. Note that collection of the local projections leads us to the global approximated piecewise polynomial representation of the geometry of the body $\mathbf{x}^{hp}$.

Analogous treatment of the admissible displacements $\mathbf{u} \in \mathcal{U}$, i.e. its restriction to V_e leads us to the local admissible displacements of the element e: $\mathbf{u}|_{V_e} \equiv \overset{e}{\mathbf{u}}$, which after introduction of the proper polynomial approximation gives: $\mathbf{u}^{hp}|_{V_e} \equiv \overset{e}{\mathbf{u}}{}^{hp}$. In this way the proper local approximation space of admissible displacements can be introduced:

$$\overset{e}{\mathbf{u}}{}^{hp} \in \mathcal{U}^{hp}(V_e) \subset \mathcal{U}(V_e). \tag{2.25}$$

The collection of the local approximations of admissible displacements gives us the global approximated piecewise representation of $\mathbf{u}^{hp}$:

$$\mathbf{u}^{hp} \in \mathcal{U}^{hp}(V) \subset \mathcal{U}(V) \tag{2.26}$$

In order to retain global continuity, i.e. $\mathbf{u}^{hp} \in C^0(\overline{V})$, of the piecewise polynomials $\mathbf{u}^{hp}$ resulting from *hp*-approximation of the field $\mathbf{u}$, we have to assume that the local fields of admissible displacements are suitably constrained so as the following consistency conditions are satisfied: $\overset{e}{\mathbf{u}}|_{S_{ef}} = \overset{f}{\mathbf{u}}|_{S_{fe}}$ for $\forall f \neq e$. It should be emphasized that the applied *hp* version of the finite element approximation can be replaced by *h* or *p* versions if necessary. Note also that our *hp* approximation is of local character, i.e. the mesh density parameter $\overset{e}{h} = diam(V_e)$ representing the diameter of the outer sphere of the element, can be different for each element. Also the local values of the transverse orders of approximation $\overset{e}{p}$ of the three-dimensional polynomial spaces $(W(V_e))^3 = (W^{\overset{e}{p}}(\overset{s}{S}_e))^2 \times W^1(x_3')$, $\overset{s}{S}_e \equiv \overset{s}{S}|_{V_e}$ suitable for the shell structures description can vary from one element to another. Some practical aspects concerning the *hp*-approximation of the 3D-based Reissner-Mindlin model can be found in [11].

Remark 2.1. It can be concluded from the above definition of the three-dimensional polynomial spaces that we use transverse approximation of varying order $\overset{e}{p}$ through the elements. On the contrary, linear order approximation of displacements in the transverse direction is applied within the elements. As a consequence of the latter assumption, the internal constraints (17) are automatically satisfied and they do not have to be introduced into the element formulation.

With all above in mind we can introduce now the finite element approximation of (23) and write it in the following form

$$B(\mathbf{u}^{hp}, \delta\mathbf{u}^{hp}) = L(\delta\mathbf{u}^{hp}) \tag{2.27}$$

for $\forall\, \delta\mathbf{u}^{hp}$ belonging to the space of admissible displacements $\mathcal{U}^{hp}$ such that

$$\mathcal{U}^{hp}(V) = \left\{ \mathbf{u}^{hp}(\mathbf{x}^{hp}) \in (H^1(V))^3 : \frac{\partial u_3'^{hp}}{\partial x_3'^{hp}} = 0, \quad u_3'^{hp} = \theta_{3i} u_i^{hp}, \right.$$
$$\left. x_3'^{hp} = \theta_{3i} x_i^{hp} \quad \text{in} : V, \quad \mathbf{u}^{hp} = 0 \quad \text{on} : Q \right\}. \tag{2.28}$$

3. A posteriori approximation error estimation

3.1. Approximation error

Local error. If we denote by $\mathbf{u}$ the exact solution resulting from the Reissner-Mindlin theory based on three-dimensional approach and $\mathbf{u}^{hp}$ represents the numerical solution resulting from finite element approximation of the Reissner-Mindlin model, then the local approximation error of the mentioned theory will be equal to

$$\mathbf{e} = \mathbf{u} - \mathbf{u}^{hp}. \tag{3.1}$$

Global characteristics of the approximation error of the Reissner-Mindlin theory. The definition presented above is point-wise as it allows finding the error at any point of the modelled structure. In order to get more general description of the approximation error let us introduce the global characteristics of the error measured in the norm of the strain energy

U equivalent to the characteristics based on the difference of the potential energies $\Pi(\mathbf{u})$ and $\Pi(\mathbf{u}^{hp})$ corresponding to the exact and approximated solutions $\mathbf{u}$ and $\mathbf{u}^{hp}$ which are unknown and known, respectively. Thus, the global approximation error can be determined as follows

$$-\frac{1}{2}\|\mathbf{e}\|_U^2 = \inf_{\mathbf{u}\in\mathcal{U}(V)} J(\mathbf{u}) = \inf_{\mathbf{u}\in\mathcal{U}(V)} \Pi(\mathbf{u}) - \Pi\left(\mathbf{u}^{hp}\right), \tag{3.2}$$

where $\mathbf{u}$ stands for the kinematically admissible displacements, while $\mathcal{U}(V)$ is the corresponding space.

Taking now advantage of the definition $\Pi(\mathbf{u}) = \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - L(\mathbf{u})$, where the bilinear $B(\mathbf{u}, \mathbf{u})$ and linear $L(\mathbf{u})$ forms are defined analogously as in (22) and of the definition concerning the approximate solution $\Pi(\mathbf{u}^{hp}) = \frac{1}{2}B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) - L(\mathbf{u}^{hp})$, in which for our Reissner-Mindlin case we have

$$\begin{aligned} B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) &= \int_V \mathbf{u}^{hp^T} \boldsymbol{\Gamma}'''^{\mathbf{T}} \mathbf{D} \boldsymbol{\Gamma}''' \mathbf{u}^{hp} dV, \\ L(\mathbf{u}^{hp}) &= \int_V \mathbf{u}^{hp^T} \mathbf{f} dV + \int_P \mathbf{u}^{hp^T} \mathbf{p} dS, \end{aligned} \tag{3.3}$$

we can determine the functional $J(\mathbf{u})$ as follows

$$\begin{aligned} J(\mathbf{u}) &= \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - L(\mathbf{u}) - \frac{1}{2}B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) + L(\mathbf{u}^{hp}) \\ &= \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - \frac{1}{2}B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) - L(\mathbf{u} - \mathbf{u}^{hp}) \\ &= \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - \frac{1}{2}B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) - B(\mathbf{u}, \mathbf{u} - \mathbf{u}^{hp}) \\ &= \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - \frac{1}{2}B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) - B(\mathbf{u}, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}^{hp}) \\ &= -\frac{1}{2}B(\mathbf{u} - \mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}) = -\frac{1}{2}B(\mathbf{e}, \mathbf{e}). \end{aligned} \tag{3.4}$$

In relation (32) we used the feature of solutions $\mathbf{u}$ and $\mathbf{u}^{hp}$, consisting in minimization of the potential energy: $B(\mathbf{u}, \mathbf{u} - \mathbf{u}^{hp}) = L(\mathbf{u} - \mathbf{u}^{hp})$, similarly to (21). Finally, let us notice that relation (3.4) expresses the difference of the potential energies or alternatively the strain energy related to the difference of the solutions.

Remark 2.2. It can be concluded from the above reasoning that the approximation error norm based on the difference of the potential energies, which could be denoted as $\|\mathbf{e}\|^2_{\Pi}$, is equivalent to the strain energy based norm introduced by the predecessors. That is why we have retained the traditional notation $\|\mathbf{e}\|^2_U$ of this norm throughout the paper.

In order to prove equivalence of our approach of the error estimation, which is based on the displacements $\mathbf{u}$, and the predecessors' approach based on the error $\mathbf{e}$ itself [8,9,10], we will prove the following theorem.

Theorem 2.1. Let the error $\mathbf{e} = \mathbf{u} - \mathbf{u}^{hp}$ be the unique minimizer of the following residual loaded functional $J(\mathbf{e})$ over $\mathcal{U}(V)$, defined in [8]

$$J(\mathbf{e}) = \frac{1}{2}B(\mathbf{e},\mathbf{e}) - L(\mathbf{e}) + B(\mathbf{u}^{hp},\mathbf{e}). \tag{3.5}$$

Then the displacement $\mathbf{u}$ is a unique minimizer of the functional $J(\mathbf{u})$ of (3.2).

Proof. The proof is straightforward and takes advantage of the definition of the error. Minimization of the functional (3.5) by taking its first variation with use of the following relation between variations of the error and displacements $\delta\mathbf{e} = \delta(\mathbf{u} - \mathbf{u}^{hp}) = \delta\mathbf{u}$ leads to

$$\begin{aligned}
\delta J(\mathbf{e}) &= \delta J(\mathbf{u} - \mathbf{u}^{hp}) \\
&= \delta\left[\frac{1}{2}B(\mathbf{u} - \mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}) - L(\mathbf{u} - \mathbf{u}^{hp}) + B(\mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp})\right] \\
&= B(\mathbf{u},\delta\mathbf{u}) - B(\mathbf{u}^{hp},\delta\mathbf{u}) - L(\delta\mathbf{u}) + B(\mathbf{u}^{hp},\delta\mathbf{u}) = B(\mathbf{u},\delta\mathbf{u}) - L(\delta\mathbf{u}) \\
&= \Pi(\delta\mathbf{u}) = \Pi\left[\delta(\mathbf{u} - \mathbf{u}^{hp})\right] = \delta\left[\Pi(\mathbf{u}) - \Pi(\mathbf{u}^{hp})\right] \\
&= \delta J(\mathbf{u}) = 0.
\end{aligned} \tag{3.6}$$

It can be seen in (3.6) that the stationary condition of (3.5) is converted into the stationary condition of (3.2), which completes the proof.

Returning to our definition (3.2) of the global approximation error, let us apply now the partitioning $\mathcal{D}$ introduced in the previous sections and divide our body into $E = E(\mathcal{D})$ finite elements. Such a partitioning leads

to division of the total potential energies into components corresponding to each finite element. These components have to take into account the work of the inter-element stress reactions. Taking them into consideration and introducing the condition of minimization of the potential energy $\Pi(\mathbf{u}^{hp})$ in the form $L(\mathbf{u}^{hp}) = B(\mathbf{u}^{hp}, \mathbf{u}^{hp})$, resulting from (23), the functional $J(\mathbf{u})$ can be written as

$$\begin{aligned} J(\mathbf{u}) &= \Pi(\mathbf{u}) - \Pi(\mathbf{u}^{hp}) = \Pi(\mathbf{u}) + \frac{1}{2}B(\mathbf{u}^{hp}, \mathbf{u}^{hp}) \\ &= \sum_{e=1}^{E} \left[\overset{e}{\Pi}(\mathbf{u}) + \frac{1}{2}\overset{e}{B}(\mathbf{u}^{hp}, \mathbf{u}^{hp}) - \int_{S_e \setminus (P \cup Q)} \mathbf{u}^T \overset{e}{\mathbf{r}}(\mathbf{u}) dS_e \right] \\ &\quad + \sum_{S_{ef}} \int_{S_{ef}} [\![\mathbf{u}]\!]^T \mathbf{r}(\mathbf{u}) dS_{ef}, \end{aligned} \tag{3.7}$$

where P and Q denote loaded and supported parts of the boundary of the body, S_e corresponds to the boundary of the element, while S_{ef}, $e \neq f$, $e, f = 1, 2, ..., E$, is the common boundary part of elements e and f: $S_{ef} = S_e \cap S_f$. In agreement with the above partitioning of the domain V into a set of subdomains V_e, let us introduce local solution spaces $\overset{e}{\mathcal{U}}$ such that

$$\begin{aligned} \overset{e}{\mathcal{U}} = \Big\{ & \mathbf{u} \in (H^1(V_e))^3 : u'_k = \overset{s}{u'}_k - \frac{x'_3}{t}(\overset{t}{u'}_k - \overset{b}{u'}_k),\ \frac{\partial u'_3}{\partial x'_3} = 0, \\ & u'_j = \theta_{ji} u_i,\ x'_3 = \theta_{3i} x_i \quad \text{in} : V_e, \quad \mathbf{u} = 0 \quad \text{on} : S_e \cap Q \Big\}. \end{aligned} \tag{3.8}$$

The logical product of the conditions in the element volumes V_e and on the parts $S_e \cap Q$ of the element boundaries gives us the following space defined on the partition $\mathcal{D}$

$$\mathcal{U}(\mathcal{D}) = \prod_{e=1}^{E(\mathcal{D})} \overset{e}{\mathcal{U}} \tag{3.9}$$

of the following property: $\mathcal{U}(V) \subset \mathcal{U}(\mathcal{D})$.

Note that formally the form (3.7) can be treated as an extension of the functional $J(\mathbf{u})$ introduced in (3.2) from the space $\mathcal{U}(V)$ to the space $\mathcal{U}(\mathcal{D})$.

In equation (3.7) the vector $[\![\mathbf{u}]\!]$ stands for the difference of the kinematically admissible displacements of the neighboring elements

$$[\![\mathbf{u}]\!] = \begin{cases} \overset{e}{\mathbf{u}} - \overset{f}{\mathbf{u}} \Leftrightarrow e > f \\ \overset{f}{\mathbf{u}} - \overset{e}{\mathbf{u}} \Leftrightarrow e < f \end{cases}, \tag{3.10}$$

where the element vectors $\overset{e}{\mathbf{u}} = \mathbf{u}|_{V_e}$ and $\overset{f}{\mathbf{u}} = \mathbf{u}|_{V_f}$ denote the displacements $\mathbf{u} \in \mathcal{U}(\mathcal{D})$ restricted to subdomains V_e and V_f. Subsequently, the unknown vectors of the inter-element stress reactions can be expressed through unknown displacements as follows

$$\mathbf{r}(\mathbf{u}) = \begin{cases} \overset{e}{\mathbf{r}}(\mathbf{u}) \Leftrightarrow e > f \\ \overset{f}{\mathbf{r}}(\mathbf{u}) \Leftrightarrow e < f \end{cases}, \tag{3.11}$$

with

$$\overset{e}{\mathbf{r}}(\mathbf{u}) = \mathbf{H}\left(\overset{e}{\boldsymbol{\nu}}\right) \overset{e}{\boldsymbol{\sigma}}(\mathbf{u}), \tag{3.12}$$

where $\overset{e}{\boldsymbol{\nu}} = (\overset{e}{\nu}_1, \overset{e}{\nu}_2, \overset{e}{\nu}_3)^T$ represents the vector of the unit outward normal to the element side S_e and $\overset{e}{\boldsymbol{\sigma}}$ is the six-component element stress vector. It can be easily verified that using the above notation the following relation hold for $\mathbf{u} \in \mathcal{U}$

$$\int_{S_e \setminus (P \cup Q)} \mathbf{u}^T \overset{e}{\mathbf{r}}(\mathbf{u})\, dS_e = \sum_{S_{ef}} \int_{S_{ef}} [\![\mathbf{u}]\!]^T \mathbf{r}(\mathbf{u})\, dS_{ef}. \tag{3.13}$$

In order to linearize the functional (3.7) we will replace the vector $\mathbf{r}(\mathbf{u})$ with the equivalent vector of the inter-element stress fluxes $\langle \mathbf{r}(\mathbf{u}^{hp}) \rangle \equiv \mathbf{r}(\mathbf{u})$ expressed by the known displacements $\mathbf{u}^{hp}$

$$\left\langle \mathbf{r}\left(\mathbf{u}^{hp}\right) \right\rangle = \begin{cases} \left\langle \overset{e}{\mathbf{r}}(\mathbf{u}^{hp}) \right\rangle \Leftrightarrow e > f \\ \left\langle \overset{f}{\mathbf{r}}(\mathbf{u}^{hp}) \right\rangle \Leftrightarrow e < f \end{cases}, \tag{3.14}$$

where

$$\left\langle \overset{e}{\mathbf{r}}\left(\mathbf{u}^{hp}\right) \right\rangle = \overset{f}{\boldsymbol{\alpha}}\, \mathbf{H}\left(\overset{e}{\boldsymbol{\nu}}\right) \overset{e}{\boldsymbol{\sigma}}\left(\mathbf{u}^{hp}\right) + \overset{e}{\boldsymbol{\alpha}}\, \mathbf{H}\left(\overset{e}{\boldsymbol{\nu}}\right) \overset{f}{\boldsymbol{\sigma}}\left(\mathbf{u}^{hp}\right). \tag{3.15}$$

In the above relation the matrix of functions describing the distribution of the equilibrated inter-element stresses is given by

$$\overset{f}{\boldsymbol{\alpha}}= [\overset{f}{\alpha}_1, \overset{f}{\alpha}_2, \overset{f}{\alpha}_3] \tag{3.16}$$

and additionally the following condition holds: $\overset{f}{\boldsymbol{\alpha}}= 1- \overset{e}{\boldsymbol{\alpha}}$, where $1 = diag[1,1,1]$. By their definition these functions have to be chosen so as to guarantee equilibration of the inter-element reaction forces. Hence not only the stresses $\overset{e}{\boldsymbol{\sigma}}\left(\mathbf{u}^{hp}\right)$ depend on the solution $\mathbf{u}^{hp}$, but also the functions $\overset{e}{\boldsymbol{\alpha}}=\overset{e}{\boldsymbol{\alpha}}\left(\mathbf{u}^{hp}\right)$.

Let us notice now that the solution $\mathbf{u}^{hp}$ is subject to constraints (28). That is why we have to use local values $\overset{e}{\boldsymbol{\alpha}}'$ of the splitting functions in (3.15) instead of the global ones

$$\overset{e}{\boldsymbol{\alpha}}=\overset{e}{\boldsymbol{\alpha}}'\left(\mathbf{u}^{hp}\right), \quad \mathbf{u}^{hp} \in \mathcal{U}^{hp}(V). \tag{3.17}$$

As a consequence, the definition (3.15) to be used in the approximation error estimation of the 3D-based Reissner-Mindlin model should be taken in the following non-standard form

$$\left\langle \overset{e}{\mathbf{r}}\left(\mathbf{u}^{hp}\right)\right\rangle = \boldsymbol{\theta}^T\left[\ (1- \overset{e}{\boldsymbol{\alpha}}')\boldsymbol{\theta}\,\mathbf{H}\left(\overset{e}{\boldsymbol{\nu}}\right)\overset{e}{\boldsymbol{\sigma}}\ (\mathbf{u}^{hp}) + \ \overset{e}{\boldsymbol{\alpha}}'\,\boldsymbol{\theta}\,\mathbf{H}\left(\overset{e}{\boldsymbol{\nu}}\right)\overset{f}{\boldsymbol{\sigma}}\ (\mathbf{u}^{hp})\right]. \tag{3.18}$$

The method of obtaining the local splitting functions will be presented in the sister paper [1] concerning implementation details.

The upper bound of the approximation error in the strain energy norm. Let us take advantage of relations (3.2) and (3.7) as well as of the definition of the inter-element stress fluxes. Then the global error of the finite element approximation of the Reissner-Mindlin theory based on three-dimensional approach is

$$-\frac{1}{2}\|\mathbf{e}\|_U^2 = \inf_{\mathbf{u}\in\mathcal{U}(V)} J(\mathbf{u})$$

$$= inf_{\mathbf{u}\in\mathcal{U}(\mathcal{D})}\left\{\sum_{e=1}^{E}\left[\overset{e}{\Pi}(\mathbf{u}) + \frac{1}{2}\,\overset{e}{B}\,(\mathbf{u}^{hp},\mathbf{u}^{hp}) - \int_{S_e\setminus(P\cup Q)}\mathbf{u}^T\langle\overset{e}{\mathbf{r}}\,(\mathbf{u}^{hp})\rangle dS_e\right]\right.$$

$$\left. + \sum_{S_{ef}}\int_{S_{ef}}[\![\mathbf{u}]\!]^T\langle\mathbf{r}(\mathbf{u}^{hp})\rangle dS_{ef}\right\}. \tag{3.19}$$

One can easily notice that the definition of the space $\mathcal{U}(\mathcal{D})$ allows an existence of displacement discontinuities on the inter-element boundaries. In order to assure nullification of the inter-element jumps of displacements appearing in the above equation and determined with relation (3.10), we can apply the Lagrange multipliers method. For this purpose we can utilize the approach elaborated in [10, 3] for three-dimensional elasticity problems. So let us introduce the Lagrange functional of the form

$$\bigwedge_{\mathbf{u}\in\mathcal{U}(\mathcal{D})}\bigwedge_{\boldsymbol{\mu}\in\mathcal{M}}\mathcal{L}(\mathbf{u},\boldsymbol{\mu})=J(\mathbf{u})-\boldsymbol{\mu}\left([\![\mathbf{u}]\!]\right), \tag{3.20}$$

where $\mathcal{M}$ is the proper space of multipliers. The principal property of the proposed Lagrangean is

$$\bigwedge_{\mathbf{u}\in\mathcal{U}(V)}\mathcal{L}(\mathbf{u},\boldsymbol{\mu})=J(\mathbf{u}). \tag{3.21}$$

In such circumstances the following relation holds:

$$\sup_{\boldsymbol{\mu}\in\mathcal{M}}\mathcal{L}(\mathbf{u},\boldsymbol{\mu})=\begin{cases} J(\mathbf{u}) \Leftrightarrow \mathbf{u}\in\mathcal{U}(V) \\ \infty \quad \Leftrightarrow \mathbf{u}\notin\mathcal{U}(V)\wedge\mathbf{u}\in\mathcal{U}(\mathcal{D}) \end{cases}. \tag{3.22}$$

Associating now the first definition of (3.19), condition (3.21), and relation (3.22) we come to the following:

$$\begin{aligned} -\frac{1}{2}\|\mathbf{e}\|_U^2 &= \inf_{\mathbf{u}\in\mathcal{U}(V)}J(\mathbf{u})=\inf_{\mathbf{u}\in\mathcal{U}(V)}\mathcal{L}(\mathbf{u},\boldsymbol{\mu})=\inf_{\mathbf{u}\in\mathcal{U}(\mathcal{D})}\sup_{\boldsymbol{\mu}\in\mathcal{M}}\mathcal{L}(\mathbf{u},\boldsymbol{\mu}) \\ &\geq \sup_{\boldsymbol{\mu}\in\mathcal{M}}\inf_{\mathbf{u}\in\mathcal{U}(\mathcal{D})}\mathcal{L}(\mathbf{u},\boldsymbol{\mu})\geq\inf_{\mathbf{u}\in\mathcal{U}(\mathcal{D})}\mathcal{L}(\mathbf{u},\boldsymbol{\mu}),\quad\forall\boldsymbol{\mu}\in\mathcal{M}. \end{aligned} \tag{3.23}$$

Choosing now the multipliers from $\mathcal{M}$ in the specific manner guaranteeing disappearance of the inter-element jumps of displacements

$$\boldsymbol{\mu}([\![\mathbf{u}]\!])=\sum_{S_{ef}}\int_{S_{ef}}[\![\mathbf{u}]\!]^T\langle\mathbf{r}(\mathbf{u}^{hp})\rangle dS_{ef} \tag{3.24}$$

and then substituting them and the definition of $J(\mathbf{u})$ contained in (3.19) into relation (3.20), and the latter in turn into (3.23), we get the following relation

$$\frac{1}{2}\|\mathbf{e}\|_U^2\geq\inf_{\mathbf{u}\in\mathcal{U}(\mathcal{D})}\sum_{e=1}^{E}\left[\overset{e}{\Pi}(\mathbf{u})+\frac{1}{2}\overset{e}{B}(\mathbf{u}^{hp},\mathbf{u}^{hp})-\int_{S_e\setminus(P\cup Q)}\mathbf{u}^T\langle\overset{e}{\mathbf{r}}(\mathbf{u}^{hp})\rangle dS_e\right], \tag{3.25}$$

or alternatively

$$\|\mathbf{e}\|_U^2 \leq -2\sum_{e=1}^{E} \inf_{\mathbf{u}\in\mathcal{U}} \left[\overset{e}{\Pi}(\overset{e}{\mathbf{u}}) + \frac{1}{2}\overset{e}{B}(\overset{e}{\mathbf{u}}^{hp}, \overset{e}{\mathbf{u}}^{hp}) - \int_{S_e\setminus(P\cup Q)} \overset{e}{\mathbf{u}}^T \left\langle \mathbf{r}(\overset{e}{\mathbf{u}}^{hp}) \right\rangle dS_e \right]. \tag{3.26}$$

The right hand side of the above inequality constitutes the upper bound estimate of the global approximation error in the strain energy norm.

The practical interpretation of the obtained result is the following. The value of the functional $J(\mathbf{u})$ obtained from solution $\mathbf{u} \in \mathcal{U}(\mathcal{D})$ being a collection of the local solutions $\overset{e}{\mathbf{u}} \in \overset{e}{\mathcal{U}}$ is always equal or greater than the corresponding value obtained from the global solution $\mathbf{u} \in \mathcal{U}(V)$.

3.2. Estimators of the approximation error

Local problems for approximation error estimation of the Reissner-Mindlin theory. It can be easily noticed that calculation of the upper bound of the global approximation error consists in finding the lower bounds of the expressions appearing under the sum in inequality (3.26) for each finite element. This task is equivalent to search for the stationary conditions of the local functionals

$$\delta \overset{e}{J}(\overset{e}{\mathbf{u}}) = \delta \left\{ -2 \left[\overset{e}{\Pi}(\overset{e}{\mathbf{u}}) + \frac{1}{2}\overset{e}{B}(\overset{e}{\mathbf{u}}^{hp}, \overset{e}{\mathbf{u}}^{hp}) - \int_{S_e\setminus(P\cup Q)} \overset{e}{\mathbf{u}}^T \langle \overset{e}{\mathbf{r}}(\overset{e}{\mathbf{u}}^{hp}) \rangle dS_e \right] \right\}$$

$$= -2 \left[\overset{e}{\Pi}(\delta \overset{e}{\mathbf{u}}) - \int_{S_e\setminus(P\cup Q)} \delta \overset{e}{\mathbf{u}}^T \langle \overset{e}{\mathbf{r}}(\overset{e}{\mathbf{u}}^{hp}) \rangle dS_e \right] = 0 \tag{3.27}$$

for $\forall \delta \overset{e}{\mathbf{u}} \in \overset{e}{\mathcal{U}}$, which after utilizing the potential energy definition analogous to that from (3.2) and (3.4), leads us to solutions of the following local problems

$$\overset{e}{B}(\overset{e}{\mathbf{u}}, \delta \overset{e}{\mathbf{u}}) - \overset{e}{L}(\delta \overset{e}{\mathbf{u}}) - \int_{S_e\setminus(P\cup Q)} \delta \overset{e}{\mathbf{u}}^T \langle \overset{e}{\mathbf{r}}(\overset{e}{\mathbf{u}}^{hp}) \rangle dS_e = 0. \tag{3.28}$$

Taking now relation (3.26) into consideration, as well as the above equation and the definition of the potential energy analogous to (3.2) and (3.4), one can notice that

$$\|\mathbf{e}\|_U^2 \leq \sum_{e=1}^{E} [\overset{e}{B}(\overset{e}{\mathbf{u}}, \overset{e}{\mathbf{u}}) - \overset{e}{B}(\overset{e}{\mathbf{u}}^{hp}, \overset{e}{\mathbf{u}}^{hp})] = \sum_{e=1}^{E} \overset{e}{\eta}^2, \tag{3.29}$$

where $\overset{e}{\eta}$ are the element approximation error indicators, while their sum constitutes the global approximation error estimator of the upper bound property.

Discretization of the local problems for the approximation error estimation. The local problems described above will be solved with the finite element approximation for which the discretization parameters can be denoted by H and P. It is obvious that the exact values of the global estimator and local indicators correspond to the case $H, P \to \infty$. However, in practice we utilize lower values of these two parameters. They can be equal to the element values of the discretization parameters p and h from the global problem ($P = p$, $H = h$), or greater than p and/or h. Taking the practical reasons into account we can write (3.28) in the form

$$\overset{e}{B}(\overset{e}{\mathbf{u}}{}^{HP}, \delta\, \overset{e}{\mathbf{u}}{}^{HP}) - \overset{e}{L}(\delta\, \overset{e}{\mathbf{u}}{}^{HP}) - \int_{S_e \setminus (P \cup Q)} \delta\, \overset{e}{\mathbf{u}}{}^{HP^T} \langle \overset{e}{\mathbf{r}}(\overset{e}{\mathbf{u}}{}^{hp}) \rangle \, dS_e = 0, \quad (3.30)$$

where $\delta\, \overset{e}{\mathbf{u}}{}^{HP} \in \overset{e}{\mathcal{U}}{}^{HP}, \overset{e}{\mathcal{U}}{}^{HP} \subset \overset{e}{\mathcal{U}}$ stands for the displacement variation, while $\overset{e}{\mathcal{U}}{}^{HP}$ denotes the space of the kinematically admissible displacements $\overset{e}{\mathbf{u}}{}^{HP}$ within the local discretized problems:

$$\begin{aligned} \overset{e}{\mathcal{U}}{}^{HP} \;=\; \Big\{ & \overset{e}{\mathbf{u}}{}^{HP} \in (H^1(V_e))^3 : \partial\, \overset{e}{u'}{}^{HP}_{3} / \partial\, \overset{e}{x}{}'^{HP}_{3} = 0,\; \overset{e}{u'}{}^{HP}_{3} = \theta_{3i} u_i^{HP}, \\ & \overset{e}{x'}{}^{HP}_{3} = \theta_{3i}\; \overset{e}{x'}{}^{HP}_{i} \text{ in} : V_e, \overset{e}{\mathbf{u}}{}^{HP} = 0 \text{ on} : S_e \cap Q \Big\}. \end{aligned} \quad (3.31)$$

In this situation the following relation describing the approximation of the element error indicator holds

$$\overset{e}{\eta}{}^2 \approx \overset{e}{\eta}{}^{HP^2} = \overset{e}{B}\Big(\overset{e}{\mathbf{u}}{}^{HP}, \overset{e}{\mathbf{u}}{}^{HP}\Big) - \overset{e}{B}\Big(\overset{e}{\mathbf{u}}{}^{HP}, \overset{e}{\mathbf{u}}{}^{HP}\Big). \quad (3.32)$$

4. Conclusions

There is a possibility of formulation of the 3D-based first order shell model equipped with three-dimensional degrees of freedom. The model can also be characterized as constrained (due to kinematic assumptions of the Reissner-Mindlin theory) six parameter (three top and three bottom displacements) model.

It is possible to construct an a posteriori global approximation error estimate based on the residual equilibration method for such a model.

It can be proved that our global estimate which is based on the difference of the potential energies corresponding to exact and approximated solutions is equivalent to the estimate based on the strain energy defined on the local error.

The proposed global estimate possesses an upper bound property, which is the result of replacement of the global exact solution by the collection of the solutions for the problems defined locally, i.e. for each finite element.

Determination of the local problems deals with taking the inter-element stress reactions into consideration, which should be in equilibrium. Equilibration of the inter-element stresses has to be performed towards the local Cartesian directions due to the internal constraints imposed on the 3D-based Reissner-Mindlin model.

The value of the global estimate can be calculated by summation of the element error indicators obtained through solution of the local problems. The unknowns of the local problems are element displacements estimating the exact solution of the problem. Subtraction of the estimations of the exact values of displacements obtained from the local problems and approximated values of displacements from solution of the global problem for each finite element gives us the value of the element error indicator.

A practical method of solution of the local problems takes advantage of the finite element approximation of these problems.

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References

1. Zboiński G. *A Posteriori Error Estimation for hp-Approximation of the 3D-Based First Order Shell Model. Part 2. Implementation Aspects.* To be published in: Applied Mathematics, Mechanics, and Informatics.

2. Zboiński G. *Hierarchical Modelling and Finite Element Approximation for Adaptive Analysis of Complex Structures* (DSc Thesis, in Polish). 520/1479/2001. Institute of Fluid Flow Machinery, Gdansk, Poland, 2001.

3. Ainsworth M., Oden J. T. *A unified approach to a posteriori error estimation using element residual methods.* Numer. Math., 65 (1993), 23-50.

4. Kelly D. W. *The self-equilibration of residuals and complementary a posteriori error estimates in the finite element method.* Int. J. Numer. Methods

Eng., 20 (1984), 1491-1506.

5. Bank R. E., Weiser A. *Some a posteriori error estimators for elliptic partial differential equations.* Math. Comput., 44 (1985), 283-301.

6. Ainsworth M., Oden J. T. *A procedure for a posteriori error estimation for h-p finite element methods.* Comput. Methods Appl. Mech. Eng., 101 (1992), 73-96.

7. Ainsworth M., Oden J. T. *A posteriori error estimation in finite element analysis.* Comput. Methods Appl. Mech. Eng., 142 (1997), 1-88.

8. Ainsworth M., Oden J. T. *A posteriori error estimators for second order elliptic systems: Part 1. Theoretical foundations and a posteriori error analysis.* Computers Math. Applic., 25 (1993), 101-113.

9. Ainsworth M., Oden J. T. *A posteriori error estimators for second order elliptic systems: Part 2. An optimal order process for calculating self-equilibrating fluxes.* Computers Math. Applic., 26 (1993), 75-87.

10. Ainsworth M., Oden J. T., Wu W. *A posteriori error estimation for hp approximation in elastostatics.* Appl. Numer. Math., 14 (1994), 23-55.

11. Zboiński G. *3D-Based hp-Adaptive First Order Shell Finite Element for Modelling and Analysis of Complex Structures.* Submitted to: International Journal for Numerical Methods in Engineering.