MATHEMATICAL PROBLEMS OF THE THEORY OF ELASTICITY OF CHIRAL MATERIALS [1]

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Abstract

The purpose of this paper is to construct explicitly fundamental matrices of solutions to the differential equations of the theory of hemitropic elastic (chiral) materials. We consider the differential equations corresponding to the cases of pseudo-oscillations and steady state oscillations and in terms of elementary functions we construct fundamental matrices satisfying the generalized Sommerfeld-Kupradze type radiation conditions. On the basis of Green's formulae we derive the general integral representations of solutions in bounded and unbounded domains by means of potential type integrals. Properties of the single- and double-layer potentials and of certain, generated by them, boundary integral (pseudodifferential) operators are studied. Applying the potential method and the theory of pseudodifferential equations we prove the uniqueness and existence theorems of solutions to the Dirichlet, Neumann and mixed boundary value problems for the pseudo-oscillation equations. Some particular results are obtained for the steady state oscillation equations.

Key words and phrases: Elasticity theory, Elastic chiral materials, Fundamental matrix, Potential theory, Pseudodifferential equations, Boundary value problems.

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[1] Dedicated to the memory of Professor Victor Kupradze on the occasion of the 100^{th} anniversary of his birth

1 Introduction

A solid which is not isotropic with respect to inversion is called *noncentrosymmetric, acentric, hemitropic*, or *chiral*. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules, as well as on a large scale, as in composites with helical or screw-shaped inclusions (for details see, e.g., [1], [20] and the references therein).

In recent years the electromagnetic field in chiral media has been the object of intensive research, see e.g. [21], [43], [44], and the references therein.

Mathematical models describing the chiral properties of elastic materials have been proposed by Aero and Kuvshinski [1], [2] (for the history of the problem see also [28], [35], [38], [48] and the references therein).

Particular problems of the elasticity theory of hemitropic continuum related to the present paper have been considered in [35], [36], [37], [48], [22], [23], [24], [20], [9].

The main goal of our investigation is to study the basic boundary value and initial boundary value problems of the noncentrosymmetric theory of elasticity for bodies of arbitrary geometrical shape. We consider separately the equilibrium equations of statics, the steady state oscillation equations (the time harmonic dependent case), and the general equations of dynamics, and develop the potential method (boundary integral equations method) to obtain the existence and uniqueness results. We construct also explicit solutions of some particular canonical problems in the form of absolutely and uniformly convergent series.

To this end, in this paper, which is the first part of our investigation, the corresponding matrices of fundamental solutions are constructed explicitly (in terms of elementary functions) and the generalized Sommerfeld-Kupradze type radiation conditions are formulated which play a crucial role to establish the uniqueness results in the case of exterior boundary value problems (BVP). Further, the boundary integral (pseudodifferential) operators generated by the single- and double-layer potentials are studied and their ellipticity and normal solvability properties are established. Based on the results obtained, the uniqueness and existence theorems of solutions to the basic BVPs of pseudo-oscillations are proved in various Hölder ($C^{k,\alpha}$), Sobolev-Slobodetski (W_p^s) and Besov ($B_{p,q}^s$) functional spaces.

For the homogeneous system of the steady state oscillation equations (in the whole space $\mathbb{R}^3$) it is shown that it possesses only the trivial solution in the class of vectors satisfying the generalized Sommerfeld-Kupradze type radiation conditions. The corresponding exterior BVPs will be considered in a forthcoming paper.

2 Basic equations and Green formulae

2.1. Constitutive equations

Let $I\!R^3$ be the three-dimensional Euclidean space and $\Omega^+ \subset I\!R^3$ be a bounded domain with a boundary $S := \partial\Omega^+$, $\overline{\Omega^+} = \Omega \cup S$; $\Omega^- = I\!R^3 \backslash \overline{\Omega^+}$. We assume that $\overline{\Omega} \in \{\overline{\Omega^+}, \overline{\Omega^-}\}$ is filled with an elastic material possessing the hemitropic properties.

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the displacement vector and the microrotation vector, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the microrotation vector in the hemitropic elasticity theory is kinematically distinct from the macrorotation vector $\frac{1}{2}$ curl u.

The tensors of the force stress $\{\tau_{pq}\}$ and the couple stress $\{\mu_{pq}\}$ in the linear theory are as follows (the constitutive equations)

$$\tau_{pq} = \tau_{pq}(U) := (\mu+\alpha)\frac{\partial u_q}{\partial x_p} + (\mu-\alpha)\frac{\partial u_p}{\partial x_q} + \lambda\delta_{pq}\operatorname{div} u$$

$$+\delta\,\delta_{pq}\operatorname{div}\omega + (\kappa+\nu)\frac{\partial \omega_q}{\partial x_p} + \kappa-\nu)\frac{\partial \omega_p}{\partial x_q} - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k, \tag{2.1}$$

$$\mu_{pq} = \mu_{pq}(U) := \delta\,\delta_{pq}\operatorname{div} u + (\kappa+\nu)\left[\frac{\partial u_q}{\partial x_p} - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\right] + \beta\,\delta_{pq}\operatorname{div}\omega$$

$$+(\kappa-\nu)\left[\frac{\partial u_p}{\partial x_q} - \sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\right] + (\gamma+\varepsilon)\frac{\partial \omega_q}{\partial x_p} + (\gamma-\varepsilon)\frac{\partial \omega_p}{\partial x_q}, \tag{2.2}$$

where $U = (u, \omega)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi-Civitá) symbol, and α, β, γ, δ, λ, μ, ν, κ, and ε are the material constants (see [1]).

The components of the vectors of the force stress $\tau^{(n)}$ and the coupled stress $\mu^{(n)}$, acting on a surface element with a normal vector $n = (n_1, n_2, n_3)$, read as

$$\tau_q^{(n)} = \sum_{p=1}^{3}\tau_{pq}n_p, \qquad \mu_q^{(n)} = \sum_{p=1}^{3}\mu_{pq}n_p, \quad q = 1, 2, 3. \tag{2.3}$$

Let us introduce the generalized stress operator (6×6 matrix differential operator)

$$T(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) \end{bmatrix}_{6\times 6},$$

$$T^{(j)} = \left[T_{pq}^{(j)}\right]_{3\times 3}, \quad j = \overline{1, 4}, \tag{2.4}$$

where $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$,

$$
\begin{aligned}
T^{(1)}_{pq}(\partial, n) &= (\mu+\alpha)\delta_{pq}\frac{\partial}{\partial n} + (\mu-\alpha)n_q\frac{\partial}{\partial x_p} + \lambda n_p\frac{\partial}{\partial x_q},\\
T^{(2)}_{pq}(\partial, n) &= (\kappa+\nu)\delta_{pq}\frac{\partial}{\partial n} + (\kappa-\nu)n_q\frac{\partial}{\partial x_p} + \delta n_p\frac{\partial}{\partial x_q} - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,\\
T^{(3)}_{pq}(\partial, n) &= (\kappa+\nu)\delta_{pq}\frac{\partial}{\partial n} + (\kappa-\nu)n_q\frac{\partial}{\partial x_p} + \delta n_p\frac{\partial}{\partial x_q},\\
T^{(4)}_{pq}(\partial, n) &= (\gamma+\varepsilon)\delta_{pq}\frac{\partial}{\partial n} + (\gamma-\varepsilon)n_q\frac{\partial}{\partial x_p}\\
&\quad +\beta n_p\frac{\partial}{\partial x_q} - 2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.
\end{aligned}
\tag{2.5}
$$

It can be easily checked that

$$(\tau^{(n)}, \mu^{(n)})^\top = T(\partial, n)\, U.$$

Denote by $T_0^{(j)}(\partial, n)$ the principal homogeneous part (6×6 matrix) of the differential operator $T(\partial, n)$, i.e.,

$$
\begin{aligned}
T_0(\partial, n) &= \begin{bmatrix} T_0^{(1)}(\partial, n) & T_0^{(2)}(\partial, n)\\ T_0^{(3)}(\partial, n) & T_0^{(4)}(\partial, n)\end{bmatrix}_{6\times 6},\\
T_0^{(j)} &= \left[T^{(j)}_{0pq}\right]_{3\times 3}, \quad j=\overline{1,4},\\
T^{(j)}_{0pq}(\partial, n) &= T^{(j)}_{pq}(\partial, n), \quad j=1,3,\\
T^{(2)}_{0pq}(\partial, n) &= T^{(2)}_{pq}(\partial, n) + 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,\\
T^{(4)}_{0pq}(\partial, n) &= T^{(4)}_{pq}(\partial, n) + 2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.
\end{aligned}
$$

We have the evident equality

$$T(\partial_x, n)U = T_0(\partial_x, n)U + 2[\alpha\, n\times\omega,\ \nu\, n\times\omega]^\top, \tag{2.6}$$

where the symbol $\times$ denotes the cross product of two vectors.

2.2. The basic equations

The equations of dynamics of the hemitropic theory of elasticity have the form

$$\sum_{p=1}^{3}\partial_p\,\tau_{pq}(x,t) + \varrho F_q(x,t) = \varrho\frac{\partial^2 u_q(x,t)}{\partial t^2},$$

$$\sum_{p=1}^{3} \partial_p \mu_{pq}(x,t) + \sum_{l,r=1}^{3} \varepsilon_{qlr} \tau_{lr}(x,t) + \varrho\, G_q(x,t) = \mathcal{I} \frac{\partial^2 \omega_q(x,t)}{\partial t^2}, \quad q = 1,2,3,$$

where t is the time variable, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the body force and body couple vectors per unit mass, ϱ is the mass density of the elastic material, and $\mathcal{I}$ is a constant characterizing the so called spin torque corresponding to the interior microrotations (i.e., the moment of inertia per unit volume).

Using the relations (2.1)-(2.2) we can rewrite the above dynamic equations in terms of the displacement and microrotation vectors:

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x,t) + (\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\, u(x,t) \\
&\quad +(\kappa+\nu)\Delta\omega(x,t) + (\delta+\kappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,\omega(x,t) \\
&\quad +2\alpha\,\mathrm{curl}\,\omega(x,t) + \varrho\, F(x,t) = \varrho\, \frac{\partial^2 u(x,t)}{\partial t^2}, \\
&(\kappa+\nu)\Delta u(x,t) + (\delta+\kappa-\nu)\,\mathrm{grad}\,\mathrm{div}\, u(x,t) + 2\alpha\,\mathrm{curl}\, u(x,t) \\
&\quad +(\gamma+\varepsilon)\Delta\omega(x,t) + (\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\omega(x,t) + 4\nu\,\mathrm{curl}\,\omega(x,t) \\
&\quad -4\alpha\,\omega(x,t) + \varrho\, G(x,t) = \mathcal{I}\, \frac{\partial^2 \omega(x,t)}{\partial t^2},
\end{aligned} \tag{2.7}$$

where Δ is the Laplace operator.

If all the quantities involved in these equations are harmonic time dependent, i.e., $u(x,t) = u(x)\exp\{-i\,t\sigma\}$, $\omega(x,t) = \omega(x)\exp\{-i\,t\sigma\}$, $F(x,t) = F(x)\exp\{-i\,t\sigma\}$, and $G(x,t) = G(x)\exp\{-i\,t\sigma\}$, with $\sigma \in I\!R^1$ and $i = \sqrt{-1}$, we obtain the *steady state oscillation equations* of the hemitropic theory of elasticity:

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\, u(x) + (\kappa+\nu)\Delta\omega(x) \\
&\quad +(\delta+\kappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,\omega(x) + 2\alpha\,\mathrm{curl}\,\omega(x) + \varrho\,\sigma^2 u(x) = -\varrho F(x), \\
&(\kappa+\nu)\Delta u(x) + (\delta+\kappa-\nu)\,\mathrm{grad}\,\mathrm{div}\, u(x) + 2\alpha\,\mathrm{curl}\, u(x) \\
&\quad +(\gamma+\varepsilon)\Delta\omega(x) + (\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\omega(x) + 4\nu\,\mathrm{curl}\,\omega(x) \\
&\quad +(\mathcal{I}\sigma^2 - 4\alpha)\omega(x) = -\varrho G(x);
\end{aligned} \tag{2.8}$$

here u, ω, F, and G are complex-valued vector functions, σ is a frequency parameter.

If $\sigma = \sigma_1 + i\,\sigma_2$ is a complex parameter with $\sigma_2 \neq 0$, then the above equations are called the *pseudo-oscillation equations*, while for $\sigma = 0$ they represent the *equilibrium equations of statics*.

Let us introduce the matrix differential operator corresponding to the system (2.8):

$$L(\partial,\sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma), & L^{(2)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma), & L^{(4)}(\partial,\sigma) \end{bmatrix}_{6\times 6}, \tag{2.9}$$

where

$$L^{(1)}(\partial,\sigma) := [(\mu+\alpha)\Delta + \varrho\sigma^2]I_3 + (\lambda+\mu-\alpha)Q(\partial),$$

$$L^{(2)}(\partial,\sigma) = L^{(3)}(\partial,\sigma) := (\kappa+\nu)\Delta I_3 + (\delta+\kappa-\nu)Q(\partial) + 2\alpha R(\partial), \tag{2.10}$$

$$L^{(4)}(\partial,\sigma) := [(\gamma+\varepsilon)\Delta + (\mathcal{I}\sigma^2 - 4\,\alpha)]I_3 + (\beta+\gamma-\varepsilon)Q(\partial) + 4\nu R(\partial).$$

Here I_k stands for the $k\times k$ unit matrix and

$$R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}_{3\times 3}, \qquad Q(\partial) := [\,\partial_k\partial_j\,]_{3\times 3}. \tag{2.11}$$

It is easy to see that

$$R(\partial)u = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix} = \operatorname{curl} u, \quad Q(\partial)\,u = \operatorname{grad}\operatorname{div} u, \tag{2.12}$$

$$\begin{aligned} &R(-\partial) = -R(\partial) = [R(\partial)]^\top, \quad Q(\partial)\,R(\partial) = R(\partial)\,Q(\partial) = 0, \\ &Q(\partial) = [Q(\partial)]^\top, \quad [R(\partial)]^2 = Q(\partial) - \Delta I_3, \quad [Q(\partial)]^2 = Q(\partial)\,\Delta. \end{aligned} \tag{2.13}$$

Due to the above notation, the equations (2.8) can be rewritten in matrix form as

$$\begin{aligned} &L(\partial,\sigma)U(x) = \Phi(x), \\ &U = (u,\omega)^\top, \quad \Phi = (\Phi^{(1)},\Phi^{(2)})^\top := (-\varrho\, F(x), -\varrho\, G(x))^\top. \end{aligned} \tag{2.14}$$

Further, let us remark that the differential operator

$$L(\partial) := L(\partial, 0) \tag{2.15}$$

corresponds to the static equilibrium case, while the differential operator

$$L_0(\partial) := \begin{bmatrix} L_0^{(1)}(\partial), & L_0^{(2)}(\partial) \\ L_0^{(3)}(\partial), & L_0^{(4)}(\partial) \end{bmatrix}_{6\times 6} \tag{2.16}$$

with

$$L_0^{(1)}(\partial) := (\mu+\alpha)\Delta\, I_3 + (\lambda+\mu-\alpha)\, Q(\partial),$$

$$L_0^{(2)}(\partial) = L_0^{(3)}(\partial) := (\kappa+\nu)\Delta\, I_3 + (\delta+\kappa-\nu)\, Q(\partial), \tag{2.17}$$

$$L_0^{(4)}(\partial) := (\gamma+\varepsilon)\Delta\, I_3 + (\beta+\gamma-\varepsilon)\, Q(\partial),$$

represents the principal homogeneous part of the operators (2.9) and (2.15).

It is evident that

$$L(\partial,\sigma)U - L(\partial)U = (\varrho\,\sigma^2\, u,\ \mathcal{I}\,\sigma^2\,\omega)^\top. \tag{2.18}$$

Let us remark that the operators $L(\partial,\sigma)$ for real σ, $L(\partial)$, and $L_0(\partial)$ are formally self-adjoint, i.e., $L(\partial,\sigma) = [L(-\partial,\sigma)]^\top$, $L(\partial) = [L(-\partial)]^\top$, $L_0(\partial) = [L_0(-\partial)]^\top$.

2.3. Green's formulae

For real-valued vectors $U := (u, \omega)^\top$, $U' := (u', \omega')^\top \in [C^2(\overline{\Omega^+})]^6$, we easily derive the following Green formula

$$\int\limits_{\Omega^+} \big[U' \cdot L(\partial)U + E(U', U)\big]\, dx = \int\limits_{\partial\Omega^+} U' \cdot T(\partial, n)U\, dS, \tag{2.19}$$

where $\partial\Omega^+$ is a piecewise smooth manifold, n is the outward unit normal vector to $\partial\Omega^+$, $E(\cdot\,,\,\cdot)$ is the so called *energy bilinear form*

$$\begin{aligned} E(U', U) = E(U, U') = \sum_{p,q=1}^{3} \{ &(\mu+\alpha)u'_{pq}u_{pq} \\ &+(\mu-\alpha)u'_{pq}u_{qp} + (\kappa+\nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) \\ &+(\kappa-\nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) + (\gamma+\varepsilon)\omega'_{pq}\omega_{pq} + (\gamma-\varepsilon)\omega'_{pq}\omega_{qp} \\ &+\delta(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}) + \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\} \end{aligned} \tag{2.20}$$

with

$$u_{pq} = \partial_p u_q - \sum_{k=1}^{3} \varepsilon_{pqk}\omega_k, \quad \omega_{pq} = \partial_p\omega_q, \quad p, q = 1, 2, 3. \tag{2.21}$$

Here and in what follows $a \cdot b$ denotes the usual scalar product of two (in general) complex vectors $a, b \in \mathbb{C}^m$:

$$a \cdot b = \sum_{j=1}^{m} a_j\, \overline{b_j},$$

where the over-bar denotes complex conjugation. The proof of the above Green formula immediately follows from the identity

$$U' \cdot L(\partial)U + E(U', U) = \sum_{p,q=1}^{3} \partial_p\, [u'_q\, \tau_{pq}(U) + \omega'_q\, \mu_{pq}(U)]. \tag{2.22}$$

From (2.20) and (2.21) we get

$$\begin{aligned} E(U, U') = &\frac{3\lambda+2\mu}{3}\left(\operatorname{div} u + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega\right)\left(\operatorname{div} u' + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega'\right) \\ &+\frac{1}{3}\left(3\beta+2\gamma - \frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\right)(\operatorname{div}\omega)(\operatorname{div}\omega') \end{aligned}$$

$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)\right]$$

$$\times\left[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega'_k}{\partial x_j}+\frac{\partial \omega'_j}{\partial x_k}\right)\right]$$

$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\left[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)\right]$$

$$\times\left[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega'_k}{\partial x_k}-\frac{\partial \omega'_j}{\partial x_j}\right)\right]$$

$$+\left(\gamma-\frac{\kappa^2}{\mu}\right)\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{1}{2}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)\left(\frac{\partial \omega'_k}{\partial x_j}+\frac{\partial \omega'_j}{\partial x_k}\right)\right.$$

$$\left.+\frac{1}{3}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)\left(\frac{\partial \omega'_k}{\partial x_k}-\frac{\partial \omega'_j}{\partial x_j}\right)\right]$$

$$+\alpha\left(\mathrm{curl}\ u+\frac{\nu}{\alpha}\,\mathrm{curl}\ \omega-2\omega\right)\cdot\left(\mathrm{curl}\ u'+\frac{\nu}{\alpha}\,\mathrm{curl}\ \omega'-2\omega'\right)$$

$$+\left(\varepsilon-\frac{\nu^2}{\alpha}\right)\ \mathrm{curl}\ \omega\ \cdot\ \mathrm{curl}\ \omega'. \tag{2.23}$$

In particular,

$$E(U,U)=\frac{3\lambda+2\mu}{3}\left(\mathrm{div}\ u+\frac{3\delta+2\kappa}{3\lambda+2\mu}\,\mathrm{div}\ \omega\right)^2$$

$$+\frac{1}{3}\left(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\right)(\mathrm{div}\,\omega)^2$$

$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)\right]^2$$

$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\left[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)\right]^2$$

$$+\left(\gamma-\frac{\kappa^2}{\mu}\right)\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{1}{2}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)^2+\frac{1}{3}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)^2\right]$$

$$+\left(\varepsilon-\frac{\nu^2}{\alpha}\right)(\text{curl }\omega)^2+\alpha\left(\text{curl }u+\frac{\nu}{\alpha}\text{ curl }\omega-2\omega\right)^2. \tag{2.24}$$

From physical considerations (positive definiteness of the potential energy (2.24) with respect to the variables (2.21)), it follows that the material constants satisfy the inequalities (cf. [2])

$$\begin{aligned}&\mu>0,\quad \alpha>0,\quad 3\lambda+2\mu>0,\quad \mu\gamma-\kappa^2>0,\quad \alpha\varepsilon-\nu^2>0,\\ &(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\kappa)^2>0.\end{aligned} \tag{2.25}$$

These inequalities imply

$$\begin{aligned}&\gamma>0,\quad \varepsilon>0,\quad \lambda+\mu>0,\quad \beta+\gamma>0,\\ &d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\kappa+\nu)^2>0,\\ &d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\kappa)^2>0.\end{aligned} \tag{2.26}$$

Lemma 2.1 *Let $U=(u,\omega)^\top\in[C^1(\Omega)]^6$ be a real-valued vector and $E(U,U)=0$ in Ω. Then*

$$u(x)=[a\times x]+b,\qquad \omega(x)=a,\quad x\in\Omega, \tag{2.27}$$

where a and b are arbitrary three-dimensional constant vectors.

Proof. It easily follows from (2.24) and (2.25). ∎

Throughout the paper L_p, W_p^s, H_p^s, and $B_{p,q}^s$ (with $s\in I\!R$, $1<p<\infty$, $1\le q\le\infty$) denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov spaces, respectively (see, e.g., [46], [47], [25]). We will use the abbreviations $W_2^s=W^s$, $H_2^s=H^s$. We recall that $H_2^s=W_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $s\in I\!R$, for any positive and non-integer t, and for any non-negative integer k.

If $U=U^{(1)}+i\,U^{(2)}$ is a complex-valued vector, where $U^{(j)}=(u^{(j)},\omega^{(j)})^\top$ $(j=1,2)$ are real-valued vectors, then

$$E(U,\overline{U})=E(U^{(1)},U^{(1)})+E(U^{(2)},U^{(2)}),$$

and, due to the positive definiteness of the energy form for real-valued vector functions, we have

$$E(U,\overline{U})\ge c_0\sum_{p,q=1}^{3}\left[(u_{pq}^{(1)})^2+(u_{pq}^{(2)})^2+(\omega_{pq}^{(1)})^2+(\omega_{pq}^{(2)})^2\right], \tag{2.28}$$

where c_0 is a positive constant depending only on the material constants, and $u_{pq}^{(j)}$ and $\omega_{pq}^{(j)}$ are defined by formulae (2.21) with $u^{(j)}$ and $\omega^{(j)}$ for u and ω.

From the positive definiteness of the energy form $E(\cdot,\cdot)$ with respect to the variables (2.21) it easily follows that there exist positive constants c_1 and c_2 such that for an arbitrary real-valued vector $U \in [C^1(\overline{\Omega^+})]^6$

$$\begin{aligned}\mathcal{B}(U,U) &:= \int\limits_{\Omega^+} E(U,U)dx \\ &\geq c_1 \int\limits_{\Omega^+} \left\{ \sum_{p,q=1}^{3} [(\partial_p u_q)^2 + (\partial_p \omega_q)^2] + \sum_{p=1}^{3} [u_p^2 + \omega_p^2] \right\} dx \\ &-c_2 \int\limits_{\Omega^+} \sum_{p=1}^{3} [u_p^2 + \omega_p^2]\, dx, \end{aligned} \tag{2.29}$$

i.e., the following Korn's type inequality holds (cf. [11], Part I, §12, [26], Ch.10)

$$\mathcal{B}(U,U) \geq c_1\, ||U||^2_{[H^1(\Omega^+)]^6} - c_2\, ||U||^2_{[H^0(\Omega^+)]^6}, \tag{2.30}$$

where $||\cdot||_{[H^s(\Omega^+)]^6}$ denotes the norm in the Sobolev space $[H^s(\Omega^+)]^6$.

These results imply that the differential operators $L(\partial,\sigma)$, $L(\partial)$, and $L_0(\partial)$ are *strongly elliptic* and the following inequality (*the accretivity condition*) holds (cf., e.g., [11], Part I, §5, [26], Ch.4, Lemma 4.5)

$$c_2'\, |\xi|^2\, |\eta|^2 \geq L_0(\xi)\eta\cdot\eta = \sum_{k,j=1}^{6} \{L_0(\xi)\}_{kj}\eta_j\, \overline{\eta_k} \geq c_1'\, |\xi|^2\, |\eta|^2 \tag{2.31}$$

with some constants $c_k' > 0$ $(k=1,2)$ for arbitrary $\xi \in \mathbb{R}^3$ and arbitrary complex vector $\eta \in \mathbb{C}^6$.

Remark 2.2 *From* (2.18)-(2.19) *it follows that*

$$\begin{aligned}&\int\limits_{\Omega^+} \Big[U'\cdot\overline{L(\partial,\sigma)U} - L(\partial,\sigma)U'\cdot\overline{U}\Big]\, dx \\ &\qquad = \int\limits_{\partial\Omega^+} \big[U'\cdot T(\partial,n)\overline{U} - T(\partial,n)U'\cdot\overline{U}\big]\, dS\end{aligned} \tag{2.32}$$

for an arbitrary complex parameter σ.

Remark 2.3 *By standard approach, Green's formula* (2.19) *can be extended to Lipschitz domains* (*see, e.g.,* [34], [26]) *and to the case of complex-valued vector functions* $U \in [W^1_p(\Omega^+)]^6$ *and* $U' \in [W^1_{p'}(\Omega^+)]^6$ *with* $1/p + 1/p' = 1$ *and* $L(\partial,\sigma)U \in [L_p(\Omega^+)]^6$ (*cf.* [25], [5], [26])

$$\int\limits_{\Omega^+} \big[U'\cdot L(\partial)U + E(U',\overline{U}\big]\, dx = \Big\langle U'\,,\,\overline{T(\partial,n)U}\Big\rangle_{\partial\Omega^+}, \tag{2.33}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ *denotes the duality between the spaces* $[B^{1/p}_{p,p}(\partial\Omega^+)]^6$ *and* $[B^{-1/p}_{p',p'}(\partial\Omega^+)]^6$*, which extends the usual* L_2*-scalar product for regular vector-functions, i.e., for* $f, g \in [L_2(S)]^6$ *we have*

$$\langle\, f\,,\,g\,\rangle_S = \sum_{k=1}^{6}\int\limits_S f_k\, g_k\, dS = (f, \overline{g})_{L_2(S)}.$$

Clearly, in this case the functional $T(\partial, n)U \in [B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ *is correctly determined by the relation* (2.33).

3 Basic fundamental matrices

3.1. Matrices of fundamental solutions of pseudo- and steady state oscillation equations

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwarz space $\mathcal{S}'(I\!\!R^3)$) which for regular summable functions f and $\widehat{f}$ read as follows

$$\begin{aligned} &\mathcal{F}_{x\to\xi}[f] = \int\limits_{I\!\!R^3} f(x)\, e^{i\,x\cdot\xi} dx = \widehat{f}(\xi),\\ &\mathcal{F}^{-1}_{\xi\to x}[\widehat{f}\,] = \frac{1}{(2\pi)^3}\int\limits_{I\!\!R^3} \widehat{f}(\xi)\, e^{-i\,x\cdot\xi} d\xi = f(x). \end{aligned} \tag{3.1}$$

Moreover, for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in \mathcal{S}'(I\!\!R^3)$ there hold

$$\mathcal{F}[\partial^\alpha f] = (-i\,\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha \widehat{f}\,] = (i\,\partial)^\alpha \mathcal{F}^{-1}[\widehat{f}\,], \tag{3.2}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\xi^\alpha = \xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$.

Denote by $\Gamma(x,\sigma) = [\Gamma_{kj}(x,\sigma)]_{6\times 6}$ the matrix of fundamental solutions of the operator $L(\partial,\sigma)$ (see (2.9)-(2.10))

$$L(\partial,\sigma)\,\Gamma(x,\sigma) = \delta(x) I_6. \tag{3.3}$$

If we represent $\Gamma(x,\sigma)$ with the help of the block matrices

$$\Gamma(x,\sigma) = \begin{bmatrix} \Gamma^{(1)}(x,\sigma) & \Gamma^{(2)}(x,\sigma)\\ \Gamma^{(3)}(x,\sigma) & \Gamma^{(4)}(x,\sigma) \end{bmatrix}_{6\times 6}, \qquad \Gamma^{(j)} = \Big[\Gamma^{(j)}_{pq}\Big]_{3\times 3}, \tag{3.4}$$

then (3.3) is equivalent to the relations

$$
\begin{aligned}
&L^{(1)}(\partial,\sigma)\Gamma^{(1)}(x,\sigma)+L^{(2)}(\partial,\sigma)\Gamma^{(3)}(x,\sigma)=I_3\,\delta(x),\\
&L^{(3)}(\partial,\sigma)\Gamma^{(1)}(x,\sigma)+L^{(4)}(\partial,\sigma)\Gamma^{(3)}(x,\sigma)=0,\\
&L^{(1)}(\partial,\sigma)\Gamma^{(2)}(x,\sigma)+L^{(2)}(\partial,\sigma)\Gamma^{(4)}(x,\sigma)=0,\\
&L^{(3)}(\partial,\sigma)\Gamma^{(2)}(x,\sigma)+L^{(4)}(\partial,\sigma)\Gamma^{(4)}(x,\sigma)=I_3\,\delta(x),
\end{aligned}
\tag{3.5}
$$

Applying the Fourier transform to the equation (3.3), and taking into consideration (3.2) and the equality $\mathcal{F}[\delta(\cdot)]=1$, we get

$$L(-i\,\xi,\sigma)\,\widehat{\Gamma}(\xi,\sigma)=I_6. \tag{3.6}$$

We assume that the frequency parameter σ is complex, in general:

$$\sigma=\sigma_1+i\,\sigma_2,\quad \sigma_1,\sigma_2\in I\!R^1. \tag{3.7}$$

We have to determine $\widehat{\Gamma}(\xi,\sigma)$ from (3.6) and afterwards, by inverting the Fourier transform, to construct explicitly the fundamental matrix $\Gamma(x,\sigma)$.

To this end, first of all we have to find $L^{-1}(-i\,\xi,\sigma)$. We set

$$
\begin{aligned}
&r:=|\xi|=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2},\\
&A(\xi)=L^{(1)}(-i\,\xi,\sigma)=[-(\mu+\alpha)r^2+\rho\sigma^2]I_3-(\lambda+\mu-\alpha)Q(\xi),\\
&B(\xi)=L^{(2)}(-i\,\xi,\sigma)=-(\kappa+\nu)r^2I_3-(\delta+\kappa-\nu)Q(\xi)-i\,2\alpha\,R(\xi),\\
&D(\xi)=L^{(4)}(-i\,\xi,\sigma)=[(\mathcal{I}\sigma^2-4\alpha)-(\gamma+\varepsilon)r^2]I_3\\
&\qquad -(\beta+\gamma-\varepsilon)Q(\xi)-i\,4\nu\,R(\xi),
\end{aligned}
\tag{3.8}
$$

where $R(\cdot)$ and $Q(\cdot)$ are defined by (2.11).

It is evident that

$$L(-i\,\xi,\sigma)=\begin{bmatrix} A & B\\ B & D\end{bmatrix}_{6\times 6}. \tag{3.9}$$

Note that, due to (2.9)-(2.13),

$$
\begin{aligned}
&L(-i\xi,\sigma)=[L(i\xi,\sigma)]^\top=\overline{L(i\xi,\overline{\sigma})},\\
&A(\xi)=A(-\xi)=A^\top(\xi),\ \ B(\xi)=B^\top(-\xi),\ \ D(\xi)=D^\top(-\xi),\\
&Q(\xi)=[Q(\xi)]^\top,\quad [R(\xi)]^\top=-R(\xi)=R(-\xi),\\
&Q(\xi)R(\xi)=R(\xi)Q(\xi)=0,[Q(\xi)]^2=r^2Q(\xi),[R(\xi)]^2=Q(\xi)-r^2\,I_3.
\end{aligned}
\tag{3.10}
$$

Therefore the matrices A, B, and D commute to each other, which allows us to apply the Schur formula (see, e.g., [12], Ch.2, §5)

$$\det L(-i\,\xi,\sigma)=\det\begin{bmatrix} A & B\\ B & D\end{bmatrix}=\det\,[AD-B^2]. \tag{3.11}$$

By (3.8) we get

$$
\begin{aligned}
AD = &\{(\mu+\alpha)(\gamma+\varepsilon)r^4 - [(\mu+\alpha)(\mathcal{I}\sigma^2-4\alpha)+(\gamma+\varepsilon)\rho\sigma^2]r^2 \\
&+\rho\sigma^2(\mathcal{I}\sigma^2-4\alpha)\}\, I_3 + \{r^2[(\mu+\alpha)(\beta+\gamma-\varepsilon)+(\lambda+\mu-\alpha)(\gamma+\varepsilon) \\
&+(\lambda+\mu-\alpha)(\beta+\gamma-\varepsilon)] - (\lambda+\mu-\alpha)(\mathcal{I}\sigma^2-4\alpha) \\
&-(\beta+\gamma-\varepsilon)\rho\sigma^2\}\, Q(\xi) + i\, 4\nu\, [(\mu+\alpha)r^2-\rho\sigma^2]\, R(\xi),
\end{aligned}
$$

$$
\begin{aligned}
B^2 = &\{(\kappa+\nu)^2\, r^4 + 4\alpha^2\, r^2\}\, I_3 + i\, 4\, \alpha\, (\kappa+\nu)\, r^2\, R(\xi) \\
&+ \left\{ \left[(\delta+\kappa-\nu)^2 + 2(\kappa+\nu)(\delta+\kappa-\nu)\right] r^2 - 4\alpha^2 \right\}\, Q(\xi).
\end{aligned}
$$

Whence

$$
AD - B^2 = aI_3 + bQ(\xi) + icR(\xi), \tag{3.12}
$$

where

$$
\begin{aligned}
a = &\, r^4[(\mu+\alpha)(\gamma+\varepsilon) - (\kappa+\nu)^2] \\
&-r^2[(\mu+\alpha)(\mathcal{I}\sigma^2-4\alpha)+(\gamma+\varepsilon)\rho\sigma^2+4\alpha^2] \\
&+\rho\sigma^2(\mathcal{I}\sigma^2-4\alpha),
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
b = &\, r^2[(\mu+\alpha)(\beta+\gamma-\varepsilon)+(\lambda+\mu-\alpha)(\gamma+\varepsilon) \\
&+(\lambda+\mu-\alpha)(\beta+\gamma-\varepsilon) - (\delta+\kappa-\nu)^2 \\
&-2(\kappa+\nu)(\delta+\kappa-\nu)] - [(\beta+\gamma-\varepsilon)\rho\sigma^2 \\
&+(\lambda+\mu-\alpha)(\mathcal{I}\sigma^2-4\alpha)-4\alpha^2],
\end{aligned} \tag{3.14}
$$

$$
c = r^2\, [4\nu(\mu+\alpha) - 4\alpha(\kappa+\nu)] - 4\nu\rho\sigma^2. \tag{3.15}
$$

By direct calculations we arrive at the equality

$$
\det\,[AD-B^2] = \begin{vmatrix} a+b\xi_1^2 & b\xi_1\xi_2 - ic\xi_3 & b\xi_1\xi_3 + ic\xi_2 \\ b\xi_1\xi_2 + ic\xi_3 & a+b\xi_2^2 & b\xi_2\xi_3 - ic\xi_1 \\ b\xi_1\xi_3 - ic\xi_2 & b\xi_2\xi_3 + ic\xi_1 & a+b\xi_3^2 \end{vmatrix}
$$

$$
\begin{aligned}
&= a^3 + a^2br^2 - bc^2[2\xi_1^2\xi_3^2 + 2\xi_2^2\xi_3^2 + 2\xi_1^2\xi_2^2 + \xi_1^4 + \xi_2^4 + \xi_3^4] - ac^2r^2 \\
&= a^3 + a^2br^2 - bc^2(\xi_1^2+\xi_2^2+\xi_3^2)^2 - ac^2r^2 \\
&= a^2(a+br^2) - c^2r^2(br^2+a) = (a+br^2)(a^2-c^2r^2).
\end{aligned}
$$

Thus

$$
\det L(-i\,\xi,\sigma) = \det[AD-B^2] = (a+br^2)(a^2-c^2r^2). \tag{3.16}
$$

Note that

$$
\begin{aligned}
a + br^2 = &\, r^4[(\lambda+2\mu)(\beta+2\gamma) - (\delta+2\kappa)^2] - r^2[(\beta+2\gamma)\rho\sigma^2 \\
&+(\lambda+2\mu)(\mathcal{I}\sigma^2-4\alpha)] + \rho\sigma^2(\mathcal{I}\sigma^2-4\alpha).
\end{aligned} \tag{3.17}
$$

On the one hand

$$a^2 - c^2r^2 = (a + cr)(a - cr), \tag{3.18}$$

where $(a \pm cr)$ are fourth order polynomials in r

$$\begin{aligned} a \pm cr = {} & r^4[(\mu + \alpha)(\gamma + \varepsilon) - (\kappa + \nu)^2] \pm r^3[4\nu(\mu + \alpha) - 4\alpha(\kappa + \nu)] \\ & -r^2[(\mu + \alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma + \varepsilon)\rho\sigma^2 + 4\alpha^2] \mp r4\nu\rho\sigma^2 \\ & +\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha). \end{aligned} \tag{3.19}$$

On the other hand

$$\begin{aligned} a^2 - c^2r^2 = {} & r^8[(\mu + \alpha)(\gamma + \varepsilon) - (\kappa + \nu)^2]^2 - r^6\{2[(\mu + \alpha)(\gamma + \varepsilon) \\ & -(\kappa + \nu)^2][(\mu + \alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma + \varepsilon)\rho\sigma^2 + 4\alpha^2] \\ & +[4\alpha(\kappa + \nu) - 4\nu(\mu + \alpha)]^2\} + r^4\{[(\mu + \alpha)(\mathcal{I}\sigma^2 - 4\alpha) \\ & +(\gamma + \varepsilon)\rho\sigma^2 + 4\alpha^2]^2 + 2\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)[(\mu + \alpha)(\gamma + \varepsilon) \\ & -(\kappa + \nu)^2] + 8\nu\rho\sigma^2[4\nu(\mu + \alpha) - 4\alpha(\kappa + \nu)]\} \\ & -r^2\{2\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)[(\mu + \alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma + \varepsilon)\rho\sigma^2 + 4\alpha^2] \\ & +16\nu^2\rho^2\sigma^4\} + \rho^2\sigma^4(\mathcal{I}\sigma^2 - 4\alpha)^2. \end{aligned} \tag{3.20}$$

Due to the evenness of the functions $a(r)$, $b(r)$, and $c(r)$ it is evident that if $r = r_0$ is a root of either the equation $a(r) + b(r)\, r^2 = 0$ or $a(r) - c(r)\, r^2 = 0$, then so is $r = -r_0$.

Denote the roots of the equation $a + br^2 = 0$ by $\pm k_1$ and $\pm k_2$. Similarly, let the roots of the equation $a^2 - r^2c^2 = 0$ be $\pm k_3$, $\pm k_4$, $\pm k_5$, and $\pm k_6$. Then

$$\begin{aligned} & a + br^2 = d_2(r^2 - k_1^2)(r^2 - k_2^2), \\ & a^2 - c^2r^2 = d_1^2(r^2 - k_3^2)(r^2 - k_4^2)(r^2 - k_5^2)(r^2 - k_6^2), \end{aligned} \tag{3.21}$$

where (for simplicity) we assume that

$$k_j \neq k_p \ \text{ for } \ j \neq p, \ \ \Im k_j \geq 0, \ \text{ and if } \ \Im k_j = 0, \ \text{ then } \ k_j > 0. \tag{3.22}$$

Note that $k_1 > 0$ and $k_2 > 0$ for $\mathcal{I}\sigma^2 - 4\alpha > 0$.

By (3.16), (3.21) and (3.22) we conclude that

$$\begin{aligned} \det L(-i\,\xi, \sigma) = \det[AD - B^2] &= (a + br^2)(a^2 - c^2r^2) \\ &= d_1^2\, d_2 \prod_{j=1}^{6}(r^2 - k_j^2). \end{aligned} \tag{3.23}$$

Now let us recall (3.5) and (3.8) to write

$$\begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \widehat{\Gamma}^{(1)}(\xi,\sigma) & \widehat{\Gamma}^{(2)}(\xi,\sigma) \\ \widehat{\Gamma}^{(3)}(\xi,\sigma) & \widehat{\Gamma}^{(4)}(\xi,\sigma) \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_3 \end{bmatrix}. \tag{3.24}$$

Whence

$$\begin{cases} A\widehat{\Gamma}^{(1)}(\xi,\sigma) + B\widehat{\Gamma}^{(3)}(\xi,\sigma) = I_3, \\ B\widehat{\Gamma}^{(1)}(\xi,\sigma) + D\widehat{\Gamma}^{(3)}(\xi,\sigma) = 0, \end{cases} \qquad \begin{cases} A\widehat{\Gamma}^{(2)}(\xi,\sigma) + B\widehat{\Gamma}^{(4)}(\xi,\sigma) = 0, \\ B\widehat{\Gamma}^{(2)}(\xi,\sigma) + D\widehat{\Gamma}^{(4)}(\xi,\sigma) = I_3. \end{cases}$$

In turn these relations yield

$$\begin{cases} (AD - B^2)\widehat{\Gamma}^{(1)}(\xi,\sigma) = D, \\ (AD - B^2)\widehat{\Gamma}^{(3)}(\xi,\sigma) = -B, \end{cases} \quad \begin{cases} (AD - B^2)\widehat{\Gamma}^{(2)}(\xi,\sigma) = -B, \\ (AD - B^2)\widehat{\Gamma}^{(4)}(\xi,\sigma) = A. \end{cases} \tag{3.25}$$

Denote (cf. (3.12))

$$M := AD - B^2 = aI_3 + bQ(\xi) + i\,cR(\xi). \tag{3.26}$$

From (3.25) it follows that

$$\begin{aligned} &\widehat{\Gamma}^{(1)}(\xi,\sigma) = M^{-1}D, \\ &\widehat{\Gamma}^{(2)}(\xi,\sigma) = \widehat{\Gamma}^{(3)}(\xi,\sigma) = -M^{-1}B, \quad \widehat{\Gamma}^{(4)}(\xi,\sigma) = M^{-1}A. \end{aligned} \tag{3.27}$$

It can be easily checked that

$$\widehat{\Gamma}(\xi,\sigma) = \begin{bmatrix} M^{-1}D & -M^{-1}B \\ -M^{-1}B & M^{-1}A \end{bmatrix} = L^{-1}(-i\,\xi,\sigma). \tag{3.28}$$

Remark that $\det M = \det[AD - B^2] = \det L(-i\,\xi,\sigma)$ is given by (3.23). Let us construct the matrix inverse to M. To this end first we construct the matrix $M^* = [M^*_{kj}]_{3\times 3}$ adjoint to M. The matrix M can be written as follows

$$M = \begin{bmatrix} a + b\xi_1^2 & b\xi_1\xi_2 - ic\xi_3 & b\xi_1\xi_3 + ic\xi_2 \\ b\xi_1\xi_2 + ic\xi_3 & a + b\xi_2^2 & b\xi_2\xi_3 - ic\xi_1 \\ b\xi_1\xi_3 - ic\xi_2 & b\xi_2\xi_3 + ic\xi_1 & a + b\xi_3^2 \end{bmatrix}. \tag{3.29}$$

Therefore we easily get

$$M^*_{11} = \begin{vmatrix} a + b\xi_2^2 & b\xi_2\xi_3 - i\,c\xi_1 \\ b\xi_2\xi_3 + i\,c\xi_1 & a + b\xi_3^2 \end{vmatrix} = a(a + br^2) - (ab + c^2)\xi_1^2,$$

$$M^*_{21} = -\begin{vmatrix} b\xi_1\xi_2 + i\,c\xi_3 & b\xi_2\xi_3 - i\,c\xi_1 \\ b\xi_1\xi_3 - i\,c\xi_2 & a + b\xi_3^2 \end{vmatrix} = -i\,(a + br^2)c\xi_3 - (ab + c^2)\xi_1\xi_2,$$

$$M^*_{31} = \begin{vmatrix} b\xi_1\xi_2 + ic\xi_3 & a + b\xi_2^2 \\ b\xi_1\xi_3 - ic\xi_2 & b\xi_2\xi_3 + ic\xi_1 \end{vmatrix} = -(ab + c^2)\xi_1\xi_3 + i(a + br^2)c\xi_2,$$

$$M^*_{12} = -\begin{vmatrix} b\xi_1\xi_2 - ic\xi_3 & b\xi_1\xi_3 + ic\xi_2 \\ b\xi_2\xi_3 + ic\xi_1 & a + b\xi_3^2 \end{vmatrix} = i(a+br^2)c\xi_3 - (ab+c^2)\xi_1\xi_2,$$

$$M^*_{22} = \begin{vmatrix} a + b\xi_1^2 & b\xi_1\xi_3 + ic\xi_2 \\ b\xi_1\xi_3 - ic\xi_2 & a + b\xi_3^2 \end{vmatrix} = a(a+br^2) - (ab+c^2)\xi_2^2,$$

$$M^*_{32} = -\begin{vmatrix} a + b\xi_1^2 & b\xi_1\xi_2 - ic\xi_3 \\ b\xi_1\xi_3 - ic\xi_2 & b\xi_2\xi_3 + ic\xi_1 \end{vmatrix} = -(ab+c^2)\xi_2\xi_3 - i(a+br^2)c\xi_1,$$

$$M^*_{13} = \begin{vmatrix} b\xi_1\xi_2 - ic\xi_3 & b\xi_1\xi_3 + ic\xi_2 \\ a + b\xi_2^2 & b\xi_2\xi_3 - ic\xi_1 \end{vmatrix} = -(ab+c^2)\xi_1\xi_3 - i(a+br^2)c\xi_2,$$

$$M^*_{23} = -\begin{vmatrix} a + b\xi_1^2 & b\xi_1\xi_3 + ic\xi_2 \\ b\xi_1\xi_2 + ic\xi_3 & b\xi_2\xi_3 - ic\xi_1 \end{vmatrix} = -(ab+c^2)\xi_2\xi_3 + i\,(a+br^2)c\xi_1,$$

$$M^*_{33} = \begin{vmatrix} a + b\xi_1^2 & b\xi_1\xi_2 - ic\xi_3 \\ b\xi_1\xi_3 + ic\xi_3 & a + b\xi_2^2 \end{vmatrix} = a(a+br^2) - (ab+c^2)\xi_3^2.$$

These formulae imply

$$M^*(\xi) = [M^*_{kj}] = a(a+br^2)I_3 - (ab+c^2)Q(\xi) - i\,c(a+br^2)R(\xi). \quad (3.30)$$

Therefore we finally have

$$M^{-1} = \frac{1}{\det M(\xi)} M^*(\xi) = \frac{1}{(a+br^2)(a^2-c^2r^2)}[a(a+br^2)I_3$$
$$-(ab+c^2)Q(\xi) - ic(a+br^2)R(\xi)]. \quad (3.31)$$

Since M^* commutes with the matrices A, B, and D, from (3.28) it follows that

$$\begin{aligned}\widehat{\Gamma}(\xi,\sigma) & \\ &= \begin{bmatrix} D(\xi) & -B(\xi) \\ -B(\xi) & A(\xi) \end{bmatrix} \begin{bmatrix} M^*(\xi) & 0 \\ 0 & M^*(\xi) \end{bmatrix} \frac{1}{(a+br^2)(a^2-c^2r^2)} \\ &= \begin{bmatrix} D(\xi) & -B(\xi) \\ -B(\xi) & A(\xi) \end{bmatrix} \begin{bmatrix} M^*(\xi) & 0 \\ 0 & M^*(\xi) \end{bmatrix} \frac{1}{d_1^2\, d_2 \prod_{j=1}^6 (r^2 - k_j^2)},\end{aligned} \quad (3.32)$$

By the inverse Fourier transform we conclude

$$\Gamma(x,\sigma) = \mathcal{F}^{-1}_{\xi\to x}[\,\widehat{\Gamma}(\xi,\sigma)\,] = \frac{1}{d_1^2\, d_2} \begin{bmatrix} D(i\partial) & -B(i\partial) \\ -B(i\partial) & A(i\partial) \end{bmatrix}$$
$$\times \begin{bmatrix} M^*(i\partial) & 0 \\ 0 & M^*(i\partial) \end{bmatrix} \mathcal{F}^{-1}\left[\frac{1}{\prod_{j=1}^6 (r^2 - k_j^2)}\right] \quad (3.33)$$

where

$$\begin{aligned} A(i\partial) &= L^{(1)}(-i\,\xi,\sigma)|_{\xi=i\partial} = L^{(1)}(\partial,\sigma), \\ B(i\partial) &= L^{(2)}(-i\,\xi,\sigma)|_{\xi=i\partial} = L^{(2)}(\partial,\sigma), \\ D(i\partial) &= L^{(4)}(-i\,\xi,\sigma)|_{\xi=i\partial} = L^{(4)}(\partial,\sigma), \end{aligned} \tag{3.34}$$

while

$$\begin{aligned} M^*(i\partial) = M^*(\xi)|_{\xi=i\partial} &= a^*(\partial)[a^*(\partial) - b^*(\partial)\Delta]I_3 \\ &+ \left[a^*(\partial)b^*(\partial) + [c^*(\partial)]^2\right] Q(\partial) + c^*(\partial)[a^*(\partial) \\ &-b^*(\partial)\Delta]R(\partial) =: \widetilde{M^*}(\partial) \end{aligned} \tag{3.35}$$

with

$$\begin{aligned} a^{(*)}(\partial) &= [(\mu+\alpha)(\gamma+\varepsilon) - (\kappa+\nu)^2]\Delta\Delta + [(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) \\ &\quad +(\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2]\Delta + \rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha), \\ b^{(*)}(\partial) &= -[(\mu+\alpha)(\beta+\gamma-\varepsilon) + (\lambda+\mu-\alpha)(\beta+2\gamma) \\ &\quad -(\delta+\kappa-\nu)^2 - 2(\kappa+\nu)(\delta+\kappa-\nu)]\Delta \\ &\quad -[(\beta+\gamma-\varepsilon)\rho\sigma^2 + (\lambda+\mu-\alpha)(\mathcal{I}\sigma^2 - 4\alpha) - 4\alpha^2], \\ c^{(*)}(\partial) &= 4\,[\alpha(\kappa+\nu) - \nu(\mu+\alpha)]\Delta - 4\nu\rho\sigma^2. \end{aligned} \tag{3.36}$$

To simplify (3.33) we apply the following representation and implications:

$$\begin{aligned} \frac{1}{\prod_{j=1}^{6}(r^2 - k_j^2)} = \sum_{j=1}^{6} \frac{p_j}{(r^2 - k_j^2)} \quad &\Longrightarrow 1 = \sum_{j=1}^{6} p_j \left[\prod_{l=1, l\neq j}^{6} (r^2 - k_l^2)\right] \\ &\Longrightarrow \quad p_j = \left[\prod_{l=1, l\neq j}^{6} (k_j^2 - k_l^2)\right]^{-1}. \end{aligned} \tag{3.37}$$

Note that, if $\Im k_j > 0$, then in the space of tempered distributions $\mathcal{S}'(I\!R^3)$ there holds

$$\begin{aligned} \mathcal{F}^{-1}\left[\frac{1}{r^2 - k_j^2}\right] &= \frac{1}{(2\pi)^3} \lim_{R\to+\infty} \int\limits_{|\xi|<R} \frac{e^{-i\,x\cdot\xi}}{|\xi|^2 - k_j^2}\, d\xi \\ &= \frac{1}{(2\pi)^3} \lim_{R\to+\infty} \int\limits_{|\xi|<R} \frac{e^{i\,x\cdot\xi}}{|\xi|^2 - k_j^2}\, d\xi = -\frac{e^{ik_j|x|}}{4\pi|x|}, \end{aligned} \tag{3.38}$$

where the limits are understood in the sense of the space $\mathcal{S}'(I\!R^3)$, while for $\Im k_j = 0$ we have (the *limiting absorption principle*)

$$\mathcal{F}^{-1}\left[\frac{1}{r^2 - k_j^2}\right] = \mathcal{F}^{-1}\left[\frac{1}{r^2 - (k_j + i\,0)^2}\right] = -\frac{e^{ik_j|x|}}{4\pi|x|}. \tag{3.39}$$

The above equalities along with (3.33) give the following form for the fundamental matrix

$$\Gamma(x,\sigma) = -\frac{1}{4\pi d_1^2\, d_2}\begin{bmatrix} L^{(4)}(\partial,\sigma) & -L^{(2)}(\partial,\sigma) \\ -L^{(2)}(\partial,\sigma) & L^{(1)}(\partial,\sigma) \end{bmatrix}$$
$$\times \begin{bmatrix} \widetilde{M}^*(\partial) & 0 \\ 0 & \widetilde{M}^*(\partial) \end{bmatrix} \Psi(x,\sigma)$$
$$= -\frac{1}{4\pi\, d_1^2\, d_2}\begin{bmatrix} L^{(4)}(\partial,\sigma)\widetilde{M}^*(\partial) & -L^{(2)}(\partial,\sigma)\widetilde{M}^*(\partial) \\ -L^{(2)}(\partial,\sigma)\widetilde{M}^*(\partial) & L^{(1)}(\partial,\sigma)\widetilde{M}^*(\partial) \end{bmatrix} \Psi(x,\sigma), \tag{3.40}$$

where

$$\Psi(x,\sigma) = \sum_{j=1}^{6} p_j\, \frac{e^{ik_j|x|}}{|x|} \tag{3.41}$$

with k_j satisfying (3.22).

From the equality

$$p_1(r^2-k_2^2)\dots(r^2-k_6^2) + p_2(r^2-k_1^2)(r^2-k_3^2)\dots(r^2-k_6^2) + \dots + p_6(r^2-k_1^2)\dots(r^2-k_5^2) = 1, \tag{3.42}$$

it follows that the numbers p_j possess the properties

$$k_1^{2m}p_1 + \dots + k_6^{2m}p_6 = 0, \;\; m=\overline{0,4}, \quad k_1^{10}p_1 + \dots + k_6^{10}p_6 = 1. \tag{3.43}$$

In view of (3.43) it follows that

$$\sum_{j=1}^{6}\frac{p_j}{|x|}e^{ik_j|x|} = \sum_{j=1}^{6}\frac{p_j}{|x|}\sum_{q=0}^{\infty}\frac{(ik_j|x|)^q}{q!} = \sum_{j=1}^{6}p_j\sum_{q=0}^{\infty}\frac{(ik_j)^q}{q!}|x|^{q-1}$$
$$= \sum_{q=0}^{\infty}\left\{\frac{|x|^{q-1}i^q}{q!}\sum_{j=1}^{6}p_jk_j^q\right\} = \sum_{q\in\{1,3,5,7,9\}}\left\{\frac{|x|^{q-1}i^q}{q!}\sum_{j=1}^{6}p_jk_j^q\right\}$$
$$-\frac{|x|^9}{10!} + \sum_{q=11}^{\infty}\sum_{j=1}^{6}\frac{|x|^{q-1}i^qp_jk_j^q}{q!} \tag{3.44}$$

which shows that the fundamental solution (3.40) has the singularity $\mathcal{O}(|x|^{-1})$ in a vicinity of the origin, since the entries of the matrix $L^{(j)}(\partial,\sigma)\,\widetilde{M}^*(\partial)$ $(j=\overline{1,4})$ are differential operators of order 10.

Remark 3.1 *Note that* (3.40) *can be written in the form*

$$\Gamma(x,\sigma) = \sum_{j=1}^{6}\widetilde{\Gamma}^{(j)}(x,\sigma), \tag{3.45}$$

where

$$\widetilde{\Gamma}^{(j)}(x,\sigma)$$
$$= -\frac{p_j}{4\pi\, d_1^2\, d_2} \begin{bmatrix} L^{(4)}(\partial,\sigma)\widetilde{M}^*(\partial) & -L^{(2)}(\partial,\sigma)\widetilde{M}^*(\partial) \\ -L^{(2)}(\partial,\sigma)\widetilde{M}^*(\partial) & L^{(1)}(\partial,\sigma)\widetilde{M}^*(\partial) \end{bmatrix} \frac{e^{ik_j|x|}}{|x|}. \qquad (3.46)$$

This representation shows that the entries of the matrix $\widetilde{\Gamma}^{(j)}(x,\sigma)$ *and its derivatives satisfy the Sommerfeld radiation conditions at infinity:*

$$\frac{\partial}{\partial|x|}[\widetilde{\Gamma}^{(j)}(x,\sigma)]_{pq} - i\,k_j[\widetilde{\Gamma}^{(j)}(x,\sigma)]_{pq} = \exp\{-\Im k_j|x|\}\,\mathcal{O}(|x|^{-2}) \qquad (3.47)$$

as $|x| \to +\infty$.

Remark 3.2 *The entries of the matrix* $\widetilde{\Gamma}^{(j)}(x,\sigma)$ *and its derivatives satisfy also the following conditions at infinity:*

$$\frac{\partial}{\partial x_l}[\widetilde{\Gamma}^{(j)}(x,\sigma)]_{pq} - i\,k_j\,\frac{x_l}{|x|}[\widetilde{\Gamma}^{(j)}(x,\sigma)]_{pq} = \exp\{-\Im k_j|x|\}\,\mathcal{O}(|x|^{-2}). \qquad (3.48)$$

These asymptotic equalities can be differentiated any times with respect to the variable x.

Remark 3.3 *Note that*

$$\Gamma(-x,\sigma) = \frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3} [L(-i\,\xi,\sigma)]^{-1}\,e^{i\,x\cdot\xi}\,d\xi = \frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3} [L(i\,\xi,\sigma)]^{-1}\,e^{-i\,x\cdot\xi}\,d\xi$$
$$= \frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3} [L^\top(-i\,\xi,\sigma)]^{-1}\,e^{-i\,x\cdot\xi}\,d\xi = [\Gamma(x,\sigma)]^\top,$$

where the above formal integrals are understood as generalized Fourier transforms.

Remark 3.4 *In the case of repeated roots* (*i.e., when* (3.22) *is violated*) *the fundamental solution can be obtained from* (3.40) *by the limiting procedure.*

Remark 3.5 *More careful analysis show that for* $\sigma = 0$

$$\partial^\alpha\,\Gamma_{kj}(x,0) = \mathcal{O}(|x|^{-1-|\alpha|})$$

for $k,j = \overline{1,6}$, *and moreover,*

$$\partial^\alpha\,\Gamma_{kj}(x,0) = \mathcal{O}(|x|^{-2-|\alpha|})$$

for either $k \geq 4$ *or* $j \geq 4$ *as* $|x| \to +\infty$.

3.2. Principal singular part of the matrix (3.40)

In this subsection we will construct explicitly the principal singular part of the fundamental matrix (3.40) of the operator $L(\partial, \sigma)$. This principal part $\Gamma_0(x)$ represents a fundamental matrix of the operator $L_0(\partial)$ defined by (2.16) and solves the equation:

$$L_0(\partial)\,\Gamma_0(x) = \delta(x)\, I_6. \tag{3.49}$$

By the same approach as above, with the help of the generalized Fourier transform, we reduce (3.49) to the equation in the space of tempered distributions $\mathcal{S}'(I\!R^3)$

$$L_0(-i\,\xi)\,\widehat{\Gamma}_0(\xi) = I_6, \tag{3.50}$$

where

$$L_0(-i\,\xi) = -L_0(\xi) = \begin{bmatrix} A_0(\xi) & B_0(\xi) \\ B_0(\xi) & D_0(\xi) \end{bmatrix}, \quad \widehat{\Gamma}_0 = \begin{bmatrix} \widehat{\Gamma}_0^{(1)} & \widehat{\Gamma}_0^{(2)} \\ \widehat{\Gamma}_0^{(3)} & \widehat{\Gamma}_0^{(4)} \end{bmatrix}, \tag{3.51}$$

$$\begin{aligned} A_0(\xi) &= -(\mu+\alpha)r^2 I_3 - (\lambda+\mu-\alpha)Q(\xi), \\ B_0(\xi) &= C_0(\xi) = -(\kappa+\nu)r^2 I_3 - (\delta+\kappa-\nu)Q(\xi), \\ D_0(\xi) &= -(\gamma+\varepsilon)r^2 I_3 - (\beta+\gamma-\varepsilon)Q(\xi). \end{aligned} \tag{3.52}$$

The equation (3.50) is equivalent to the relations

$$\begin{aligned} &A_0\widehat{\Gamma}_0^{(1)} + B_0\widehat{\Gamma}_0^{(3)} = I_3, \quad B_0\widehat{\Gamma}_0^{(1)} + D_0\widehat{\Gamma}_0^{(3)} = 0, \\ &A_0\widehat{\Gamma}_0^{(2)} + B_0\widehat{\Gamma}_0^{(4)} = 0, \quad B_0\widehat{\Gamma}_0^{(2)} + D_0\widehat{\Gamma}_0^{(4)} = I_3, \end{aligned} \tag{3.53}$$

whence it follows that

$$\begin{aligned} &(A_0D_0 - B_0^2)\widehat{\Gamma}_0^{(1)} = D_0, \quad (A_0D_0 - B_0^2)\widehat{\Gamma}_0^{(3)} = -B_0, \\ &(A_0D_0 - B_0^2)\widehat{\Gamma}_0^{(4)} = A_0, \quad (A_0D_0 - B_0^2)\widehat{\Gamma}_0^{(2)} = -B_0. \end{aligned} \tag{3.54}$$

Therefore

$$\widehat{\Gamma}_0 = \begin{bmatrix} M_0^{-1}D_0 & -M_0^{-1}B_0 \\ -M_0^{-1}B_0 & M_0^{-1}A_0 \end{bmatrix}, \tag{3.55}$$

where $M_0 = A_0D_0 - B_0^2$. Note that (cf. (3.16))

$$\det M_0 = \det[A_0D_0 - B_0^2] = \det[a_0 I_3 + b_0 Q(\xi)]$$

with (see (2.26))

$$\begin{aligned} a_0 &= [(\mu+\alpha)(\gamma+\varepsilon) - (\kappa+\nu)^2]r^4 = d_1 r^4, \\ b_0 &= [(\lambda+2\mu)(\beta+\gamma-\varepsilon) + (\gamma+\varepsilon)(\lambda+\mu-\alpha) - (\delta+\kappa-\nu)^2 \\ &\qquad -2(\kappa+\nu)(\delta+\kappa-\nu)]r^2 = (d_2 - d_1)r^2. \end{aligned} \tag{3.56}$$

Simple calculations lead to the equality

$$\det M_0 = \det[a_0 I_3 + b_0 Q(\xi)] = (a_0 + b_0 r^2) a_0^2 = d_1^2\, d_2\, r^{12}. \tag{3.57}$$

Moreover, the matrix $M_0^*(\xi)$, adjoint to $M_0(\xi)$, reads as follows (cf. (3.30))

$$M_0^*(\xi) = d_1\, r^6\, [d_2\, r^2\, I_3 - (d_2 - d_1) Q(\xi)]. \tag{3.58}$$

Therefore, the matrix $M_0^{-1}(\xi)$, inverse to $M_0(\xi)$, has the form

$$M_0^{-1}(\xi) = \frac{1}{\det M_0^1(\xi)}\, M_0^*(\xi) = \frac{1}{d_1\, d_2\, r^6} [d_2\, r^2\, I_3 - (d_2 - d_1) Q(\xi)]. \tag{3.59}$$

Taking into consideration that

$$\begin{aligned} M_0^{-1}(\xi) A_0(\xi) = \frac{-1}{d_1\, d_2\, r^4} \big[& d_2(\mu + \alpha) r^2 I_3 \\ & + \{\lambda \delta_2 + (2d_1 - d_2)\mu - d_2 \alpha\} Q(\xi) \big], \end{aligned} \tag{3.60}$$

$$\begin{aligned} M_0^{-1}(\xi) B_0(\xi) = \frac{-1}{d_1 d_2 r^4} \big[& d_2(\kappa + \nu) r^2 I_3 \\ & + \{\delta d_1 + \kappa(2d_1 - d_2) - d_2 \nu\} Q(\xi) \big], \end{aligned} \tag{3.61}$$

$$\begin{aligned} M_0^{-1}(\xi) D_0(\xi) = \frac{-1}{d_1 d_2 r^4} \big[& d_2(\gamma + \varepsilon) r^2 I_3 \\ & + \{\beta d_1 + \gamma(2d_1 - d_2) - d_2 \varepsilon\} Q(\xi) \big], \end{aligned} \tag{3.62}$$

from (3.55) we get

$$\widehat{\Gamma}_0(\xi) = -\frac{1}{d_1} \Lambda_1 \frac{1}{r^2} - \frac{1}{d_1 d_2} \Lambda_2 \frac{1}{r^4}, \tag{3.63}$$

where

$$\Lambda_1 = \begin{bmatrix} (\gamma + \varepsilon) I_3 & -(\kappa + \nu) I_3 \\ -(\kappa + \nu) I_3 & (\mu + \alpha) I_3 \end{bmatrix}, \tag{3.64}$$

$$\Lambda_2 =$$

$$\begin{bmatrix} [\beta d_1 + (2d_1 - d_2)\gamma - d_2 \varepsilon] Q(x) & -[\delta d_1 + (2d_1 - d_2)\kappa - d_2 \nu] Q(x) \\ -[\delta d_1 + (2d_1 - d_2)\kappa - d_2 \nu] Q(x) & [\lambda d_1 + (2d_1 - d_2)\mu - d_2 \alpha] Q(x) \end{bmatrix} \tag{3.65}$$

We arrive at the equality

$$\begin{aligned} \Gamma_0(x) = \mathcal{F}^{-1}[\widehat{\Gamma}_0(\xi)] &= -\frac{1}{d_1} \Lambda_1 \mathcal{F}^{-1} \left[\frac{1}{r^2} \right] - \frac{1}{d_1 d_2} \mathcal{F}^{-1} \left[\Lambda_2 \frac{1}{r^4} \right] \\ &= -\frac{1}{8\pi d_1 d_2 |x|} \left\{ \begin{bmatrix} A^{(1)} & A^{(2)} \\ A^{(2)} & A^{(3)} \end{bmatrix} - \begin{bmatrix} B^{(1)}(x) & B^{(2)}(x) \\ B^{(2)}(x) & B^{(3)}(x) \end{bmatrix} \right\} \end{aligned} \tag{3.66}$$

with

$$\begin{aligned}
A^{(1)} &= [d_2(\gamma+\varepsilon)+d_1(\beta+2\gamma)]I_3,\\
A^{(2)} &= -[d_2(\kappa+\nu)+d_1(\delta+2\kappa)]I_3,\\
A^{(3)} &= [d_2(\mu+\alpha)+d_1(\lambda+2\mu)]I_3,\\
B^{(1)}(x) &= [d_1(\beta+2\gamma)-d_2(\gamma+\varepsilon)]|x|^{-2}Q(x),\\
B^{(2)}(x) &= -[d_1(\delta+2\kappa)-d_2(\kappa+\nu)]|x|^{-2}Q(x),\\
B^{(3)}(x) &= [d_1(\lambda+2\mu)-d_2(\mu+\alpha)]|x|^{-2}Q(x).
\end{aligned}$$

We can easily see that the entries of the matrix $\Gamma_0(x)$ are homogeneous functions of order -1 and in a vicinity of the origin (i.e., for small $|x|$)

$$\partial^\alpha\,[\Gamma(x,\sigma)-\Gamma_0(x)] = \mathcal{O}(|x|^{-|\alpha|}) \tag{3.67}$$

for an arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and an arbitrary complex number σ, which shows that $\Gamma_0(x)$ is a principal singular part of the matrix $\Gamma(x,\sigma)$. Actually $\Gamma_0(x)$ is a kernel function of a parametrix operator for the differential operator $L(\partial,\sigma)$ (it does not matter whether (3.22) holds or not).

3.3. Special representation of the principal singular part

In this subsection we derive some formulae which will help us to calculate the principal symbol matrices of the boundary integral (pseudodifferential) operators generated by the single- and double-layer potentials (see Section 4).

We have

$$\Gamma_0(x) = -\frac{1}{8\pi^3}\int_{\mathbb{R}^3}[L_0(\xi)]^{-1}\,e^{-i\,x\cdot\xi}\,d\xi = -\frac{1}{8\pi^3}\int_{\mathbb{R}^3}[L_0(\xi)]^{-1}\,e^{i\,x\cdot\xi}\,d\xi, \tag{3.68}$$

where the above formal integrals are understood as generalized Fourier transforms, i.e.,

$$\Gamma_0(x) = -\mathcal{F}^{-1}[L_0^{-1}(\xi)] = -\frac{1}{8\pi^3}\mathcal{F}[L_0^{-1}(\xi)].$$

We recall that $L_0(\xi)$ is a positive definite matrix for $\xi\in\mathbb{R}^3\setminus\{0\}$.

Let $E=[e_{kj}]_{3\times3}:\mathbb{R}^3\to\mathbb{R}^3$ be an orthogonal matrix with $\det E=1$:

$$E\,E^\top = E^\top\,E = I_3. \tag{3.69}$$

Then

$$\begin{aligned}
\Gamma_0(Ex) &= -\frac{1}{8\pi^3}\int_{\mathbb{R}^3}[L_0(\xi)]^{-1}e^{-i\,E\,x\cdot\xi}\,d\xi = -\frac{1}{8\pi^3}\int_{\mathbb{R}^3}[L_0(E\,\xi)]^{-1}e^{-i\,x\cdot\xi}\,d\xi\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}e^{-i\,\widetilde{x}\cdot\widetilde{\xi}}\left\{-\frac{1}{2\pi}\int_{\mathbb{R}^1}[L_0(E\,\xi)]^{-1}e^{-i\,x_3\xi_3}\,d\xi_3\right\}\,d\widetilde{\xi},
\end{aligned} \tag{3.70}$$

where $\widetilde{x} = (x_1, x_2)$, $\widetilde{\xi} = (\xi_1, \xi_2)$, i.e.,

$$\Gamma_0(E\,x) = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}} \left[-\frac{1}{2\pi} \int_{I\!R^1} [L_0(E\,\xi)]^{-1} e^{-ix_3\xi_3}\, d\xi_3 \right]. \tag{3.71}$$

This implies (due to the Cauchy integral theorem for analytic functions)

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\Gamma_0(E\,x)] = -\frac{1}{2\pi} \int_{I\!R^1} [L_0(E\,\xi)]^{-1} e^{-i\,x_3\xi_3}\, d\xi_3$$

$$= \begin{cases} -\frac{1}{2\pi} \int_{\ell^+} [L_0(E\,\xi)]^{-1} e^{-i\,x_3\xi_3}\, d\xi_3 & \text{for} \quad x_3 \le 0, \\ \frac{1}{2\pi} \int_{\ell^-} [L_0(E\,\xi)]^{-1} e^{-i\,x_3\xi_3}\, d\xi_3 & \text{for} \quad x_3 \ge 0, \end{cases} \tag{3.72}$$

where ℓ^+ [resp. ℓ^-] is a closed simple curve in the upper [resp. lower] half-plane of the complex ξ_3-plane ($\xi_3 = \xi_3' + i\xi_3''$) enveloping all the roots (with respect to ξ_3) of the equation $\det L_0(E\,\xi) = 0$ with positive [resp. negative] imaginary parts. Clearly, (3.72) does not depend on the shape of ℓ^+ [resp. ℓ^-].

The integration in (3.72) is performed counter clockwise. It can easily be shown that the entries of the matrix (3.72) are homogeneous functions in $\widetilde{\xi}$ of order -1.

From (3.72) it follows that the matrix $\left[-\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}\,[\Gamma_0(E\,x)]|_{x_3=0}\right]_{6\times 6}$ is positive definite for arbitrary $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$ due to the accretivity condition (2.31). This matrix represents the principal homogeneous symbol (modulo a positive constant multiplier) of the single-layer potentials associated with the matrices $\Gamma(\cdot\,, \sigma)$ and $\Gamma_0(\,\cdot\,)$ (see Sections 4 and 6).

4 General integral representations

4.1. Representation formulae for bounded domains

In what follows we assume that the boundary $S = \partial\Omega^\pm$ is C^{k,α_0}-smooth, with integer $k \ge 1$ and $0 < \alpha_0 \le 1$, and $n(x)$ stands for the outward unit normal vector to Ω^+ at the point $x \in S$. The symbols $[\,\cdot\,]^\pm$ denote the limits on S from $\Omega^\pm$.

Let us introduce the generalized single- and double-layer potentials, and the Newton type volume potential

$$V^{(\sigma)}(\varphi)(x) = \int_S \Gamma(x-y,\sigma)\,\varphi(y)\,dS_y, \quad x \in I\!R^3 \setminus S, \tag{4.1}$$

$$W^{(\sigma)}(\varphi)(x) = \int_S [T(\partial_y, n(y))\Gamma(y-x,\sigma)]^\top\,\varphi(y)\,dS_y, \quad x \in I\!R^3 \setminus S, \tag{4.2}$$

$$N^{(\sigma)}_\Omega(\psi)(x) = \int_\Omega \Gamma(x-y,\sigma)\,\psi(y)\,dy, \quad x \in I\!R^3, \tag{4.3}$$

where $T(\partial, n)$ is the stress operator of the theory of hemitropic elasticity (see (2.4)), $\Gamma(\cdot, \sigma)$ is the fundamental matrix given by (3.40), $\varphi = (\varphi_1, \cdots, \varphi_6)^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1, \cdots, \psi_6)^\top$ is defined on $\Omega \in \{\Omega^+, \Omega^-\}$.

Due to Remark 3.3 and the equality

$$\begin{aligned} L_{kj}(\partial_x, \sigma) & \left([T(\partial_y, n(y))\Gamma(y-x, \sigma)]^T \right)_{jp} \\ & = L_{kj}(\partial_x) T_{pq}(\partial_y, n(y))\Gamma_{qj}(y-x, \sigma) \\ & = T_{pq}(\partial_y, n(y)) L_{kj}(\partial_x)\Gamma_{qj}(y-x, \sigma) \\ & = T_{pq}(\partial_y, n(y)) L_{kj}(\partial_x)\Gamma_{jq}(x-y, \sigma) = 0, \;\; x \neq y, \end{aligned}$$

it can easily be checked that the potentials defined by (4.1) and (4.2) are C^∞-smooth in the domain $I\!R^3 \setminus S$ and solve the homogeneous equations (2.8) ($F = 0$, $G = 0$) for an arbitrary L_p-summable vector function φ. The volume potential $N^{(\sigma)}_{\Omega^+}(\psi)$ solves the non-homogeneous equation $L(\partial, \sigma)U(x) = \psi(x)$ in Ω^+ for $\psi \in [C^{0,\alpha_0}(\Omega^+)]^6$.

The single- and double-layer potentials, and the volume potential constructed with the help of the fundamental matrix $\Gamma_0(\cdot)$ (the principal singular part of the matrix $\Gamma(\cdot, \sigma)$) will be denoted by $V_0(\varphi)$ and $W_0(\varphi)$, and $N_{0,\Omega}(\psi)$, respectively. Clearly, $V_0(\varphi)$ and $W_0(\varphi)$ solve the homogeneous equation $L_0(\partial)U(x) = 0$ for an arbitrary L_p-summable vector function φ.

By standard arguments we can prove the following assertions (cf. [33], Ch. I, Lemma 2.1; Ch. II, Lemma 8.2).

Theorem 4.1 *Let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^6$. Then there holds the following integral representation formula*

$$\begin{aligned} W^{(\sigma)}([U]^+)(x) - V^{(\sigma)}([TU]^+)(x) + N^{(\sigma)}_{\Omega^+}(L(\partial, \sigma)U)(x) \\ = \begin{cases} U(x) & \text{for} \;\; x \in \Omega^+, \\ 0 & \text{for} \;\; x \in \Omega^-. \end{cases} \end{aligned} \tag{4.4}$$

Theorem 4.2 *Let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^6$. Then there holds the following integral representation formula*

$$\begin{aligned} W_0([U]^+)(x) - V_0([T_0 U]^+)(x) + N_{0,\Omega^+}(L_0(\partial)U)(x) \\ = \begin{cases} U(x) & \text{for} \;\; x \in \Omega^+, \\ 0 & \text{for} \;\; x \in \Omega^-. \end{cases} \end{aligned} \tag{4.5}$$

Note that these theorems can be extended to the case $U \in [H^1(\Omega^+)]^6$ with $L(\partial, \sigma)U \in [L_2(\Omega^+)]^6$ and $L_0(\partial)U \in [L_2(\Omega^+)]^6$ by a standard approach (cf., e.g., [25], [5], [26]).

4.2. Sommerfeld-Kupradze type radiation conditions and representation formulae for unbounded domains

In this subsection we will derive integral representation formulae in the case of unbounded domains with compact boundary. To this end we introduce a special class of radiating functions.

We say that a C^1-smooth vector function $U = (U_1, \cdots, U_6)^\top$ satisfies the *generalized Sommerfeld-Kupradze type radiation conditions* in Ω^- if

$$U(x) = \sum_{j=1}^{6} U^{(j)}(x) \quad \text{in} \quad \Omega^-, \tag{4.6}$$

where

$$U^{(j)} = (U_1^{(j)}, \cdots, U_6^{(j)})^\top, \;\; \Delta U_p^{(j)}(x) + k_j^2\, U_p^{(j)}(x) = 0, \;\; j,p = 1, \cdots, 6, \tag{4.7}$$

with k_j satisfying (3.22), and for sufficiently large $|x|$ there hold the relations:

$$\begin{gathered} U_p^{(j)}(x) = \exp\{-\Im k_j |x|\}\, \mathcal{O}(|x|^{-1}), \\ \frac{\partial}{\partial x_l} U_p^{(j)} - i\, k_j\, \hat{x}_l\, U_p^{(j)}(x) = \exp\{-\Im k_j |x|\}\, \mathcal{O}(|x|^{-2}), \;\; j,p = 1, \cdots, 6, \end{gathered} \tag{4.8}$$

where $\hat{x} = x/|x|$ and $\hat{x}_l = x_l/|x|$, $l = 1, 2, 3$.

Denote the above described class of vectors by $SK(\Omega^-)$. Vector functions of this class will be referred to as *radiating vectors*. Due to Remarks 3.1 and 3.2 it is evident that the columns of the fundamental matrix $\Gamma(x, \sigma)$ are radiating vectors.

We recall that $\Im k_j \geq 0$, and if $\Im k_j = 0$, then $k_j > 0$ (see (3.22)). Therefore,

$$k_j + k_p \neq 0 \;\; \text{for} \;\; j, p = 1, \cdots, 6. \tag{4.9}$$

Now we are in the position to prove the following

Theorem 4.3 *Let $U \in [C^1(\overline{\Omega^-})]^6$ be a regular radiating solution of the homogeneous equation $L(\partial, \sigma)U(x) = 0$ in Ω^-. Then there holds the following integral representation formula*

$$-W^{(\sigma)}([U]^-)(x) + V^{(\sigma)}([TU]^-)(x) = \begin{cases} U(x) & \text{for} \quad x \in \Omega^-, \\ 0 & \text{for} \quad x \in \Omega^+. \end{cases} \tag{4.10}$$

Proof. Let R be a sufficiently large positive number such that $\overline{\Omega^+} \subset B(O, R)$, where $B(O, R)$ is the ball of radius R centered at the origin O. Denote $\Omega_R^- := \Omega^- \cap B(O, R)$. Let $x \in \Omega^-$ be an arbitrary point and choose

R such that $x \in \Omega_R^-$. Write the integral representation formula (4.4) for $U(x)$ in the domain Ω_R^-

$$U(x) = -W^{(\sigma)}([U]^-)(x) + V^{(\sigma)}([TU]^-)(x) + \Psi(x, R), \tag{4.11}$$

with

$$\Psi(x, R) := \int_{\Sigma_R} \Big\{ \big[T(\partial_y, \hat{y})\Gamma(y - x, \sigma)\big]^\top U(y) - \Gamma(x - y, \sigma) T(\partial_y, \hat{y}) U(y) \Big\}\, d\Sigma_R, \tag{4.12}$$

where Σ_R is the boundary of $B(O, R)$ and $\hat{y} = y/|y|$ is the outward normal to Σ_R.

Further, let

$$\widetilde{U}(x) := U(x) + W^{(\sigma)}([U]^-)(x) - V^{(\sigma)}([TU]^-)(x). \tag{4.13}$$

From (4.11) we then have

$$\widetilde{U}(x) = \Psi(x, R), \quad x \in \Omega_R^-. \tag{4.14}$$

Note that the left-hand side expression $\widetilde{U}(x)$ does not depend on R.

Let us integrate the last equality with respect to R over the interval $(R_1, 2R_1)$ and divide by R_1 where R_1 a sufficiently large number. We get

$$\widetilde{U}(x) = \frac{1}{R_1} \int_{R_1}^{2R_1} \Psi(x, R)\, dR. \tag{4.15}$$

In what follows we show that for a radiating solution U, the right-hand side expression in (4.15) tends to zero as $R_1 \to 0$.

To this end, note that for a fixed x and sufficiently large $|y|$ we have (see (2.6) and Remark 3.2)

$$T(\partial_y, \hat{y})U(y) = \sum_{j=1}^{6} \{ i\, k_j\, T_0(\hat{y}, \hat{y}) U^{(j)}(y) + \mathcal{A}(\hat{y})\, U^{(j)}(y) \} + \mathcal{O}(R^{-2}), \tag{4.16}$$

$$\begin{aligned} T(\partial_y, \hat{y})\Gamma(y - x, \sigma) = \sum_{j=1}^{6} \Big\{ & i\, k_j\, T_0(\hat{y}, \hat{y}) \widetilde{\Gamma}^{(j)}(y - x, \sigma) \\ & + \mathcal{A}(\hat{y}) \widetilde{\Gamma}^{(j)}(y - x, \sigma) \Big\} + \mathcal{O}(R^{-2}), \end{aligned} \tag{4.17}$$

where (see (2.6))

$$\mathcal{A}(\hat{y}) = \begin{bmatrix} [0]_{3\times 3} & 2\alpha R(\hat{y}) \\ [0]_{3\times 3} & 2\nu R(\hat{y}) \end{bmatrix}_{6\times 6} \tag{4.18}$$

since U and $\Gamma(\cdot,\sigma)$ are radiating. Evidently,

$$\mathcal{A}(\hat y)\,U^{(j)}(y) = 2[\alpha\,\hat y\times\omega^{(j)}(y),\,\nu\,\hat y\times\omega^{(j)}(y)]^\top.$$

Therefore, from (4.15) and (4.12) it follows that

$$\begin{aligned}\widetilde U(x) = \frac{1}{R_1}\int_{R_1}^{2R_1} dR\int\limits_{\Sigma_R}\sum_{j,q=1}^{6}\Big\{\Big[i\,k_j\,T_0(\hat y,\hat y)\widetilde\Gamma^{(j)}(y-x,\sigma)\\ +\mathcal{A}(\hat y)\widetilde\Gamma^{(j)}(y-x,\sigma)\Big]^\top U^{(q)}(y) - [\widetilde\Gamma^{(j)}(y-x,\sigma)]^\top[ik_q\,T_0(\hat y,\hat y)U^{(q)}(y)\\ +\mathcal{A}(\hat y)\,U^{(q)}(y)]\Big\}\,d\Sigma_R + \mathcal{O}(R_1^{-1}).\end{aligned}\tag{4.19}$$

To show that the right-hand side in (4.19) tends to sero as $R_1\to\infty$ it suffices to prove that

$$\psi_{jq}(R_1) = \frac{1}{R_1}\int_{R_1}^{2R_1} dR\int\limits_{\Sigma_1} h^{(j)}(R\hat y)g^{(q)}(R\hat y)R^2\,d\Sigma_1 \to 0 \tag{4.20}$$

as $R_1\to 0$, where

$$\begin{aligned}&h^{(j)}(R\hat y)=\mathcal{O}(R^{-1}),\quad \tfrac{\partial}{\partial R}h^{(j)}(R\hat y) - i\,k_j\,h^{(j)}(R\hat y) = \mathcal{O}(R^{-2}),\\ &g^{(q)}(R\hat y)=\mathcal{O}(R^{-1}),\quad \tfrac{\partial}{\partial R}g^{(q)}(R\hat y) - i\,k_q\,h^{(q)}(R\hat y) = \mathcal{O}(R^{-2}).\end{aligned}\tag{4.21}$$

Note that $h^{(j)}(R\hat y) = \widetilde\Gamma^{(j)}_{ps}(R\hat y - x)$ and $g^{(q)}(R\hat y) = U^{(q)}_m(R\hat y)$ satisfy the above relations due to the radiation conditions (4.8).

Taking into consideration that $k_j + k_q \neq 0$ we get

$$\begin{aligned}&h^{(j)}(R\hat y)\,g^{(q)}(R\hat y)\\ &\quad= \frac{1}{i(k_j+k_q)}[ik_jh^{(j)}(R\hat y)\,g^{(q)}(R\hat y) + h^{(j)}(R\hat y)ik_q\,g^{(q)}(R\hat y)]\\ &\quad= \frac{1}{i(k_j+k_q)}\left[\frac{\partial h^{(j)}(R\hat y)}{\partial R}\,g^{(q)}(R\hat y) + h^{(j)}(R\hat y)\,\frac{\partial g^{(q)}(R\hat y)}{\partial R}\right] + \mathcal{O}(R^{-3}).\end{aligned}$$

Therefore from (4.20) with the help of the integration by parts formula we derive

$$\begin{aligned}\psi_{jq}(R_1) &= \frac{1}{R_1}\int_{\Sigma_1} d\Sigma\int_{R_1}^{2R_1}\frac{R^2}{i(k_j+k_q)}\frac{\partial}{\partial R}[h^{(j)}(R\hat y)\,g^{(q)}(R\hat y)]dR + \mathcal{O}(R_1^{-1})\\ &= \frac{1}{i(k_j+k_q)R_1}\int_{\Sigma_1}\Big\{[R^2\,h^{(j)}(R\hat y)\,g^{(q)}(R\hat y)]_{R_1}^{2R_1}\end{aligned}$$

$$-\int_{R_1}^{2R_1} h^{(j)}(R\hat{y})\, g^{(q)}(R\hat{y}) 2R\, dR \Big\}\, d\Sigma_1$$

$$= \frac{1}{i(k_j + k_q)R_1} \int_{\Sigma_1} \mathcal{O}(1)\, d\Sigma_1 = \mathcal{O}(R_1^{-1}) \to 0 \quad \text{as} \quad R_1 \to +\infty.$$

Thus $\psi_{jq}(R_1) \to)$ as $R_1 \to +\infty$, which shows that the right-hand side in (4.19) tends to zero as $R_1 \to +\infty$. In turn this yields $\widetilde{U}(x) = 0$, whence the proof of the equality (4.10) follows for $x \in \Omega^-$.

The proof for the case $x \in \Omega^+$ may be verbatim performed. ∎

4.3. Uniqueness result for the whole space

Here we prove the following

Theorem 4.4 *Let k_j $(j = \overline{1,6})$ satisfy the conditions* (3.22)*, and let U be a radiating solution of the homogeneous equation $L(\partial, \sigma)U(x) = 0$ in $I\!R^3$. Then U vanishes identically in $I\!R^3$.*

Proof. Fix a point $x \in I\!R^3$ and choose a positive number R such that $|x| < R$. Due to Theorem 4.1 we can write the general integral representation formula for the domain $B(O,R)$ at the point x

$$U(x) = \int_{\Sigma_R} \Big\{ [T(\partial_y, \hat{y})\Gamma(y - x, \sigma)]^\top U(y)$$
$$-\Gamma(x - y, \sigma)\, T(\partial_y, \hat{y})U(y) \Big\}\, d\Sigma_R. \tag{4.22}$$

Applying the same arguments as in the proof of Theorem 4.4 (i.e., by taking the integral mean value over the interval $(R_1, 2R_1)$) we can show quite analogously that the integrals in the right-hand side of (4.22) tend to zero as $R_1 \to \infty$. This completes the proof since x is an arbitrary point. ∎

5 Auxiliary BVPs

In this section we consider some auxiliary BVPs for the operator $L_0(\partial)$ (see (2.16)), which will help us to establish Fredholm properties of the boundary integral (pseudodifferential) operators generated by the potentials (4.1) and (4.2).

Problem $(I_0)_f^\pm$. Find a regular solution $U \in [C^1(\overline{\Omega^\pm})]^6$ to the differential equation

$$L_0(\partial)U(x) = 0, \quad x \in \Omega^\pm, \tag{5.1}$$

satisfying the Dirichlet type boundary condition

$$[U(x)]^{\pm} = f(x), \quad x \in S = \partial\Omega^{\pm}, \tag{5.2}$$

where f is a given vector function on S.

Problem $(II_0)^{\pm}_{f}$. Find a regular solution $U \in [C^1(\overline{\Omega^{\pm}})]^6$ of the differential equation (5.1) satisfying the Robin type boundary condition

$$[T_0(\partial, n)U(x)]^{\pm} \pm d\,[U(x)]^{\pm} = f(x), \quad x \in S, \tag{5.3}$$

where f is a given vector function on S and d is a given non-negative constant.

In addition, in the case of exterior domain Ω^-, we assume that

$$U(x) = \mathcal{O}(|x|^{-1}), \quad \partial_j U(x) = \mathcal{O}(|x|^{-2}), \quad j = 1,2,3 \quad \text{as} \quad |x| \to +\infty. \tag{5.4}$$

Clearly, when $d = 0$ in (5.3) we get the Neumann type BVP.

We have the following Green's formulae for arbitrary real-valued vector functions $U := (u,\omega)^{\top}, U' := (u',\omega')^{\top} \in [C^2(\overline{\Omega^{\pm}})]^6$ satisfying the conditions (5.4)

$$\int\limits_{\Omega^{\pm}} \big[L_0(\partial)U \cdot U' + E_0(U,U')\big]\,dx = \pm \int\limits_{\partial\Omega^{\pm}} T_0(\partial,n)U \cdot U'\,dS, \tag{5.5}$$

where n is the outward unit normal vector to $\partial\Omega^+$,

$$\begin{aligned}
E_0(U,U') &= E_0(U',U) \\
&= \frac{3\lambda+2\mu}{3}\left(\operatorname{div} u + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega\right)\left(\operatorname{div} u' + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega'\right) \\
&+\frac{1}{3}\left(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\right)(\operatorname{div}\omega)(\operatorname{div}\omega') \\
&+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)\right] \\
&\qquad\times\left[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega'_k}{\partial x_j}+\frac{\partial \omega'_j}{\partial x_k}\right)\right] \\
&+\frac{\mu}{3}\sum_{k,j=1}^{3}\left[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)\right] \\
&\qquad\times\left[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega'_k}{\partial x_k}-\frac{\partial \omega'_j}{\partial x_j}\right)\right]
\end{aligned}$$

$$+\left(\gamma-\frac{\kappa^2}{\mu}\right)\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{1}{2}\left(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\right)\left(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\right)\right.$$

$$\left.+\frac{1}{3}\left(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\right)\left(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\right)\right]$$

$$+\alpha\left(\mathrm{curl}\ u+\frac{\nu}{\alpha}\,\mathrm{curl}\ \omega\right)\cdot\left(\mathrm{curl}\ u'+\frac{\nu}{\alpha}\,\mathrm{curl}\ \omega'\right)$$

$$-\left(\varepsilon-\frac{\nu^2}{\alpha}\right)\mathrm{curl}\ \omega\cdot\mathrm{curl}\ \omega'. \tag{5.6}$$

In particular,

$$E_0(U,U)=\frac{3\lambda+2\mu}{3}\left(\mathrm{div}\ u+\frac{3\delta+2\kappa}{3\lambda+2\mu}\,\mathrm{div}\ \omega\right)^2$$

$$+\frac{1}{3}\left(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\right)(\mathrm{div}\,\omega)^2$$

$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\right)\right]^2$$

$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\left[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\right)\right]^2$$

$$+\left(\gamma-\frac{\kappa^2}{\mu}\right)\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{1}{2}\left(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\right)^2+\frac{1}{3}\left(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\right)^2\right]$$

$$+\left(\varepsilon-\frac{\nu^2}{\alpha}\right)(\mathrm{curl}\ \omega)^2+\alpha\left(\mathrm{curl}\ u+\frac{\nu}{\alpha}\,\mathrm{curl}\ \omega\right)^2. \tag{5.7}$$

Due to the relations (2.24) we easily derive that the equality $E_0(U,U)=0$ implies

$$U(x)=(b',b'')^\top, \tag{5.8}$$

where b' and b'' are arbitrary three-dimensional constant vectors.

The following uniqueness results hold true.

Theorem 5.1 *The homogeneous BVPs* $(I_0)_0^\pm$, $(II_0)_0^\pm$ *with* $d>0$, *and* $(II_0)_0^-$ *with* $d=0$ *have only the trivial solutions.*

The homogeneous BVP $(II_0)_0^+$ *with* $d=0$ *has the general solution of the form* (5.8).

Proof. It follows from (5.5) and (5.7). ∎

6 Properties of potentials and boundary pseudodifferential operators

The jump and mapping properties of the above introduced single- and double-layer potentials and the corresponding boundary integral (pseudodifferential) operators in the Hölder ($C^{k+\alpha}$), Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces can be studied by standard methods (see, e.g., [18], [10], [29], [3], [30], [31], [7], [8], [32], [33], and [26]).

We will use the following abbreviations (when no confusion can be caused):

(a) if all elements of a vector $v = (v_1, ..., v_m)^\top$ (a matrix $N = [N_{kj}]_{m\times n}$) belong to one and the same space X, we will write $v \in X$ ($N \in X$) instead of $v \in X^m$ ($N \in X_{m\times n}$);

(b) if $K: \ X_1 \times X_2 \times \cdots \times X_m \to Y_1 \times Y_2 \times \cdots \times Y_n$ and $X_1 = X_2 = \cdots = X_m$, $\ Y_1 = Y_2 = \cdots = Y_n$, we will write $K: \ X \to Y$ instead of $K: \ X^m \to Y^n$.

Theorem 6.1 *Let $S \in C^{k+1,\alpha_0}$ where $k \geq 0$ is an integer, $0 < \alpha_0 \leq 1$, and let $0 < \gamma_0 < \alpha_0$. Then the operators*

$$
\begin{aligned}
V^{(\sigma)},\ V_0 \quad &: \quad C^{k,\gamma_0}(S) \to C^{k+1,\gamma_0}(\overline{\Omega^\pm}),\\
W^{(\sigma)},\ W_0 \quad &: \quad C^{k,\gamma_0}(S) \to C^{k,\gamma_0}(\overline{\Omega^\pm}),
\end{aligned}
\tag{6.1}
$$

are bounded.

For any $g \in C^{k,\gamma_0}(S)$ and any $x \in S$

$$
\begin{aligned}
[V^{(\sigma)}(g)(x)]^\pm &= V^{(\sigma)}(g)(x) = \mathcal{H}^{(\sigma)}\, g(x),\\
[V_0(g)(x)]^\pm &= V_0(g)(x) = \mathcal{H}_0\, g(x),
\end{aligned}
\tag{6.2}
$$

$$
\begin{aligned}
[T(\partial_x, n(x))V^{(\sigma)}(g)(x)]^\pm &= [\mp 2^{-1} I_6 + \mathcal{K}^{(\sigma)}]g(x),\\
[T_0(\partial_x, n(x))V_0(g)(x)]^\pm &= [\mp 2^{-1} I_6 + \mathcal{K}_0]g(x),
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
[W^{(\sigma)}(g)(x)]^\pm &= [\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)*}]\, g(x),\\
[W_0(g)(x)]^\pm &= [\pm 2^{-1} I_6 + \mathcal{K}_0^*]\, g(x),
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
[T(\partial_x, n(x))W^{(\sigma)}(g)(x)]^+ &= [T(\partial_x, n(x))W^{(\sigma)}(g)(x)]^-\\
&= \mathcal{L}^{(\sigma)}\, g(x),\\
[T_0(\partial_x, n(x))W_0(g)(x)]^+ &= [T_0(\partial_x, n(x))W_0(g)(x)]^-\\
&= \mathcal{L}_0\, g(x),\ k \geq 1,
\end{aligned}
\tag{6.5}
$$

where

$$\begin{aligned}&\mathcal{H}^{(\sigma)}\, g(x) := \int_S \Gamma(x-y,\sigma)\, g(y)\, dS_y,\\ &\mathcal{H}_0\, g(x) := \int_S \Gamma_0(x-y)\, g(y)\, dS_y,\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\mathcal{K}^{(\sigma)}\, g(x) := \int_S T(\partial_x, n(x))\Gamma(x-y,\sigma)\, g(y)\, dS_y,\\ &\mathcal{K}_0\, g(x) := \int_S T_0(\partial_x, n(x))\Gamma_0(x-y)\, g(y)\, dS_y,\end{aligned} \tag{6.7}$$

$$\begin{aligned}&\mathcal{K}^{(\sigma)*}\, g(x) := \int_S [T(\partial_y, n(y))\Gamma(y-x,\sigma)]^\top\, g(y)\, dS_y,\\ &\mathcal{K}_0^*\, g(x) := \int_S [T_0(\partial_y, n(y))\Gamma_0(y-x]^\top\, g(y)\, dS_y,\end{aligned} \tag{6.8}$$

$$\begin{aligned}&\mathcal{L}^{(\sigma)}\, g(x)\\ &\quad := \lim_{\Omega^\pm \ni z\to x\in S} T(\partial_z, n(x)) \int_S [T(\partial_y, n(y))\Gamma(y-z,\sigma)]^\top\, g(y)\, dS_y,\\ &\mathcal{L}_0\, g(x)\\ &\quad := \lim_{\Omega^\pm \ni z\to x\in S} T_0(\partial_z, n(x)) \int_S [T_0(\partial_y, n(y))\Gamma_0(y-z)]^\top\, g(y)\, dS_y.\end{aligned} \tag{6.9}$$

Proof. The proof of the boundedness and smoothness results is quite similar to the proof of Theorems 4.1-4.3, 5.1, 5.2, 7.1, 7.2, 8.4 in the reference [18], Ch. V, and Theorems 4.1, 4.2, 10.1, and 10.2 in the reference [33], Ch. I-II.

To establish the jump relations (6.2)-(6.5), in addidtion to the above mentioned technique developed in [18], Ch. V, and [33], Ch. I-II, we need the following equality

$$\int_S [T_0(\partial_y, n(y))\Gamma_0(y-x)]^\top\, dS_y = \begin{cases} I_6 & \text{for} \quad x\in\Omega^+,\\ 2^{-1}I_6 & \text{for} \quad x\in S,\\ 0 & \text{for} \quad x\in\Omega^-,\end{cases}$$

which is an analogue of the well-known Gauss formula for harmonic functions. ∎

Theorem 6.2 *The operators $V^{(\sigma)}$, V_0, $W^{(\sigma)}$, and W_0 can be extended by continuity to the bounded mappings*

$$\begin{aligned} V^{(\sigma)},\ V_0 &: H^{-\frac12}(S)\to H^1(\Omega^+) \quad [H^{-\frac12}(S)\to H^1_{loc}(\Omega^-)],\\ W^{(\sigma)},\ W_0 &: H^{\frac12}(S)\to H^1(\Omega^+) \quad [H^{\frac12}(S)\to H^1_{loc}(\Omega^-)].\end{aligned}$$

The jump relations (6.2)-(6.5) *on S remain valid for the extended operators in the corresponding functional spaces.*

Proof. It is quite similar to the proof of the analogous theorems in [7], [8], and [26], Ch. 6. ∎

Theorem 6.3 *Let S, k, γ_0, and α_0 be as in Theorem 6.1. Then the operators*

$$\mathcal{H} \; : \; C^{k,\gamma_0}(S) \to C^{k+1,\gamma_0}(S) \quad [\,H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S)\,], \tag{6.10}$$

$$\mathcal{K} \; : \; C^{k,\gamma_0}(S) \to C^{k,\gamma_0}(S) \quad [\,H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)\,], \tag{6.11}$$

$$\mathcal{K}^* \; : \; C^{k,\gamma_0}(S) \to C^{k,\gamma_0}(S) \quad [\,H^{\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S)\,], \tag{6.12}$$

$$\mathcal{L} \; : \; C^{k+1,\gamma_0}(S) \to C^{k,\gamma_0}(S) \quad [\,H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)\,] \tag{6.13}$$

are bounded, where

$$\mathcal{H} \in \{\mathcal{H}^{(\sigma)},\, \mathcal{H}_0\}, \quad \mathcal{K} \in \{\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)},\, \pm 2^{-1}I_6 + \mathcal{K}_0\},$$
$$\mathcal{L} \in \{\mathcal{L}^{(\sigma)},\, \mathcal{L}_0\}, \quad \mathcal{K}^* \in \{\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)*},\, \pm 2^{-1}I_6 + \mathcal{K}_0^*\}.$$

Moreover,

(i) *The principal homogeneous symbol matrices of the operators $\pm 2^{-1}I_6 + \mathcal{K}_0$ and $\pm 2^{-1}I_6 + \mathcal{K}_0^*$ are nondegenerate, while the principal homogeneous symbol matrices of the operators $\mathcal{H}_0$ and $\mathcal{L}_0$ are positive definite; the operators $\mathcal{H}_0$, $\pm 2^{-1}I_6 + \mathcal{K}_0$, $\pm 2^{-1}I_6 + \mathcal{K}_0^*$, and $\mathcal{L}_0$ are elliptic pseudodifferential operators of order -1, 0, 0, and 1, respectively;*

(ii) *the operators $\pm 2^{-1}I_6 + \mathcal{K}_0$ and $\pm 2^{-1}I_6 + \mathcal{K}_0^*$ are mutually adjoint singular integral operators of normal type with index equal to zero. The operators $\mathcal{H}_0$, $2^{-1}I_6 + \mathcal{K}_0$ and $2^{-1}I_6 + \mathcal{K}_0^*$ are invertible. The inverse of $\mathcal{H}_0$*

$$\mathcal{H}_0^{-1} : C^{k+1,\gamma_0}(S) \to C^{k,\gamma_0}(S) \quad [H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)]$$

is a singular integro-differential operator;

(iii) *the $\mathcal{L}_0$ and $\mathcal{L}^{(\sigma)}$ are singular integro-differential operators and the following equalities hold in appropriate functional spaces:*

$$\begin{aligned} &\mathcal{K}_0^*\mathcal{H}_0 = \mathcal{H}_0\mathcal{K}_0, \quad \mathcal{L}_0\mathcal{K}_0^* = \mathcal{K}_0\mathcal{L}_0, \\ &\mathcal{H}_0\mathcal{L}_0 = -4^{-1}I_6 + (\mathcal{K}_0^*)^2, \quad \mathcal{L}_0\mathcal{H}_0 = -4^{-1}I_6 + \mathcal{K}_0^2, \end{aligned} \tag{6.14}$$

$$\begin{aligned} &\mathcal{K}^{(\sigma)*}\mathcal{H}^{(\sigma)} = \mathcal{H}^{(\sigma)}\mathcal{K}^{(\sigma)}, \;\; \mathcal{L}^{(\sigma)}\mathcal{K}^{(\sigma)*} = \mathcal{K}^{(\sigma)}\mathcal{L}^{(\sigma)}, \\ &\mathcal{H}^{(\sigma)}\mathcal{L}^{(\sigma)} = -4^{-1}I_6 + (\mathcal{K}^{(\sigma)*})^2, \;\; \mathcal{L}^{(\sigma)}\mathcal{H}^{(\sigma)} = -4^{-1}I_6 + (\mathcal{K}^{(\sigma)})^2; \end{aligned} \tag{6.15}$$

(iv) *The operators $-\mathcal{H}_0$ and $\mathcal{L}_0$ are self-adjoint and non-negative elliptic pseudodifferential operators with positive definite principal symbol matrices and with index equal to zero:*

$$\langle -\mathcal{H}_0 h, h\rangle_S \geq 0, \qquad \langle g, \mathcal{L}_0 g\rangle_S \geq 0, \tag{6.16}$$

$$\forall h \in C^{\gamma_0}(S),\; \forall g \in C^{1,\gamma_0}(S), \;\; [\,\forall h \in H^{-\frac{1}{2}}(S),\; \forall g \in H^{\frac{1}{2}}(S)\,],$$

with equality only for $h = 0$ *and for*

$$g = (b', b'')^\top \tag{6.17}$$

respectively, where $b', b'' \in I\!R^3$ *are arbitrary constant vectors; here* $\langle \cdot \,, \, \cdot \rangle_S$ *denotes the duality between the spaces* $\left[H^{\frac{1}{2}}(S) \right]^6$ *and* $\left[H^{-\frac{1}{2}}(S) \right]^6$ *which extends the usual* $[\, L_2(S)\,]^6$*-scalar product;*

(v) *a general solution of the homogeneous equations* $[-2^{-1}I_6 + \mathcal{K}_0^*]g = 0$ *and* $\mathcal{L}_0\, g = 0$ *is given by* (6.17) (*i.e.,* $\ker \mathcal{L}_0 = \ker(-2^{-1}I_6 + \mathcal{K}_0^*)$ *is a six dimensional null-space*).

Proof. The mapping properties (6.10)-(6.13) can be shown by the standard approach developed in, e.g., [18], Ch. V, [33], Ch. I-II, [26], Ch. 6.

The items (i)-(v) of the theorem can be established by invoking the results of Subsection 3.3 and Section 5, and applying the same arguments as in [4], [33], Ch. I, §§ 4 – 6, [26], Ch. 6. ∎

Corollary 6.4 *Let* S, k, γ_0, *and* α_0 *be as in Theorem* 6.1. *Then*

(i) *the operators* $\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)}$ *and* $\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)*}$ *are mutually adjoint singular integral operators of normal type with index equal to zero;*

(ii) *the operators* $-\mathcal{H}^{(\sigma)}$ *and* $\mathcal{L}^{(\sigma)}$ *are elliptic pseudodifferential operators of order* -1 *and* $+1$, *respectively, with index equal to zero and with positive definite principal symbol matrices.*

Proof. It is a straightforward consequence of Theorem 6.3 (i) and (iv) since the pseudodifferential operators with superscript σ, defined by the formulae (6.6)-(6.9), are compact pertubations of the corresponding operators with subscript 0 given by the same formulae, due to the relation (3.67). ∎

Applying the general theory of pseudodifferential operators and equations on smooth manifolds without boundary (see, e.g., [45], [10], [6], [40], [41], [42] and the references therein) we can prove the following assertion.

Theorem 6.5 *Let* $V^{(\sigma)}$, V_0, $W^{(\sigma)}$, W_0, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{K}^*$, *and* $\mathcal{L}$ *be as in Theorems* 6.2 *and* 6.3. *The boundary integral* (*pseudodifferential*) *operators* (6.1) *and* (6.10)-(6.13) *can be extended continuously to the following bounded operators*

$$V^{(\sigma)},\ V_0 : B^s_{p,p}(S) \to H_p^{s+1+\frac{1}{p}}(\Omega^+)\ \ [B^s_{p,p}(S) \to H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)], \tag{6.18}$$

$$: B^s_{p,q}(S) \to B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)\ \ [B^s_{p,q}(S) \to B_{p,q,loc}^{s+1+\frac{1}{p}}(\Omega^-)], \tag{6.19}$$

$$W^{(\sigma)},\ W_0 : B^s_{p,p}(S) \to H_p^{s+\frac{1}{p}}(\Omega^+)\ \ [B^s_{p,p}(S) \to H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)], \tag{6.20}$$

$$: B^s_{p,q}(S) \to B^{s+\frac{1}{p}}_{p,q}(\Omega^+) \ \ [B^s_{p,q}(S) \to B^{s+\frac{1}{p}}_{p,q,loc}(\Omega^-)]. \tag{6.21}$$

$$\mathcal{H} : H^s_p(S) \to H^{s+1}_p(S) \quad [B^s_{p,q}(S) \to B^{s+1}_{p,q}(S)], \tag{6.22}$$

$$\mathcal{K},\, \mathcal{K}^* : H^s_p(S) \to H^s_p(S) \quad [B^s_{p,q}(S) \to B^s_{p,q}(S)], \tag{6.23}$$

$$\mathcal{L} : H^{s+1}_p(S) \to H^s_p(S) \quad [B^{s+1}_{p,q}(S) \to B^s_{p,q}(S)], \tag{6.24}$$

where $s \in I\!R$, $1 < p < \infty$, $1 \leq q \leq \infty$, $S \in C^\infty$.

The null-spaces of the operators (6.22)-(6.24) *are invariant with respect to p, q, and s.*

7 Basic boundary value problems of pseudo-oscillations

7.1. Formulation of the basic BVPs

Throughout this section we assume (if not otherwise stated)

$$\begin{aligned} &\sigma = \sigma_1 + i\sigma_2,\ \sigma_1 \in I\!R,\ \sigma_2 > 0; \\ &k_j \neq k_p \text{ for } j \neq p,\ \Im k_j > 0 \text{ for } j,p = \overline{1,6}. \end{aligned} \tag{7.1}$$

We shall investigate the following BVPs:
Find a solution $U \in (u,\omega)^\top$ to the differential equation

$$L(\partial,\sigma)U(x) = \Phi(x) \ \text{ in } \ \Omega^\pm \tag{7.2}$$

satisfying one of the following boundary conditions on $S = \partial\Omega^\pm$:

Problem $(I^{(\sigma)})^\pm$ (the Dirichlet problem):

$$[U(x)]^\pm = f(x), \quad x \in S; \tag{7.3}$$

Problem $(II^{(\sigma)})^\pm$ (the Neumann problem):

$$[T(\partial,n)U(x)]^\pm = F(x), \quad x \in S; \tag{7.4}$$

Problem $(III^{(\sigma)})^\pm$ (a mixed problem):

$$[U(x)]^\pm = f_D(x), \quad x \in S_D, \tag{7.5}$$

$$[T(\partial,n)U(x)]^\pm = F_N(x), \quad x \in S_N, \tag{7.6}$$

where S_D and S_N are two open, disjoint parts of S and $\overline{S_D} \cup \overline{S_N} = S$.

We look for either a regular solution (to Problems $(I^{(\sigma)})^{\pm}$ and $(II^{(\sigma)})^{\pm}$)

$$U \in C^1(\overline{\Omega^{\pm}}) \cap C^2(\Omega^{\pm}) \tag{7.7}$$

or a weak solution (to Problems $(I^{(\sigma)})^{\pm}$, $(II^{(\sigma)})^{\pm}$), and $(III^{(\sigma)})^{\pm}$)

$$U \in H^1(\Omega^+) \quad \left[U \in H^1_{loc}(\Omega^-) \right]. \tag{7.8}$$

Note that, due to the strong ellipticity property of the operator $L(\partial, \sigma)$ and the restriction (7.1), an arbitrary polynomially bounded solution of the equation (7.2) in Ω^-, actually decays exponentially as $|x| \to +\infty$ if $\operatorname{supp} \Phi$ is compact (see (3.45)-(3.46)).

In the case of regular setting, the vector function Φ and the boundary data f and F belong to some Hölder spaces

$$\Phi \in C^{0,\alpha_0}(\overline{\Omega^{\pm}}),\ \operatorname{supp} \Phi \ \text{ is compact},\ f \in C^{1,\alpha_0}(S),\ F \in C^{0,\alpha_0}(S), \tag{7.9}$$

while in the case of weak formulation they belong to the functional spaces

$$\begin{aligned} & f \in H^{\frac{1}{2}}(S), \quad F \in H^{-\frac{1}{2}}(S), \\ & f_D \in H^{\frac{1}{2}}(S_D), \quad F_N \in H^{-\frac{1}{2}}(S_N), \\ & \Phi \in \tilde{H}^{-1}(\Omega^+) \quad \left[\Phi \in \tilde{H}^{-1}_{comp}(\Omega^-) \right], \end{aligned} \tag{7.10}$$

where for $\Omega \subset I\!\!R^3$ and $\mathcal{M} \subset S$

$$\begin{aligned} & \tilde{H}^s_p(\Omega) := \left\{ \Phi \in H^s_p(I\!\!R^3) : \operatorname{supp} \Phi \subset \overline{\Omega} \right\}, \\ & H^s_p(\mathcal{M}) := \left\{ f|_{\mathcal{M}} : f \in H^s_p(S) \right\}, \\ & B^s_{p,q}(\mathcal{M}) := \left\{ f|_{\mathcal{M}} : f \in B^s_{p,q}(S) \right\}, \\ & \tilde{H}^s_p(\mathcal{M}) := \{ f \in H^s_p(S) : \operatorname{supp} f \subset \overline{\mathcal{M}} \}, \\ & \tilde{B}^s_{p,q}(\mathcal{M}) := \{ f \in B^s_{p,q}(S) : \operatorname{supp} f \subset \overline{\mathcal{M}} \}, \end{aligned}$$

where $f|_{\mathcal{M}}$ denotes the restriction of f to $\mathcal{M}$.

As it is well known, even for C^{∞}-smooth domains and C^{∞}-smooth data, solutions to mixed BVPs do not belong to the space $C^{0,\alpha_0}(\overline{\Omega^{\pm}})$ with $\alpha_0 \geq 1/2$, in general. Solutions or their derivatives have singularities at the collision curves of changing boundary conditions. Therefore, we will investigate the mixed BVP $(III^{(\sigma)})^{\pm}$ in the weak formulation and along with the correct solvability in the corresponding functional spaces we will establish Hölder C^{0,α_0}-continuity of solutions (with exponent $\alpha_0 < 1/2$).

Note that in the case of weak setting of the above BVPs, i.e., when $U \in H^1(\Omega^{\pm})$ and the conditions (7.10) hold, the Dirichlet conditions (7.3) and (7.5) are understood in the usual trace sense, while the Neumann conditions

(7.4) and (7.6) are understood in the functional sense with $[T(\partial,n)U]^{\pm} \in H^{-\frac{1}{2}}(S)$ defined by the relation

$$\langle\, [T(\partial,n)U]_S^{\pm}\,,\, [U']_S^{\pm}\,\rangle_S := \pm \int\limits_{\Omega^{\pm}} E(U,U')\,dx$$

for all $U' \in H^1(\Omega^{\pm})$, where $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$.

7.2. Uniqueness theorem

In this subsection we assume that Ω^+ and Ω^- are Lipschitz domains.

Theorem 7.1 *The homogeneous versions of Problems $(I^{(\sigma)})^{\pm}$, $(II^{(\sigma)})^{\pm}$, and $(III^{(\sigma)})^{\pm}$ have only the trivial solution in the space $H^1(\Omega^{\pm})$.*

Proof. Let $U \in H^1(\Omega^+)$ solve one of the interior homogeneous boundary value problem mentioned in the theorem. We apply Green's formula (2.33) with U and $U' = U$. We obtain (see (2.12))

$$\int\limits_{\Omega^+} [\,E(U,\overline{U}) - \varrho\,\overline{\sigma}^2\,|u|^2 - \mathcal{I}\,\overline{\sigma}^2\,|\omega|^2]\,dx = 0, \tag{7.11}$$

since $\left\langle [U]_S^+\,,\,\overline{[T(\partial,n)U]_S^+}\right\rangle_S = 0$ due to the homogeneous boundary conditions. Note that $E(U,\overline{U}) \geq 0$ due to (2.23).

Separating the imaginary part we get from (7.11)

$$2\,\sigma_1\,\sigma_2 \int\limits_{\Omega^+} [\,\varrho\,|u|^2 + \mathcal{I}\,|\omega|^2]\,dx = 0,$$

whence $u = 0$ and $\omega = 0$ in Ω^+ follow if $\sigma_1 \neq 0$.

For $\sigma_1 = 0$ we have from (7.11)

$$\int\limits_{\Omega^+} [\,E(U,\overline{U}) + \varrho\,\sigma_2^2\,|u|^2 + \mathcal{I}\,\sigma_2^2\,|\omega|^2]\,dx = 0,$$

which implies $U = 0$ in Ω^+. Thus the conclusion of the theorem holds for the interior problems.

The proof for the exterior BVPs in Ω^- is quite similar, since $U \in H^1(\Omega^-)$ and we can write Green's formula

$$\int\limits_{\Omega^-} [U' \cdot L(\partial)U + \,E(U',\overline{U})\,]\,dx = -\left\langle [U']_S^-\,,\,\overline{[T(\partial,n)U]_S^-}\right\rangle_S \tag{7.12}$$

with arbitrary $U' \in H^1(\Omega^-)$. ∎

Remark 7.2 *It is evident that the nonhomogeneous BVPs* $(I^{(\sigma)})^{\pm}$, $(II^{(\sigma)})^{\pm}$, $(III^{(\sigma)})^{\pm}$ *have at most one solution in the spaces* $H^1(\Omega^{\pm})$ *and* $C^1(\overline{\Omega^{\pm}})$.

7.3. Existence results

In what follows, without loss of generality we assume that $\Phi = 0$ in (7.2), since a corresponding particular solution $U_{\pm}^{(p)}$ can be written explicitly as a volume potential (see (4.3))

$$U_{\pm}^{(p)}(x) := N_{\Omega^{\pm}}^{(\sigma)}(\Phi)(x) = \int\limits_{\Omega^{\pm}} \Gamma(x-y,\sigma)\,\Phi(y)\,dy, \quad x \in \Omega^{\pm}, \tag{7.13}$$

where $\Gamma(\,\cdot\,,\sigma)$ is the fundamental matrix (3.40) of the operator $L(\partial,\sigma)$.

7.3.1. Problems $(I^{(\sigma)})^{\pm}$ and $(II^{(\sigma)})^{\pm}$

First we consider the regular case, i.e.,

$$\begin{aligned} &S = \partial\Omega^{\pm} \in C^{k,\alpha_0},\ f \in C^{1,\gamma_0}(S), \\ &F \in C^{0,\gamma_0}(S),\ 0 < \gamma_0 < \alpha_0 \le 1,\ k \ge 2. \end{aligned} \tag{7.14}$$

We look for a solution to Problem $(I^{(\sigma)})^{\pm}$ in the form of the double-layer potential (see (4.2))

$$U(x) = W^{(\sigma)}(g)(x), \quad x \in \Omega^{\pm}, \tag{7.15}$$

while a solution to Problem $(II^{(\sigma)})^{\pm}$ we seek in the form of the single-layer potential (see (4.1))

$$U(x) = V^{(\sigma)}(h)(x), \quad x \in \Omega^{\pm}, \tag{7.16}$$

where $g \in [C^{1,\gamma_0}(S)]^6$ and $h \in [C^{0,\gamma_0}(S)]^6$ are sought for densites.

Invoking Theorem 6.1 and the boundary conditions (7.3) and (7.4) we see that the BVPs $(I^{(\sigma)})^{\pm}$ and $(II^{(\sigma)})^{\pm}$ are reduced to the singular integral equations (see (6.8) and (6.7)), respectively:

$$\Big[\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)*}\Big]\, g(x) = f(x), \quad x \in S, \tag{7.17}$$

and

$$\Big[\mp 2^{-1} I_6 + \mathcal{K}^{(\sigma)}\Big]\, h(x) = F(x), \quad x \in S. \tag{7.18}$$

Lemma 7.3 *Let* S, k, α_0, *and* γ_0 *be as in* (7.14). *The operators*

$$\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)}\ :\ C^{0,\gamma_0}(S) \to C^{0,\gamma_0}(S) \tag{7.19}$$

$$\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)*}\ :\ C^{0,\gamma_0}(S) \to C^{0,\gamma_0}(S) \tag{7.20}$$

are invertible.

Proof. Let us show the invertibility of the operator $2^{-1}I_6 + \mathcal{K}^{(\sigma)}$. By Corollary 6.4 it is a singular integral operator of normal type with index zero. Due to the general theory of singular integral equations (see, e.g., [18], Ch. IV; [27], Ch. XII-XIV) it remains to show the injectivity of the operator in question. Therefore we have to prove that the homogeneous equation

$$[2^{-1}I_6 + \mathcal{K}^{(\sigma)}]\, h(x) = 0, \quad x \in S \tag{7.21}$$

possesses only the trivial solution. This can be shown by standard arguments.

Indeed, let h be a solution to (7.21) and construct the single-layer potential

$$U(x) = V^{(\sigma)}(h)(x). \tag{7.22}$$

This vector solves the homogeneous BVP $(II^{(\sigma)})^-$ and therefore vanishes in Ω^- due to Theorem 7.1. Since the single-layer potential is continuous in $I\!\!R^3$, the vector (7.22) solves the interior homogeneous BVP $(I^{(\sigma)})^+$ and due to Theorem 7.1 vanishes in Ω^+ as well. Recall that (see (6.3))

$$h = [TV^{(\sigma)}(h)]_S^- - [TV^{(\sigma)}(h)]_S^+.$$

Therefore $h = 0$. Thus the operator

$$2^{-1}I_6 + \mathcal{K}^{(\sigma)} \ : \ C^{0,\gamma_0}(S) \to C^{0,\gamma_0}(S)$$

is injective and, consequently, it is invertible.

Clearly, the same property holds also for the formally adjoint operator $2^{-1}I_6 + \mathcal{K}^{(\sigma)*}$.

The proof for the operators $-2^{-1}I_6 + \mathcal{K}^{(\sigma)}$ and $-2^{-1}I_6 + \mathcal{K}^{(\sigma)*}$ is word for word. ∎

Remark 7.4 *Applying the well known embedding theorems for singular integral equations* (*see, e.g.,* [18], Ch. IV, §6)*, in the same way as above we can show that the operators*

$$\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)}, \ \ \pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)*} \ : \ L_2(S) \to L_2(S)$$

are also invertible.

Moreover, since (7.17) *and* (7.18) *are singular integral equations of normal type* (*i.e., the corresponding principal symbol matrices are nondegenerate*) *we have that, if S is as in* (7.14) *and $f, F \in C^{k-1,\gamma_0}(S)$, then the solutions $g, h \in C^{k-1,\gamma_0}(S)$* (*see, e.g.,* [18], Ch. IV, §6).

Now we can formulate the basic existence results which immediately follow from Theorem 7.1, Lemma 7.3, and Remark 7.4.

Theorem 7.5 *Let S and f be as in* (7.14) *with $k = 2$. Then Problems $(I^{(\sigma)})^{\pm}$ (with $\Phi = 0$) are uniquely solvable in the space of regular vector-functions. Moreover, the solutions belong to the space $C^{1,\gamma_0}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm})$ and they can be represented by the double-layer potential* (7.15) *where the density vector $g \in C^{1,\gamma_0}(S)$ solves the corresponding integral equation* (7.17)*.*

Theorem 7.6 *Let S and F be as in* (7.14) *with $k = 1$. Then Problems $(II^{(\sigma)})^{\pm}$ (with $\Phi = 0$) are uniquely solvable in the space of regular vector-functions. Moreover, the solutions belong to the space $C^{1,\gamma_0}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm})$ and they can be represented by the single-layer potential* (7.16) *where the density vector $h \in C^{0,\gamma_0}(S)$ solves the corresponding integral equation* (7.18)*.*

Theorem 6.3 and Corollary 6.4 give us possibility to represent solutions of Problems $(I^{(\sigma)})^{\pm}$ and $(II^{(\sigma)})^{\pm}$ by means of single-layer and doble-layer potentials, respectively.

Theorem 7.7 *Let S and f be as in Theorem* 7.5*. Then the unique regular solution of Problems $(I^{(\sigma)})^{\pm}$ (with $\Phi = 0$) can be represented in the form of single-layer potential $U(x) = V^{(\sigma)}(h)(x)$ $(x \in \Omega^{\pm})$ where the density vector $h \in C^{0,\gamma_0}(S)$ solves the integral equation*

$$\mathcal{H}^{(\sigma)}\, h(x) = f(x), \quad x \in S. \tag{7.23}$$

Proof. It can easily be shown that the operator

$$\mathcal{H}^{(\sigma)} \ : \ C^{0,\gamma_0}(S) \to C^{1,\gamma_0}(S) \tag{7.24}$$

is injective. In what follows we will show that (7.24) is surjective. To this end let us apply the operator $\mathcal{L}^{(\sigma)}$ to both sides of (7.23) to obtain the equation

$$\mathcal{L}^{(\sigma)}\, \mathcal{H}^{(\sigma)}\, h(x) = \mathcal{L}^{(\sigma)}\, f(x), \quad x \in S, \tag{7.25}$$

and prove that (7.23) and (7.25) are equivalent. Due to Theorem 6.3 and Corollary 6.4, $\mathcal{L}^{(\sigma)}\, \mathcal{H}^{(\sigma)} = -4^{-1} I_6 + (\mathcal{K}^{(\sigma)})^2$ is a singular integral operator of normal type with index zero and by Lemma 7.3 the operator

$$\mathcal{L}^{(\sigma)}\, \mathcal{H}^{(\sigma)} \ : \ C^{0,\gamma_0}(S) \to C^{0,\gamma_0}(S) \tag{7.26}$$

is invertible. Therefore equation (7.25) is solvable in the space $C^{0,\gamma_0}(S)$ for an arbitrary $f \in C^{1,\gamma_0}(S)$.

By standard arguments, with the help of Theorem 7.1, it can be shown that $\mathcal{L}^{(\sigma)}\varphi(x) = 0$ on S implies $\varphi(x) = 0$ on S.

Thus, if h solves equation (7.25) then it solves also equation (7.23), and vice versa, i.e., equations (7.23) and (7.25) are equivalent.

From the above arguments it follows that the operator (7.24) is inverible, which completes the proof. ∎

Theorem 7.8 *Let S and F be as in Theorem 7.6. Then the unique regular solution of Problems $(II^{(\sigma)})^{\pm}$ (with $\Phi = 0$) can be represented in the form of double-layer potential $U(x) = W^{(\sigma)}(g)(x)$ $(x \in \Omega^{\pm})$ where the density vector $g \in C^{1,\gamma_0}(S)$ solves the pseudodifferential (singular integro-differential) equation*

$$\mathcal{L}^{(\sigma)}\, g(x) = F(x), \quad x \in S. \tag{7.27}$$

Proof. We have to show that (7.27) is uniquely solvable for an arbitrary $F \in C^{0,\gamma_0}(S)$. As we have mentioned in the proof of Theorem 7.7, the operator

$$\mathcal{L}^{(\sigma)} \;:\; C^{1,\gamma_0}(S) \to C^{0,\gamma_0}(S) \tag{7.28}$$

is injective.

Next we establish that (7.28) is surjective. To this end let us apply the operator $\mathcal{H}^{(\sigma)}$ to both sides of (7.28) to obtain the equation

$$\mathcal{H}^{(\sigma)}\, \mathcal{L}^{(\sigma)}\, g(x) = \mathcal{H}^{(\sigma)}\, F(x), \quad x \in S. \tag{7.29}$$

By Theorems 6.3 and Corollary 6.4 we have that the operator $\mathcal{H}^{(\sigma)}\,\mathcal{L}^{(\sigma)} = -4^{-1}I_6 + (\mathcal{K}^{(\sigma)*})^2$ is a singular integral operator of normal type with index zero and by Lemma 7.3 the operator

$$\mathcal{H}^{(\sigma)}\,\mathcal{L}^{(\sigma)} \;:\; C^{1,\gamma_0}(S) \to C^{1,\gamma_0}(S)$$

is invertible.

Therefore equation (7.29) is solvable in the space $C^{1,\gamma_0}(S)$ for an arbitrary $F \in C^{0,\gamma_0}(S)$. Since the operator (7.24) is bijective, we conclude that equations (7.28) and (7.29) are equivalent. In turn this implies that (7.28) is surjective. Thus the operator (7.28) is bijective, which completes the proof. ∎

By standard arguments we can extend the above existence results to the case of the weak setting (cf. [7], [8], [16], [17], [13], [13], [26]). We have the following theorems.

Theorem 7.9 *The operators*

$$\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)*} \quad : \quad H^{\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \tag{7.30}$$

$$\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)} \quad : \quad H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S), \tag{7.31}$$

$$\mathcal{H}^{(\sigma)} \quad : \quad H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \tag{7.32}$$

$$\mathcal{L}^{(\sigma)} \quad : \quad H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S), \tag{7.33}$$

are invertible.

Moreover, if $S \in C^\infty$ *then the operators*

$$\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)*} \quad : \quad H^s_p(S) \to H^s_p(S) \quad [B^s_{p,q}(S) \to B^s_{p,q}(S)],$$

$$\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)} \quad : \quad H^s_p(S) \to H^s_p(S) \quad [B^s_{p,q}(S) \to B^s_{p,q}(S)],$$

$$\mathcal{H}^{(\sigma)} \quad : \quad H^s_p(S) \to H^{s+1}_p(S) \quad [B^s_{p,q}(S) \to B^{s+1}_{p,q}(S)],$$

$$\mathcal{L}^{(\sigma)} \quad : \quad H^{s+1}_p(S) \to H^s_p(S) \quad [B^{s+1}_{p,q}(S) \to B^s_{p,q}(S)],$$

are invertible for $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

Proof. It is a straightforward consequence of Theorems 6.2, 6.3, 6.5 and 7.1. ∎

Applying these results we can easily prove the following assertions.

Theorem 7.10 *Let* $f \in H^{\frac{1}{2}}(S)$. *Then there exists a unique solution* $U \in H^1(\Omega^\pm)$ *of Problem* $(I^{(\sigma)})^\pm$ *which can be represented by a double-layer potential* $U(x) = W^{(\sigma)}(g)(x)$ $(x \in \Omega^\pm)$ *where the density vector* $g \in H^{\frac{1}{2}}(S)$ *solves the pseudodifferential equation on* S

$$[\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)*}]\, g = f$$

(the sign "+" corresponds to the domain Ω^+ *and "−" to the domain* Ω^-*).*

Moreover, the solution vector U *can also be represented by a single-layer potential* $U(x) = V^{(\sigma)}(h)(x)$ $(x \in \Omega^\pm)$ *where the density vector* $h \in H^{-\frac{1}{2}}(S)$ *solves the uniquely solvable pseudodifferential equation* $\mathcal{H}^{(\sigma)}\, h = f$.

Theorem 7.11 *Let* $F \in H^{-\frac{1}{2}}(S)$. *Then there exists a unique solution* $U \in H^1(\Omega^\pm)$ *of Problem* $(II^{(\sigma)})^\pm$ *which can be represented by a single-layer potential* $U(x) = V^{(\sigma)}(h)(x)$ $(x \in \Omega^\pm)$ *where the density vector* $h \in H^{-\frac{1}{2}}(S)$ *solves the pseudodifferential equation on* S

$$[\mp 2^{-1} I_6 + \mathcal{K}^{(\sigma)}]\, h = f$$

(the sign "−" corresponds to the domain Ω^+ *and "+" to the domain* Ω^-*).*

Moreover, the solution vector U *can also be represented by a double-layer potential* $U(x) = W^{(\sigma)}(g)(x)$ $(x \in \Omega^\pm)$ *where the density vector* $g \in H^{\frac{1}{2}}(S)$ *solves the uniquely solvable pseudodifferential equation* $\mathcal{L}^{(\sigma)}\, g = F$.

Remark 7.12 *Theorem* 7.10 *implies that a unique solution to Problem* $(I^{(\sigma)})^{\pm}$ *can be represented as a single-layer potential*

$$U(x) = V^{(\sigma)}\left([\mathcal{H}^{(\sigma)}]^{-1} f\right)(x), \quad x \in \Omega^{\pm}, \tag{7.34}$$

where $f = [U]^{+}$ *and* $[\mathcal{H}^{(\sigma)}]^{-1}$ *is a pseudodifferential operator inverse to the operator* (7.32). *Clearly, the principal homogeneous symbol matrix of the operator* $[\mathcal{H}^{(\sigma)}]^{-1}$ *is negative definite* (*cf. Corollary* 6.4).

Moreover, if $f \in B^{1-1/p}_{p,q}(S)$ *with* $1 < p < \infty$ *and* $1 \le q \le \infty$, *Problem* $(I^{(\sigma)})^{\pm}$ *has a unique solution in the space* $[B^{1}_{p,q}(\Omega^{\pm})]^{6}$ *which is representable in the form* (7.34) *where* $[\mathcal{H}^{(\sigma)}]^{-1}$ *is a pseudodifferential operator inverse to the operator* $\mathcal{H}^{(\sigma)} : B^{-1/p}_{p,q}(S) \to B^{1-1/p}_{p,q}(S)$. *The proof is quite similar to the proof of Theorem* 12.10 *in* [13].

7.3.2. Some results from the theory of pseudodifferential equations on manifolds with boundary.

In this subsection we shall present some principal results from the theory of elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces. They can be found in [10], [39], [13, [40], [41], [42], [6], and will be the main tools for proving existence theorems for the mixed problems.

Let $\mathcal{M} \in C^{\infty}$ be a compact, n-dimensional, nonselfintersecting, two-sided manifold with boundary $\partial\mathcal{M} \in C^{\infty}$ and let $\mathcal{A}$ be a strongly elliptic $m \times m$ matrix pseudodifferential operator of order $\alpha \in I\!R$ on $\overline{\mathcal{M}}$. Denote by $\sigma_{\mathcal{A}}(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \mathcal{M}$, $\xi \in I\!R^{n} \setminus \{0\}$) and associate with $\sigma_{\mathcal{A}}$ the $m \times m$ matrix function

$$\mathcal{A}_{0\eta}(x,\xi) = |\xi|^{-\alpha}\sigma_{\mathcal{A}}(x, |\xi'|\eta, \xi_n), \tag{7.35}$$

where $\eta \in S^{n-2} \subset I\!R^{n-1}$ with S^{n-2} the unit sphere in $I\!R^{n-1}$ and $\xi' = (\xi_1, ..., \xi_{n-1})$.

It is known that the matrix $\mathcal{A}_{0\eta}$ in (7.35) admits the factorization

$$\mathcal{A}_{0\eta}(x,\xi) = \mathcal{A}^{-}_{\eta}(x,\xi)D(\eta,x,\xi)\mathcal{A}^{+}_{\eta}(x,\xi) \quad \text{for } x \in \partial\mathcal{M},$$

where $[\mathcal{A}^{-}_{\eta}(x,\xi)]^{\pm 1}$ and $[\mathcal{A}^{+}_{\eta}(x,\xi)]^{\pm 1}$ are matrices, which are homogeneous of degree 0 in ξ and admit analytic bounded continuations with respect to ξ_n into the lower and upper complex half-planes, respectively. $D(\eta,x,\xi)$ is a bounded lower triangular matrix with entries of the form

$$\left(\frac{\xi_n - i\,|\xi'|}{\xi_n + i\,|\xi'|}\right)^{\delta_j(x)}, \; j = 1, \cdots, m,$$

on the main diagonal; here

$$\delta_j(x) = (2\pi\, i)^{-1} \ln \lambda_j(x),\ \ j = 1, \cdots, m,$$

where $\lambda_1(x), \cdots,\ \lambda_m(x)$ are the eigenvalues of the matrix

$$\tilde{\sigma}_{\mathcal{A}}(x) = [\sigma_{\mathcal{A}}(x, 0, \cdots, 0, -1)]^{-1}[\sigma_{\mathcal{A}}(x, 0, \cdots, 0, +1)].$$

The branch in the logarithmic function is chosen with regard to the inequality $1/p-1 < \mathrm{Re}\,\delta_j(x) < 1/p,\ j = 1, \cdots, m$. The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system.

Note that if $\sigma_{\mathcal{A}}(x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in I\!\!R^n \setminus \{0\}$, then

$$\mathrm{Re}\,\delta_j(x) = 0 \quad \text{for} \quad j = 1, \cdots, m, \tag{7.36}$$

since, in this case, the eigenvalues of the matrix $\tilde{\sigma}_{\mathcal{A}}(x)$ are positive numbers for any $x \in \overline{\mathcal{M}}$ (see [15], Lemma 6.4). The Fredholm properties of such operators are characterized by the following lemma (see [6], [40], [41], [42]).

Lemma 7.13 *Let $s \in I\!\!R$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\alpha \in I\!\!R$ having a positive definite principal homogeneous symbol matrix, i.e.,*

$\sigma_{\mathcal{A}}(x, \xi)\zeta \cdot \zeta \ge c_0|\zeta|^2$ for $x \in \overline{\mathcal{M}},\ \xi \in I\!\!R^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^m$, where c_0 is a positive constant.

Then the operators

$$A \ :\ \tilde{H}^s_p(\mathcal{M}) \to H^{s-\alpha}_p(\mathcal{M}), \tag{7.37}$$

$$\ :\ \tilde{B}^s_{p,q}(\mathcal{M}) \to B^{s-\alpha}_{p,q}(\mathcal{M}), \tag{7.38}$$

are Fredholm operators with index zero if

$$1/p - 1 < s - \alpha/2 < 1/p. \tag{7.39}$$

Moreover, the null-spaces and indices of the operators (7.37), (7.38) *are the same* (*for all values of the parameter $q \in [1, +\infty]$*) *provided p and s satisfy the inequality* (7.39).

7.3.3. Mixed Problem $(III^{(\sigma)})^{\pm}$

In this subsection we assume that (cf. (7.10))

$$\Phi = 0, \quad f_D \in H^{\frac{1}{2}}(S_D), \quad F_N \in H^{-\frac{1}{2}}(S_N). \tag{7.40}$$

Denote by $f^{(e)}$ some fixed extension of the vector-function f_D from S_D onto S preserving the functional space $H^{\frac{1}{2}}(S)$:

$$f^{(e)} \in H^{\frac{1}{2}}(S), \quad r_{S_D} f^{(e)} = f_D; \tag{7.41}$$

here and in what follows $r_{\mathcal{M}}$ is the restriction operator to $\mathcal{M}$.

Evidently, an arbitrary esxtension f of f_D onto the whole of S which preserves the functional space can be then represented as

$$f = f^{(e)} + \varphi \quad \text{with} \quad \varphi \in \tilde{H}^{\frac{1}{2}}(S_N). \tag{7.42}$$

For definiteness, first we consider Problem $(III^{(\sigma)})^+$ (Problem $(III^{(\sigma)})^-$ can be considered quite similarly).

In accordance with Theorem 7.10 and Remark 7.12 we can look for a solution in the form

$$U(x) = V^{(\sigma)}\Big([\mathcal{H}^{(\sigma)}]^{-1}\,(f^{(e)} + \varphi)\Big)(x), \tag{7.43}$$

where $\varphi \in \tilde{H}^{\frac{1}{2}}(S_N)$ is an unknown vector function.

It is easy to chek that the Dirichlet condition (7.5) on S_D is satisfied automatically. It remains only to satisfy the Neumann condition (7.6) on S_N which leads to the pseudodifferential equation

$$[-2^{-1}I_6 + \mathcal{K}^{(\sigma)}]\,[\mathcal{H}^{(\sigma)}]^{-1}\,(f^{(e)} + \varphi) = F_N \tag{7.44}$$

on the open subsurface S_N for the unknown vector function φ.

Let

$$\mathcal{N}^{(\sigma)} := [-2^{-1}I_6 + \mathcal{K}^{(\sigma)}]\,[\mathcal{H}^{(\sigma)}]^{-1}, \tag{7.45}$$

$$F^{(0)} := F_N - r_{S_N}\mathcal{N}^{(\sigma)} f^{(e)} \in H^{-\frac{1}{2}}(S_N). \tag{7.46}$$

Equation (7.44) can be then rewritten in the form

$$r_{S_N}\mathcal{N}^{(\sigma)}\varphi = F^{(0)} \quad \text{on} \quad S_N. \tag{7.47}$$

It can be seen that the operator $\mathcal{N}^{(\sigma)}$ has the mapping property

$$\mathcal{N}^{(\sigma)} \,:\, H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S). \tag{7.48}$$

Moreover, there hold the following assertions.

Lemma 7.14 *The operator $\mathcal{N}^{(\sigma)}$ is a strongly elliptic pseudodifferential operator of order* 1 *with a positive definite principal homogeneous symbol matrix.*

Proof. It is evident that the principal homogeneous symbol matrices of the operators (7.45) and its main singular part $\mathcal{N}_0 := [-2^{-1}I_6 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}$ are the same. Note that the operators with subscript 0 are generated by the potentials with the kernel matrix $\Gamma_0(\,\cdot\,)$ which is a fundamental matrix of the operator $L_0(\partial)$ (see (2.16)) and represents a principal singular part of the matrix $\Gamma(\,\cdot\,,\sigma)$ (see (3.45)).

We write Green's formula (5.5) in Ω^+ for real-valued vector functions $U(x) = U'(x) = V_0(\mathcal{H}_0^{-1}\,g)(x)$ with $g \in H^{\frac{1}{2}}(S)$ to obtain

$$\big\langle [T_0(\partial,n)V_0(\mathcal{H}_0^{-1}\,g)]^+\,,\,[V_0(\mathcal{H}_0^{-1}\,g)]^+\big\rangle_S = \int\limits_{\Omega^+} E_0\left(V_0(\mathcal{H}_0^{-1}\,g), V_0(\mathcal{H}_0^{-1}\,g)\right)\,dx$$

which implies (see (5.7) and Lemma 6.1)

$$\big\langle [-2^{-1}I_6 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}\,g\,,\,g\big\rangle_S \geq 0. \tag{7.49}$$

Since the principal homogeneous symbol matrices of the operators $-2^{-1}I_6 + \mathcal{K}_0$ and $\mathcal{H}_0^{-1}$ are nondegenerate (see Theorem 6.3) and g is an arbitrary vector function of the space $[H^{\frac{1}{2}}(S)]^6$, it follows from (7.49) that the principal homogeneous symbol matrix of the composition of these operators (i.e., of the operator $\mathcal{N}_0$) is positive definite. ∎

Lemma 7.15 *The operators*

$$r_{S_N}\,\mathcal{N}^{(\sigma)} \;:\; \tilde{H}_p^s(S_N) \to H_p^{s-1}(S_N), \tag{7.50}$$

$$: \;\tilde{B}_{p,q}^s(S_N) \to B_{p,q}^{s-1}(S_N), \tag{7.51}$$

are invertible if

$$1/p - 1/2 < s < 1/p + 1/2. \tag{7.52}$$

Moreover, the operators (7.50) *and* (7.51) *have the trivial null-spaces and zero indices* (*for all values of the parameter* $q \in [1, +\infty]$) *provided* p *and* s *satisfy the inequality* (7.52).

Proof. The mapping properties (7.50) and (7.51) follow from Lemma 7.13 with $\alpha = 1$, since $\mathcal{N}^{(\sigma)}$ is a pseudodifferential operator of order 1 with a positive definite homogeneous symbol matrix due to Theorem 7.14.

To prove the invertibility of the operators (7.50) and (7.51) we first consider the case $p = 2$, $s = 1/2$, and $q = 2$, and show that the null space of the operator

$$r_{S_N}\,\mathcal{N}^{(\sigma)} : \tilde{H}^{\frac{1}{2}}(S_N) = \tilde{B}_{2,2}^{\frac{1}{2}}(S_N) \to H^{-\frac{1}{2}}(S_N) = \tilde{B}_{2,2}^{-\frac{1}{2}}(S_N)$$

is trivial, i.e., the equation

$$r_{S_N} \mathcal{N}^{(\sigma)} \varphi = 0 \quad \text{on} \quad S_N \tag{7.53}$$

admits only the trivial solution ($\varphi = 0$) in the space $\tilde{H}^{\frac{1}{2}}(S_N)$.

Indeed, let $\varphi \in \tilde{H}^{\frac{1}{2}}(S_N)$ be any solution of the homogeneous equation (7.53). It is evident that the vector

$$U(x) = V^{(\sigma)} \left([\mathcal{H}^{(\sigma)}]^{-1} \varphi \right)(x), \quad x \in \Omega^+$$

belongs to the space $H^1(\Omega^+) = W_2^1(\Omega^+)$ and solves the homogeneous mixed Problem $(III^{(\sigma)})^{\pm}$. Therefore, $U(x) = V^{(\sigma)} \left([\mathcal{H}^{(\sigma)}]^{-1} \varphi \right)(x) = 0$ for $x \in \Omega^+$, due to Theorem 7.1 and, consequently, $[U(x)]^+ = \varphi(x) = 0$ for $x \in S$. Since the principal singular part of the operator $\mathcal{N}^{(\sigma)}$ is self-adjoint (due to the positive definiteness of the principal homogeneous symbol matrix of $\mathcal{N}^{(\sigma)}$) we conclude that the index of $\mathcal{N}^{(\sigma)}$ is equal to zero and thus, by Lemma 7.13 the operator

$$r_{S_N} \mathcal{N}^{(\sigma)} \ : \ \tilde{H}^{\frac{1}{2}}(S_N) \to H^{-\frac{1}{2}}(S_N)$$

is invertible. Now Lemma 7.13 completes the proof. ∎

Theorem 7.16 *Let the conditions* (7.40) *be fulfilled. Then Problem* $(III^{(\sigma)})^+$ *has a unique solution representable in the form of* (7.43) *where* $\varphi \in \tilde{H}^{\frac{1}{2}}(S_N)$ *is defined by the uniquely solvable pseudodifferential equation* (7.47).

Proof. It follows from Theorem 7.1 and Lemma 7.15. ∎

Corollary 7.17 *Let* $4/3 < p < 4$ *and*

$$\Phi = 0, \quad f_D \in B_{p,p}^{1-1/p}(S_D), \quad F_N \in B_{p,p}^{-1/p}(S_N). \tag{7.54}$$

Then Problem $(III^{(\sigma)})^+$ *has a unique solution* $U \in W_p^1(\Omega^+)$ *which is representable in the form of* (7.43) *where* $f^{(e)} \in B_{p,p}^{1-1/p}(S)$ *is some fixed extension of the vector function* $f_D \in B_{p,p}^{1-1/p}(S_D)$ *from* S_D *onto* S *preserving the functional space* $B_{p,p}^{1-1/p}(S)$ *and* $\varphi \in \tilde{B}_{p,p}^{1-1/p}(S_N)$ *is defined by the uniquely solvable pseudodifferential equation*

$$r_{S_N} \mathcal{N}^{(\sigma)} \varphi = F^{(0)} \quad \text{on} \quad S_N \tag{7.55}$$

with

$$F^{(0)} := F_N - r_{S_N} \mathcal{N}^{(\sigma)} f^{(e)} \in B_{p,p}^{-1/p}(S_N).$$

Proof. First we note that in accordance with Lemma 7.13 equation (7.55) is uniquely solvable for $s = 1 - 1/p$ and $4/3 < p < 4$, where the last equality follows from the inequality (7.52). This implies the solvability of Problem $(III^{(\sigma)})^+$ in the space $W_p^1(\Omega^+)$ with $p \in (4/3, 4)$.

Next we show the uniqueness of solution in the space $W_p^1(\Omega^+)$ for arbitrary $p \in (4/3, 4)$ (for $p = 2$ it has been proved in Theorem 7.1). Let $U \in W_p^1(\Omega^+)$ be some solution of the homogeneous Problem $(III^{(\sigma)})^+$. Clearly, then

$$[U]^+ \in \tilde{B}_{p,p}^{1-1/p}(S_N). \tag{7.56}$$

By Remark 7.12 we have the representation

$$U(x) = V^{(\sigma)}\left([\mathcal{H}^{(\sigma)}]^{-1}[U]^+\right)(x), \quad x \in \Omega^+.$$

Since U satisfies the homogeneous Neumann condition (7.6) on S_N, we have

$$r_{S_N} \mathcal{N}^{(\sigma)}[U]^+ = 0 \quad \text{on} \quad S_N,$$

whence $[U]^+ = 0$ on S follows due to the inclusion (7.56), Lemma 7.15, and the inequality $4/3 < p < 4$. Therefore, $U = 0$ in Ω^+. ∎

Further we prove the main regularity result for a solution to Problem $(III^{(\sigma)})^+$.

Theorem 7.18 *Let the conditions* (7.54) *and*

$$4/3 < p < 4, \ 1 < t < \infty, \ 1 \le q \le \infty, \ 1/t - 1/2 < s < 1/t + 1/2, \tag{7.57}$$

be fulfilled, and let $U \in W_p^1(\Omega^+)$ be the unique solution to the mixed problem $(III^{(\sigma)})^+$.

In addition to (7.54),

i) *if*

$$f_D \in B_{t,t}^s(S_D), \ F_N \in B_{t,t}^{s-1}(S_N), \tag{7.58}$$

then

$$U \in H_t^{s+1/t}(\Omega^+); \tag{7.59}$$

ii) *if*

$$f_D \in B_{t,q}^s(S_D), \ F_N \in B_{t,q}^{s-1}(S_N), \tag{7.60}$$

then

$$U \in B_{t,q}^{s+1/t}(\Omega^+); \tag{7.61}$$

iii) *if*

$$f_D \in C^{\alpha_0}(S_D), \ \ F_N \in B^{\alpha_0-1}_{\infty,\infty}(S_N), \ \ \alpha_0 > 0, \tag{7.62}$$

then

$$U \in C^{\beta_0}(\overline{\Omega^+}) \ \text{with any} \ \beta_0 \in (0, \alpha_1), \ \ \alpha_1 := \min\{\alpha_0, 1/2\}. \tag{7.63}$$

Proof. Applying Corollary 7.17, Lemma 7.15, the inclusions(7.54) and (7.58) [resp. (7.60)] along with the inequalities (7.57), we conclude from (7.58) that $\varphi \in \tilde{B}^s_{t,t}(S_N)$ [resp. $\varphi \in \tilde{B}^s_{t,q}(S_N)$] since $F_0 \in B^{s-1}_{t,t}(S_N)$ [resp. $F_0 \in B^{s-1}_{t,q}(S_N)$].

Note that $f^{(e)} \in B^s_{t,t}(S)$ [resp. $f^{(e)} \in B^s_{t,q}(S)$] is some extension of the vector f_D onto the whole of S. Therefore, by Theorem 6.5 and the representation formula (7.43) the inclusion (7.59) [resp. (7.61)] follows.

To prove (iii) we use the following embeddings (see, e.g., [46], [47])

$$C^{\alpha_0}(\mathcal{S}) = B^{\alpha_0}_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha_0-\varepsilon_0}_{\infty,1}(\mathcal{S}) \subset B^{\alpha_0-\varepsilon_0}_{\infty,q}(\mathcal{S})$$

$$\subset B^{\alpha_0-\varepsilon_0}_{t,q}(\mathcal{S}) \subset C^{\alpha_0-\varepsilon_0-k/t}(\mathcal{S}), \tag{7.64}$$

where ε_0 is an arbitrary small positive number, $\mathcal{S} \subset I\!R^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < t < \infty$, $\alpha_0 - \varepsilon_0 - k/t > 0$, α_0 and $\alpha_0 - \varepsilon_0 - k/t$ are not integers. From (7.62) and the embeddings (7.64) the condition (7.61) follows with any $s \le \alpha_0 - \varepsilon_0$.

Bearing in mind (7.57) and taking t sufficiently large and ε_0 sufficiently small, we may put $s = \alpha_0 - \varepsilon_0$ if

$$1/t - 1/2 < \alpha_0 - \varepsilon_0 < 1/t + 1/2, \tag{7.65}$$

and $s \in (1/t - 1/2, 1/t + 1/2)$ if

$$1/t + 1/2 < \alpha_0 - \varepsilon_0. \tag{7.66}$$

By (7.61) the solution U belongs then to $B^{s+1/t}_{t,q}(\Omega^+)$ with $s + 1/t = \alpha_0 - \varepsilon_0 + 1/t$ if (7.65) holds, and with $s + 1/t \in (2/t - 1/2, 2/t + 1/2)$ if (7.66) holds. In the last case we can take $s + 1/t = 2/t + 1/2 - \varepsilon_0$. Therefore, we have either $U \in B^{\alpha_0-\varepsilon_0+1/t}_{t,q}(\Omega^+)$, or $U \in B^{1/2+2/t-\varepsilon_0}_{t,q}(\Omega^+)$ in accordance with the inequalities (7.65) and (7.66). The last embedding in (7.64) (with $k = 3$) yields that either $U \in C^{\alpha_0-\varepsilon_0-2/t}(\overline{\Omega^+})$, or $U \in C^{1/2-\varepsilon_0-1/t}(\overline{\Omega^+})$ which lead to the inclusion

$$U \in C^{\alpha_1-\varepsilon_0-2/t}(\overline{\Omega^+}), \tag{7.67}$$

where $\alpha_1 := \min\{\alpha_0, 1/2\}$. Since t is sufficiently large and ε_0 is sufficiently small, the embedding (7.67) completes the proof. ∎

By the same arguments it can be shown that the uniqueness, existence and regularity results, similar to the above ones, hold also true for a solution to Problem $(III^{(\sigma)})^-$. We note only that the solution is representable again in the form (7.43) where $f^{(e)}$ is the same as above, and φ solves (in various functional spaces) the pseudodifferential equation

$$r_{S_N} \mathcal{N}_-^{(\sigma)} \varphi = F_-^{(0)} \quad \text{on} \quad S_N, \tag{7.68}$$

where

$$\mathcal{N}_-^{(\sigma)} := [2^{-1} I_6 + \mathcal{K}^{(\sigma)}]\, [\mathcal{H}^{(\sigma)}]^{-1}, \tag{7.69}$$

$$F_-^{(0)} := F_N - r_{S_N} \mathcal{N}_-^{(\sigma)} f^{(e)}. \tag{7.70}$$

The operator $\mathcal{N}_-^{(\sigma)}$ has the same properties as $\mathcal{N}^{(\sigma)}$ described above.

7.4. Basic BVPs of statics

For the interior and exterior BVPs of statics, i.e., for $\sigma = 0$, we have the analogous uniqueness and existence results.

For illustration we will consider the interior and exterior BVPs $(I^{(0)})_f^\pm$ and $(II^{(0)})_F^\pm$ (see (7.2), (7.3), (7.4) with $\sigma = 0$ and $\Phi = 0$), where in the case of exterior problems we assume that

$$\partial^\alpha U_k = \begin{cases} \mathcal{O}(|x|^{-1-|\alpha|}) & \text{for} \quad k = 1, 2, 3, \\ \mathcal{O}(|x|^{-2-|\alpha|}) & \text{for} \quad k = 4, 5, 6, \end{cases} \tag{7.71}$$

as $|x| \to +\infty$ for multi-indeces $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = 0, 1$.

Taking into account Remark 3.5 we easily conclude that the corresponding single- and double-layer potentials $V^{(0)}(g)$ and $W^{(0)}(g)$ satisfy the following decay conditions at infinity

$$\partial^\alpha [V^{(0)}(g)]_k, \partial^\alpha [W^{(0)}(g)]_k = \begin{cases} \mathcal{O}(|x|^{-1-|\alpha|}) & \text{for} \quad k = 1, 2, 3, \\ \mathcal{O}(|x|^{-2-|\alpha|}) & \text{for} \quad k = 4, 5, 6, \end{cases} \tag{7.72}$$

as $|x| \to +\infty$ for arbitrary multi-indeces $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Therefore with the help of the results obtained in Section 6 by the same method as above we can prove the following assertions (with appropriate slight modifications).

Theorem 7.19 *The homogeneous versions of the BVPs $(I^{(0)})_0^\pm$ and $(II^{(0)})_0^-$ have only the trivial solution in the space of regular vector functions $C^1(\overline{\Omega^\pm})$, while the problem $(II^{(0)})_0^+$ has the vector* (2.27) *as a general solution.*

Proof. The proof for the interior problems follows from Green's identity (2.19) and Lemma 2.1, while for the exterior domains the uniqueness is a consequence of the requirement (7.71) and the formula (with $U = U'$)

$$\int_{\Omega^-} \big[U' \cdot L(\partial)U + E(U', U)\big]\, dx = - \int_{\partial\Omega^-} [U']^- \cdot [T(\partial, n)U]^- \, dS, \qquad (7.73)$$

which is valid for $U, U' \in H^1_{loc}(\Omega^-)$ satisfying the conditions (7.71) and $L(\partial)U \in L_1(\Omega^-)$. ∎

Theorem 7.20 *Let S and f be as in* (7.14) *with* $k = 2$. *Then Problem $(I^{(0)})^+_f$ is uniquely solvable in the space of regular vector-functions. Moreover, the solution belongs to the space $C^{1,\gamma_0}(\overline{\Omega^+}) \cap C^\infty(\Omega^+)$ and it can be represented by the double-layer potential $U(x) = W^{(0)}(g)(x)$, $x \in \Omega^+$, where the density vector $g \in C^{1,\gamma_0}(S)$ is defined by the uniquely solvable integral equation*

$$\Big[\, 2^{-1} I_6 + \mathcal{K}^{(0)*} \Big]\, g(x) = f(x), \quad x \in S.$$

Proof. The proof immediately follows from Theorem 6.1 and the fact that the mapping

$$2^{-1} I_6 + \mathcal{K}^{(0)*} \,:\, C^{1,\gamma_0}(S) \to C^{1,\gamma_0}(S)$$

is an isomorphism. ∎

Theorem 7.21 *Let S and f be as in* (7.14) *with* $k = 2$. *Then Problem $(I^{(0)})^-_f$ is uniquely solvable in the space of regular vector-functions. Moreover, the solution belongs to the space $C^{1,\gamma_0}(\overline{\Omega^-}) \cap C^\infty(\Omega^-)$ and it can be represented by the linear combination of the single- and double-layer potentials $U(x) = W^{(0)}(g)(x) + V^{(0)}(g)(x)$, $x \in \Omega^-$, where the density vector $g \in C^{1,\gamma_0}(S)$ is defined by the uniquely solvable integral equation*

$$\Big[-2^{-1} I_6 + \mathcal{K}^{(0)*} + \mathcal{H}^{(0)} \Big]\, g(x) = f(x), \quad x \in S.$$

Proof. We can easily show that the mapping

$$-2^{-1} I_6 + \mathcal{K}^{(0)*} + \mathcal{H}^{(0)} \,:\, C^{1,\gamma_0}(S) \to C^{1,\gamma_0}(S)$$

is an isomorphism. Whence the proof follows. ∎

Theorem 7.22 *Let S and F be as in* (7.14) *with* $k = 1$. *Then Problem $(II^{(0)})^-_F$ is uniquely solvable in the space of regular vector-functions. Moreover, the solution belongs to the space $C^{1,\gamma_0}(\overline{\Omega^-}) \cap C^\infty(\Omega^-)$ and it can be* *represented by the single-layer potential* $U(x) = V^{(0)}(h)(x)$, $x \in \Omega^-$, *where the density vector* $h \in C^{1,\gamma_0}(S)$ *is defined by the uniquely solvable integral equation*

$$\left[2^{-1} I_6 + \mathcal{K}^{(0)}\right] h(x) = F(x), \quad x \in S.$$

Proof. The proof follows from Theorem 6.1 and the fact that the mapping

$$2^{-1} I_6 + \mathcal{K}^{(0)} \,:\, C^{1,\gamma_0}(S) \to C^{1,\gamma_0}(S)$$

is an isomorphism. ∎

To deal with the interior Neumann problem we proceed as follows.

Denote by $X_\Omega\{\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(6)}\}$ the linear span of vectors of rigid displacements in a region Ω, where, for definiteness, we assume that (cf. (2.27))

$$\begin{aligned} &\chi^{(1)} = (0, -x_3, x_2, 1, 0, 0)^\top, && \chi^{(4)} = (1, 0, 0, 0, 0, 0)^\top, \\ &\chi^{(2)} = (x_3, 0, -x_1, 0, 1, 0)^\top, && \chi^{(5)} = (0, 1, 0, 0, 0, 0)^\top, \\ &\chi^{(3)} = (-x_2, x_1, 0, 0, 0, 1)^\top, && \chi^{(6)} = (0, 0, 1, 0, 0, 0)^\top. \end{aligned} \tag{7.74}$$

The restriction of the space $X_\Omega\{\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(6)}\}$ onto the boundary $S = \partial\Omega$ we denote by $X_S\{\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(6)}\}$. Clearly the vectors $\{\chi^{(j)}\}_{j=1}^{6}$ are linearly independent in the both spaces X_Ω and X_S.

Theorem 7.23 *The linear span* $X_S\{\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(6)}\}$ *represents the null space of the operator* $-2^{-1} I_6 + \mathcal{K}^{(0)*}$.

Proof. It can be cheked that each vector $\chi^{(j)}$ solves the homogeneous differential equation $L(\partial)\chi^{(j)}(x) = 0$ in Ω^+ and $T(\partial, n)\chi^{(j)}(x) = 0$ on S. Threfore by the general integral representation formula (see (4.4) with $\sigma = 0$) we establish that the vectors $\chi^{(j)} \in X_S$ solve the homogeneous integral equation

$$[\,-2^{-1} I_6 + \mathcal{K}^{(0)*}\,]\,\chi^{(j)} = 0, \quad j = \overline{1, 6}.$$

Further it can be shown that the dimension of the null space of the adjoint operator $-2^{-1} I_6 + \mathcal{K}^{(0)}$ equals to 6, since the corresponding nontrivial linearly independent solutions of the adjoint homogeneous equation are related to the nontrivial linearly independent solutions of Problem $(II^{(0)})_0^+$, i.e., to the vectors of the space X_{Ω^+}.

On the other hand, the index of the operator in question is zero (due to Theorem 6.3.(ii)), which completes the proof. ∎

Theorem 7.24 *The interior Neumann problems* $(II^{(0)})^+_F$ *is solvable if and only if*

$$\int_S F \cdot \chi^{(j)}\, dS = 0, \quad j = \overline{1,6}, \tag{7.75}$$

and solutions can be represented by the single-layer potential $U(x) = V^{(0)}(h)(x)$, $x \in \Omega^+$, *where the density vector* $h \in C^{1,\gamma_0}(S)$ *solves the integral equation*

$$\left[-2^{-1} I_6 + \mathcal{K}^{(0)}\right] h(x) = F(x), \quad x \in S. \tag{7.76}$$

A solution vector U *is defined modulo a rigid displacement* $\chi \in X_{\Omega^+}$*, while* TU *is determined uniquely.*

Proof. The proof is standard.

The necessity of the conditions (7.75) follows from Green's formula (2.19) with $U'(x) = \chi^{(j)}(x)$, $x \in \Omega^+$.

Sufficiency follows from the general theory of singular integral equations of normal type, since the solvability conditions for the equation (7.76) coincide with (7.75). ∎

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