# ON THE INVESTIGATION OF STATIC HIERARCHIC MODEL FOR ELASTIC RODS [1]

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*Abstract*

In the present paper static one-dimensional hierarchical model for elastic cusped rod is constructed. The corresponding boundary value problem is studied and the uniqueness and existence of its solution in suitable weighted Sobolev spaces is proved. The convergence of the sequence of approximate solutions restored from the solutions of one-dimensional problems to the solution of original three-dimensional problem is proved and under regularity conditions the rate of approximation is estimated.

*Key words and phrases*: Mathematical modelling of linearly elastic cusped rods, *a priori* error estimation.

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The construction and the intensive investigation of the lower-dimensional mathematical models of bodies with negligible thickness or width in comparison with the other geometric dimensions arise with the wide use of structures of such type in the practice ([1, 2]). One of the methods of constructing hierarchic models for elastic prismatic shells was proposed by I. Vekua in [3]. Note, that in [3] initial boundary value problems were considered in the spaces of sufficiently smooth functions and convergence of the sequence of approximate solutions to the exact solution of the three-dimensional problem was not investigated. In static case the existence and uniqueness of the solution to reduced two-dimensional problem obtained in [3] in Sobolev spaces first were investigated in [4]. The rate of approximation of the solution to the original three-dimensional problem by vector-functions restored from the solution of reduced problems in $C^k$ spaces was estimated in [5]. Later, various lower-dimensional models were constructed and investigated in [6-10].

[1] Dedicated to the memory of Professor Victor Kupradze on the occasion of the $100^{th}$ anniversary of his birth

The present work is devoted to study of static boundary value problem for cusped rods in the linear theory of elasticity. Generalizing an idea of I. Vekua, one-dimensional hierarchic model of rods was obtained in [11]. In the nondegenerate case, i.e. in the case of strictly positive rod thickness and width, hierarchic model of static boundary value problem was constructed in [12], where the existence and uniqueness of the solution to obtained one-dimensional model is investigated and the relation of the model to the original problem is studied. In this paper we construct the hierarchic one-dimensional model of elastic cusped rod in static equilibrium and investigate the corresponding boundary value problem in weighted Sobolev spaces. Moreover, we prove the convergence of the sequence of vector-functions constructed by means of the solutions to reduced problems and estimate the rate of approximation.

Let us consider an elastic rod with initial configuration $\overline{\Omega} \subset \mathbb{R}^3$, which consists of homogeneous isotropic material with Lamé constants $\lambda, \mu$ and Lipschitz domain $\Omega$ of the following form

$$\Omega = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ h_\alpha^-(x_3) < x_\alpha < h_\alpha^+(x_3),\ x_3 \in I, \alpha = 1, 2 \right\},$$

where $I = (d_1, d_2)$, $d_2 > d_1$, $h_\alpha^\pm \in C^0(\overline{I}) \cap C^1(I)$, $h_\alpha^+(x_3) > h_\alpha^-(x_3)$, for $x_3 \in (d_1, d_2]$, $\alpha = 1, 2$. Denote by $\Gamma_2 = \{x \in \overline{\Omega}; x_3 = d_2\}$ the upper face of the rod and the rest part of the boundary $\Gamma = \partial\Omega$ - by $\Gamma_1 = \Gamma \backslash \Gamma_2$. Assume, that the rod is subjected to applied body forces with density $\boldsymbol{f} = (f_i)$, the upper face $\Gamma_2$ is clamped and on the surface $\Gamma_1$ the surface forces with density $\boldsymbol{\tau} = (\tau_i)$ are acting, $f_i$, $\tau_i$ are components of the body and surface forces, respectively, $i = \overline{1, 3}$. Throughout the paper we assume that the indices $i, j, p, q$ vary in the set $\{1, 2, 3\}$. The partial derivative with respect to $i$-th argument $\partial/\partial x_i$ we denote by $\partial_i$. For any Lipschitz domain $D \subset \mathbb{R}^s$, $s \in \mathbb{N}$, we denote by $H^k(D)$ the Sobolev space of order $k \in \mathbb{N}$ based on $L^2(D)$, $H^0(D) = L^2(D)$, $H_0^k(D)$ is the closure of the set $C_0^\infty(D)$ of the infinitely differentiable functions with compact support in $D$ in the space $H^k(D)$, and $H^{1/2}(\Gamma_1)$ is the Sobolev space on the part of the boundary $\Gamma_1 \subset \partial D$, which is an element of Lipschitz dissection of $\partial D$ ([13]). The spaces of vector-functions we denote by $\mathbf{H}^k(D) = [H^k(D)]^3$, $\mathbf{H}^{1/2}(\Gamma_1) = [H^{1/2}(\Gamma_1)]^3$, $k \in \mathbb{N}$.

The variational formulation of the static problem of the three-dimensional theory of linearized elasticity is the following: find the vector-function $\boldsymbol{u} = (u_i) \in V(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega);\quad \boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma_2\}$, which for all $\boldsymbol{v} \in V(\Omega)$ satisfies equation

$$\int\limits_\Omega (\lambda e_{pp}(\boldsymbol{u}) e_{qq}(\boldsymbol{v}) + 2\mu e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}))\, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{\tau}, \boldsymbol{v} \rangle_{\Gamma_1}, \qquad (1)$$

where summation convention with respect to the repeated indices is used, $e_{ij}(\boldsymbol{v}) = 1/2(\partial_i v_j + \partial_j v_i)$, $\boldsymbol{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{\tau} \in \mathbf{H}^{-1/2}(\Gamma_1)$, $\widetilde{\mathbf{H}}^{-1}(\Omega)$ and $\mathbf{H}^{-1/2}(\Gamma_1)$ are the dual spaces of the Sobolev spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma_1)$, respectively ([13]), and $\langle .,. \rangle_\Omega$, $\langle .,. \rangle_{\Gamma_1}$ denote the duality between the corresponding spaces.

The three-dimensional problem (1) has a unique solution $\boldsymbol{u}$ if Lamé constants $\mu > 0$, $3\lambda + 2\mu > 0$, and then $\boldsymbol{u}$ is also a solution to the following minimization problem: find $\boldsymbol{u} \in V(\Omega)$ such that

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in V(\Omega)} J(\boldsymbol{v}), \qquad J(\boldsymbol{v}) = \frac{1}{2} B(\boldsymbol{v}, \boldsymbol{v}) - L(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V\,(\Omega)\,,$$

where $B(.,.)$ is the bilinear form with respect to $\boldsymbol{u}$ and $\boldsymbol{v}$ in the left side of equation (1) and $L(.)$ is the linear form in the right side of (1).

In order to reduce the problem (1) to one-dimensional problem let us consider the subspace $V_{\mathbf{N}_1\mathbf{N}_2}(\Omega) \subset V\,(\Omega)$, $\mathbf{N}_\alpha = (N_\alpha^1, N_\alpha^2, N_\alpha^3)$, $\alpha = 1,2$, of vector-functions, the $i$-th components of which are polynomials of degree $N_1^i$ with respect to the variable $x_1$ and of degree $N_2^i$ with respect to the variable $x_2$, i.e.

$$V_{\mathbf{N}_1\mathbf{N}_2}(\Omega) = \{\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} \in \mathbf{H}^1(\Omega); v_{\mathbf{N}_1\mathbf{N}_2 i} = \sum_{k_1^i=0}^{N_1^i} \sum_{k_2^i=0}^{N_2^i} \frac{1}{h_1 h_2}(k_1^i + \frac{1}{2})(k_2^i + \frac{1}{2}) \overset{k_1^i k_2^i}{v_i} P_{k_1^i}(\omega_1) P_{k_2^i}(\omega_2), (h_1 h_2)^{-1/2} \overset{k_1^i k_2^i}{v_i} \in L^2(I), \boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} = \mathbf{0} \text{ on } \Gamma_2, i = \overline{1,3}\},$$

where $\omega_\alpha = (x_\alpha - \overline{h}_\alpha)/h_\alpha$, $h_\alpha = (h_\alpha^+ - h_\alpha^-)/2$, $\overline{h}_\alpha = (h_\alpha^+ + h_\alpha^-)/2$, $\alpha = 1,2$, and $P_k$ denotes the Legendre polynomial of degree $k$. Considering the problem (1) on $V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$ we obtain the following variational problem: the unknown is the vector-function $\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2} = (w_{\mathbf{N}_1\mathbf{N}_2 i}) \in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$,

$$w_{\mathbf{N}_1\mathbf{N}_2 i} = \sum_{k_1^i=0}^{N_1^i} \sum_{k_2^i=0}^{N_2^i} \frac{1}{h_1 h_2} \left(k_1^i + \frac{1}{2}\right) \left(k_2^i + \frac{1}{2}\right) \overset{k_1^i k_2^i}{w_i} P_{k_1^i}(\omega_1) P_{k_2^i}(\omega_2),$$

which satisfies equation

$$B(\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}, \boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}) = L(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}), \qquad \forall \boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} \in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega). \tag{2}$$

From the definition of the space $V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$, taking into account $h_\alpha^\pm \in C^1(I)$ and $h_\alpha > 0$ in $I$ $(\alpha = 1,2)$, it follows that for each vector-function $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} \in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$ the functions $\overset{k_1^i k_2^i}{v_i}$ belong to the space $H^1$ in the interior of the interval $I$, i.e. $\overset{k_1^i k_2^i}{v_i} \in H^1_{loc}(I)$, $0 \le k_\alpha^i \le N_\alpha^i$, $\alpha = 1,2$, $i = \overline{1,3}$. Moreover, since $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} \in \mathbf{H}^1(\Omega)$, applying the properties of the Legendre polynomials ([14]), for the vector-function $\vec{v}_{\mathbf{N}_1\mathbf{N}_2}$ with components $\overset{k_1^i k_2^i}{v_i}$ (i.e. $\vec{v}_{\mathbf{N}_1\mathbf{N}_2} = (\overset{00}{v_1}, ..., \overset{N_1^i N_2^i}{v_1}, \overset{00}{v_2}, ..., \overset{N_1^2 N_2^2}{v_2}, \overset{00}{v_3}, ..., \overset{N_1^3 N_2^3}{v_3})^T$) we have, that

$$\|\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{\mathbf{H}^1(\Omega)} = \sum_{i=1}^{3}\sum_{s_1^i=0}^{N_1^i}\sum_{s_2^i=0}^{N_2^i}(s_1^i+\frac{1}{2})(s_2^i+\frac{1}{2})\Bigg[\|(h_1h_2)^{-1/2}\overset{s_1^i s_2^i}{v_i}\|^2_{L^2(I)}+$$

$$+\sum_{\alpha=1}^{2}\|\sum_{k_\alpha^i=s_\alpha^i}^{N_\alpha^i}(k_\alpha^i+\frac{1}{2})(1-(-1)^{k_\alpha^i+s_\alpha^i})(h_1h_2)^{-1}h_\alpha^{-1/2}[(2-\alpha)]\overset{k_1^i s_2^i}{v_i}+$$

$$+(\alpha-1)\overset{s_1^i k_2^i}{v_i}]\|^2_{L^2(I)}+\|(h_1h_2)^{-1/2}\left((\overset{s_1^i s_2^i}{v_i})'-\sum_{\alpha=1}^{2}h_\alpha^{-1}h'_\alpha(s_\alpha^i+1)\overset{s_1^i s_2^i}{v_i}-\right.$$

$$-\sum_{\alpha=1}^{2}\sum_{k_\alpha^i=s_\alpha^i+1}^{N_\alpha^i}(k_\alpha^i+\frac{1}{2})\left((h_\alpha^+)'-(-1)^{k_\alpha^i-s_\alpha^i}(h_\alpha^-)'\right)h_\alpha^{-1}$$

$$\left.((2-\alpha)\overset{k_1^i s_2^i}{v_i}+(\alpha-1)\overset{s_1^i k_2^i}{v_i})\right)\|^2_{L^2(I)}\Bigg]<\infty,$$

where the prime denotes differentiation with respect to the argument and we assume that the sum is equal to zero if its upper limit is less than the lower one. Note that the square root of the last expression is the norm in the space $[H^1_{loc}(I)]^{N_{1,2}^{1,2,3}}$, $N_{1,2}^{1,2,3} = \sum_{i=1}^{3} N_1^i N_2^i + 3$, which we denote by $\|.\|_*$.

So, the problem (2) is equivalent to the following problem: find $\vec{w}_{\mathbf{N}_1\mathbf{N}_2} \in \vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I) = \{\vec{v}_{\mathbf{N}_1\mathbf{N}_2} = (\overset{k_1^i k_2^i}{v_i}) = [H^1_{loc}(I)]^{N_{1,2}^{1,2,3}}\ ;\ \|\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\|^2_* < \infty,\ \overset{k_1^i k_2^i}{v_i} = 0$ for $x_3 = d_2,\ k_\alpha^i = \overline{0, N_\alpha^i}, \alpha = 1,2,\ i = \overline{1,3}\}$, which satisfies equation

$$B_{\mathbf{N}_1\mathbf{N}_2}(\vec{w}_{\mathbf{N}_1\mathbf{N}_2}, \vec{v}_{\mathbf{N}_1\mathbf{N}_2}) = L_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2}), \qquad \forall \vec{v}_{\mathbf{N}_1\mathbf{N}_2} \in \vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I), \qquad (3)$$

where $B_{\mathbf{N}_1\mathbf{N}_2}$, $L_{\mathbf{N}_1\mathbf{N}_2}$ are the forms $B$ and $L$ on the subspace $V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$ rewritten in terms of $\vec{w}_{\mathbf{N}_1\mathbf{N}_2}$, $\vec{v}_{\mathbf{N}_1\mathbf{N}_2}$.

Note that in the definition of the space $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$ condition $\overset{k_1^i k_2^i}{v_i} = 0$ for $x_3 = d_2$ is understood on the trace sense, since for the vector-functions from the space $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$ we can define the trace on the end $x_3 = d_2$ of $I$. Indeed, if $\vec{v}_{\mathbf{N}_1\mathbf{N}_2} \in \vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$, then the corresponding $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} \in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega) \subset \mathbf{H}^1(\Omega)$ and therefore for the functions $\overset{k_1^i k_2^i}{v_i}$ ($k_\alpha^i = \overline{0, N_\alpha^i}$, $\alpha = 1,2$, $i = \overline{1,3}$) we define the trace operator $tr_*$ by

$$tr_*(\overset{k_1^i k_2^i}{v_i}) = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} tr(v_{\mathbf{N}_1\mathbf{N}_2 i})|_{\Gamma_2} P_{k_1^i}(\omega_1)P_{k_2^i}(\omega_2)dx_1dx_2.$$

Thus, we have reduced the original three-dimensional static problem (1) for linearly elastic cusped rod to one-dimensional problem, for which the following existence and uniqueness theorem is valid.

**Theorem 1.** *If Lamé constants* $\mu > 0$, $3\lambda + 2\mu > 0$, $\boldsymbol{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega)$ *and* $\boldsymbol{\tau} \in \mathbf{H}^{-1/2}(\Gamma_1)$, *then the obtained one-dimensional problem (3) has a unique solution* $\vec{w}_{\mathbf{N}_1\mathbf{N}_2}$, *which also is a solution to the following minimization problem: find* $\vec{w}_{\mathbf{N}_1\mathbf{N}_2} \in \vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$, *such that*

$$J_{\mathbf{N}_1\mathbf{N}_2}(\vec{w}_{\mathbf{N}_1\mathbf{N}_2}) = \inf_{\vec{v}_{\mathbf{N}_1\mathbf{N}_2} \in \vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)} J_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2}),$$
$$J_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2}) = \frac{1}{2} B_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2}, \vec{v}_{\mathbf{N}_1\mathbf{N}_2}) - L_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2}).$$

**Proof.** First, let us prove that the space $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$ is complete. Let $\{\vec{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2}\}_{l=1}^{\infty}$ be a Cauchy sequence in $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$, i.e.

$$\|\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2} - \boldsymbol{v}^{(m)}_{\mathbf{N}_1\mathbf{N}_2}\|_* \to 0, \qquad \text{as } l, m \to \infty.$$

From the definition of the norm $\|.\|_*$ we infer, that $\{\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2}\}_{l=1}^{\infty}$ is a Cauchy sequence in the space $V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$, where $\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2} = (v^{(l)}_{\mathbf{N}_1\mathbf{N}_2 i})$,

$$v^{(l)}_{\mathbf{N}_1\mathbf{N}_2 i} = \sum_{k_1^i=0}^{N_1^i} \sum_{k_2^i=0}^{N_2^i} \frac{1}{h_1 h_2} \left(k_1^i + \frac{1}{2}\right) \left(k_2^i + \frac{1}{2}\right) \overset{k_1^i k_2^i (l)}{v_i} P_{k_1^i}(\omega_1) P_{k_2^i}(\omega_2), \; i = \overline{1,3}.$$

Hence, there exists $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} \in \mathbf{H}^1(\Omega)$ such that $\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2} \to \boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}$ in $\mathbf{H}^1(\Omega)$, as $l \to \infty$, and, consequently, $tr\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2} \to tr\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}$ in $\mathbf{H}^{1/2}(\partial\Omega)$,

$$\overset{rs(l)}{v}_{\mathbf{N}_1\mathbf{N}_2 i} = \int\limits_{h_1^-}^{h_1^+} \int\limits_{h_2^-}^{h_2^+} v^{(l)}_{\mathbf{N}_1\mathbf{N}_2 i} P_r(\omega_1) P_s(\omega_2) dx_1 dx_2 \to$$
$$\to \overset{rs}{v}_{\mathbf{N}_1\mathbf{N}_2 i} = \int\limits_{h_1^-}^{h_1^+} \int\limits_{h_2^-}^{h_2^+} v_{\mathbf{N}_1\mathbf{N}_2 i} P_r(\omega_1) P_s(\omega_2) dx_1 dx_2,$$

in the space $L^2(I)$, as $l \to \infty$, for all $r, s \in \mathbb{N} \cup \{0\}$. Since $\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2} \in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$, we have that $tr\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2} = \mathbf{0}$ on $\Gamma_2$ and $\overset{k_1^i k_2^i (l)}{v}_{\mathbf{N}_1\mathbf{N}_2 i} = 0$, for all $k_1^i > N_1^i$, $k_2^i > N_2^i$. Therefore, $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2} = \mathbf{0}$ on $\Gamma_2$ and $\overset{k_1^i k_2^i}{v}_{\mathbf{N}_1\mathbf{N}_2 i} = 0$, for all $k_1^i > N_1^i$, $k_2^i > N_2^i$,

$i=1,2,3$. So, the $i$-th component of the limit vector-function $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}$ is of the following form:

$$v_{\mathbf{N}_1\mathbf{N}_2 i}=\sum_{k_1^i=0}^{N_1^i}\sum_{k_2^i=0}^{N_2^i}\frac{1}{h_1h_2}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right)\overset{k_1^ik_2^i}{v_i}P_{k_1^i}(\omega_1)P_{k_2^i}(\omega_2),\, i=\overline{1,3},$$

and, hence, $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}=(v_{\mathbf{N}_1\mathbf{N}_2 i})\in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$. Thus, the corresponding vector-function $\vec{v}_{\mathbf{N}_1\mathbf{N}_2}=(\overset{00}{v_1},...,\overset{N_1^1N_2^1}{v_1},\overset{00}{v_2},...,\overset{N_1^2N_2^2}{v_2},\overset{00}{v_3},...,\overset{N_1^3N_2^3}{v_3})^T\in\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$, since $\|\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\|_*=\|\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}\|_{\mathbf{H}^1(\Omega)}<\infty$, $\overset{k_1^ik_2^i}{v_i}=\overset{k_1^ik_2^i}{v}_{\mathbf{N}_1\mathbf{N}_2 i}\in H^1_{loc}(I)$, $\overset{k_1^ik_2^i}{v}_{\mathbf{N}_1\mathbf{N}_2 i}=0$ for $x_3=d_2$, $0\le k_\alpha^i\le N_\alpha^i$, $\alpha=1,2$, $i=\overline{1,3}$. Moreover,

$$\left\|\vec{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2}-\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\right\|_*=\left\|\boldsymbol{v}^{(l)}_{\mathbf{N}_1\mathbf{N}_2}-\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}\right\|_{\mathbf{H}^1(\Omega)}\to 0,\quad \text{as } l\to\infty.$$

Hence, $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$ is a Hilbert space with respect to the scalar product defined by the norm $\|.\|_*$.

Since the bilinear form $B$ is coercive on $V(\Omega)$, then it is coercive on the subspace $V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)\subset V(\Omega)$, and, consequently, the bilinear form $B_{\mathbf{N}_1\mathbf{N}_2}$ is coercive on the space $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$, i.e., for all $\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\in\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$,

$$B_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2},\vec{v}_{\mathbf{N}_1\mathbf{N}_2})=B(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2},\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2})\ge\alpha\,\|\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{\mathbf{H}^1(\Omega)}=\alpha\,\|\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\|^2_*.$$

From the conditions of the theorem it follows that the linear form $L$ is continuous and therefore $L_{\mathbf{N}_1\mathbf{N}_2}$ is continuous too, i.e., for all $\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\in\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$,

$$L_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2})=L(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2})\le c\,\|\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{\mathbf{H}^1(\Omega)}=c\,\|\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\|^2_*.$$

Thus, all the conditions of Lax-Milgram lemma are fulfilled and, hence, the formulated theorem is proved. □

So, we have reduced the three-dimensional problem (1) to one-dimensional problem (3) and prove that it has a unique solution. Now we study the relation of the obtained model of rod to its original model, i.e. investigate convergence of the sequence $\{\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}\}$, where $\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}\in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$ corresponds to the solution $\vec{w}_{\mathbf{N}_1\mathbf{N}_2}$ of the one-dimensional problem (3), to the exact solution of the three-dimensional problem. The convergence result and estimate of the rate of convergence is given in the next theorem, but before we formulate it, let us introduce the following anisotropic weighted Sobolev space

$$\mathbf{H}^{s,s,1}_{h^\pm_{1,2}}(\Omega)=\{\boldsymbol{v}\in\mathbf{H}^1(\Omega);\ \partial_\alpha^{k-1}\boldsymbol{v}\in\mathbf{H}^1(\Omega),\ (h_\alpha^\pm)'\partial_\alpha^k\boldsymbol{v}\in\mathbf{L}^2(\Omega),\ 1\le k\le s\},$$

where $\alpha = 1,2$, $s \in \mathbb{N}$, equipped with the norm

$$\|\boldsymbol{v}\|^2_{\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)} = \sum_{k=1}^{s}\sum_{\alpha=1}^{2}\left(\left\|\partial_\alpha^{k-1}\boldsymbol{v}\right\|^2_{\mathbf{H}^1(\Omega)} + \left\|(h^+_\alpha)'\partial^k_\alpha\boldsymbol{v}\right\|^2_{\mathbf{L}^2(\Omega)} + \right.$$
$$\left. + \left\|(h^-_\alpha)'\partial^k_\alpha\boldsymbol{v}\right\|^2_{\mathbf{L}^2(\Omega)}\right).$$

Note that $\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)$ is a Hilbert space. Indeed, any Cauchy sequence $\{\boldsymbol{v}_n\}_{n\geq 1}$ in the space $\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)$ is a Cauchy sequence in the space $\mathbf{H}^1(\Omega)$ and, consequently, $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathbf{H}^1(\Omega)$, as $n \to \infty$. Moreover, $\partial^{k-1}_\alpha\boldsymbol{v}_n \to \partial^{k-1}_\alpha\boldsymbol{v}$ in $\mathbf{H}^1(\Omega)$, as $n\to\infty$, $k=\overline{1,s}$. Since $h^\pm_1$, $h^\pm_2 \in C^1(I)$, we have that $h^\pm_1, h^\pm_2 \in C^1(\overline{I}_1)$, where $I_1$ is any subinterval of $I$, $\overline{I}_1 \subset I$, and hence we obtain

$$(h^\pm_\alpha)'\partial^k_\alpha\boldsymbol{v}_n \to (h^\pm_\alpha)'\partial^k_\alpha\boldsymbol{v} \qquad \text{in } \mathbf{L}^2(\Omega_1), \text{ as } n\to\infty, \alpha=1,2, \tag{4}$$

where $k=\overline{1,s}$, $\Omega_1$ is any subdomain of $\Omega, \overline{\Omega}_1 \subset \Omega$. From the definition of the space $\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)$ it follows, that the sequence $\{(h^\pm_\alpha)'\partial^k_\alpha\boldsymbol{v}_n\}_{n\geq1}$ converges in $\mathbf{L}^2(\Omega)$, and, taking into account (4), we have $(h^\pm_\alpha)'\partial^k_\alpha\boldsymbol{v}_n \to (h^\pm_\alpha)'\partial^k_\alpha\boldsymbol{v}$ in $\mathbf{L}^2(\Omega)$, as $n\to\infty$, $k=\overline{1,s}, \alpha=1,2$, and thus the space $\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)$ is complete.

**Theorem 2.** *If all the conditions of Theorem 1 are fulfilled, then the vector-function* $\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2} = (w_{\mathbf{N}_1\mathbf{N}_2 i})$,

$$w_{\mathbf{N}_1\mathbf{N}_2 i} = \sum_{k^i_1=0}^{N^i_1}\sum_{k^i_2=0}^{N^i_2}\frac{1}{h_1h_2}\left(k^i_1+\frac12\right)\left(k^i_2+\frac12\right)\overset{k^i_1k^i_2}{w_i}\, P_{k^i_1}(\omega_1)P_{k^i_2}(\omega_2),\ i=\overline{1,3},$$

*restored from the solution* $\vec{w}_{\mathbf{N}_1\mathbf{N}_2} = (\overset{00}{w_1},..., \overset{N^1_1N^1_2}{w_1}, \ldots, \overset{00}{w_3}, ..., \overset{N^3_1N^3_2}{w_3})^T$ *of one-dimensional problem (3) tends to the solution* $\boldsymbol{u}$ *of the three-dimensional problem (1) in the space* $\mathbf{H}^1(\Omega)$, *as* $N^i_\alpha\to\infty$, $i=\overline{1,3}$, $\alpha=1,2$. *Moreover, if* $\boldsymbol{u}\in\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)$, $s\geq 2$, *then*

$$\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{\mathbf{H}^1(\Omega)} \leq \left(\frac{1}{N_1^{2s-3}}+\frac{1}{N_2^{2s-3}}\right)\theta(\Omega,\Gamma_2,h^\pm_1,h^\pm_2,\mathbf{N}_1,\mathbf{N}_2),$$

*where* $N_\alpha = \min\limits_{1\leq i\leq 3}\{N^i_\alpha\}$, $\theta(\Omega,\Gamma_2,h^\pm_1,h^\pm_2,\mathbf{N}_1,\mathbf{N}_2)\to 0$, *as* $N^i_\alpha\to\infty$, $i=\overline{1,3}$, $\alpha=1,2$. *If, in addition,* $\|\boldsymbol{u}\|_{\mathbf{H}^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)}\leq c$, *c is independent of* $h^{\max}_\alpha =$

$\max_{x_3\in\bar{I}} h_\alpha(x_3)$, $\alpha=1,2$, *then*

$$\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{E(\Omega)} \le \left(\frac{(h_1^{\max})^{2(s-1)}}{N_1^{2s-3}}+\frac{(h_2^{\max})^{2(s-1)}}{N_2^{2s-3}}\right)\overline{\theta}(\mathbf{N}_1,\mathbf{N}_2),$$

*where* $\overline{\theta}(\mathbf{N}_1,\mathbf{N}_2)\to 0$, *as* $N_1,N_2\to\infty$, $\|\boldsymbol{v}\|_{E(\Omega)}=\sqrt{B^\Omega(\boldsymbol{v},\boldsymbol{v})}$, $\boldsymbol{v}\in V(\Omega)$.

**Proof.** According to the Theorem 1, the solution $\vec{w}_{\mathbf{N}_1\mathbf{N}_2}$ of the problem (3) minimizes the functional $J_{\mathbf{N}_1\mathbf{N}_2}$ on the space $\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$, i.e.,

$$\frac{1}{2}B_{\mathbf{N}_1\mathbf{N}_2}(\vec{w}_{\mathbf{N}_1\mathbf{N}_2},\vec{w}_{\mathbf{N}_1\mathbf{N}_2})-L_{\mathbf{N}_1\mathbf{N}_2}(\vec{w}_{\mathbf{N}_1\mathbf{N}_2})\le$$
$$\le\frac{1}{2}B_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2},\vec{v}_{\mathbf{N}_1\mathbf{N}_2})-L_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2}),\ \ \forall\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\in\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I). \qquad (5)$$

Since for all $\vec{v}_{\mathbf{N}_1\mathbf{N}_2}\in\vec{V}_{\mathbf{N}_1\mathbf{N}_2}(I)$,

$$B_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2},\vec{v}_{\mathbf{N}_1\mathbf{N}_2})=B(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2},\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}),\qquad L_{\mathbf{N}_1\mathbf{N}_2}(\vec{v}_{\mathbf{N}_1\mathbf{N}_2})=L(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}),$$

where $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}=(v_{\mathbf{N}_1\mathbf{N}_2i})$, $v_{\mathbf{N}_1\mathbf{N}_2i}=\sum_{k_1^i=0}^{N_1^i}\sum_{k_2^i=0}^{N_2^i}\frac{1}{h_1h_2}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right)\overset{k_1^ik_2^i}{v_i}$ $P_{k_1^i}(\omega_1)P_{k_2^i}(\omega_2)$, $i=\overline{1,3}$, then from (5) we obtain

$$B(\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2},\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2})\le B(\boldsymbol{u},\boldsymbol{u})-2L(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2})+B(\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2},\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}),$$

and, consequently, for all $\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}\in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$,

$$B(\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2},\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2})\le B(\boldsymbol{u}-\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2},\boldsymbol{u}-\boldsymbol{v}_{\mathbf{N}_1\mathbf{N}_2}). \qquad (6)$$

From the last inequality it follows, that the vector-function $\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}$ approximates the solution $\boldsymbol{u}$ of the original problem. Indeed, by trace theorems for Sobolev spaces ([13]), for any $\boldsymbol{v}\in\mathbf{H}^1(\Omega)$, $\boldsymbol{v}=\mathbf{0}$ on $\Gamma_2$, there exists continuation $\tilde{\boldsymbol{v}}\in\mathbf{H}^1_0(\Omega_1)$ of $\boldsymbol{v}$, where $\Omega_1$ is a Lipschitz domain, $\Omega_1\supset\Omega$, $\partial\Omega_1\supset\Gamma_0$. From the density of $C_0^\infty(\Omega_1)$ in $H_0^1(\Omega_1)$, we obtain that the set of infinitely differentiable vector-functions on $\Omega$, which are equal to zero on $\Gamma_2$, is dense in the space $V(\tilde{\Omega}_1)$, $\tilde{\Omega}_1=\{x\in\mathbb{R}^3;\tilde{h}_\alpha^-(x_3)<x_\alpha<\tilde{h}_\alpha^+(x_3),x_3\in I,\alpha=1,2\}$, where the functions $\tilde{h}_\alpha^-,\tilde{h}_\alpha^+$ are such, that $\tilde{h}_\alpha^-(x_3)\le h_\alpha^-(x_3)\le h_\alpha^+(x_3)\le\tilde{h}_\alpha^+(x_3)$, $x_3\in\bar{I}$, $\tilde{h}_\alpha^-(d_1)<\tilde{h}_\alpha^+(d_1)$ and $\Omega\subset\tilde{\Omega}_1\subset\Omega_1$. Since the union $\bigcup_{\mathbf{N}_1,\mathbf{N}_2\ge\mathbf{0}}V_{\mathbf{N}_1\mathbf{N}_2}(\tilde{\Omega}_1)$ of the spaces $V_{\mathbf{N}_1\mathbf{N}_2}(\tilde{\Omega}_1)$ for all $N_\alpha^i\ge 0$, $i=\overline{1,3}$, $\alpha=1,2$, is dense in $V(\tilde{\Omega}_1)$ ([12]), we have that $\bigcup_{\mathbf{N}_1,\mathbf{N}_2\ge\mathbf{0}}V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$ is dense in $V(\Omega)$, and due to coerciveness of the bilinear form $B$, from the inequality (6) it follows that $\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2} \to \boldsymbol{u}$ in the space $\mathbf{H}^1(\Omega)$, as $N_1^1$, $N_2^1$,...,$N_1^3$, $N_2^3 \to \infty$.

Let us estimate the rate of approximation of $\boldsymbol{u}$ by $\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}$, if $\boldsymbol{u}$ satisfies additional regularity conditions of the theorem. By means of the solution $\boldsymbol{u}$ of the three-dimensional problem we construct vector-function $\boldsymbol{u}_{\mathbf{N}_1\mathbf{N}_2}$, the $i$-th component $u_{\mathbf{N}_1\mathbf{N}_2 i}$ of which is the sum of the first $(N_1^i+1)(N_2^i+1)$ terms of $u_i$ Fourier-Legendre series expansion with respect to the variables $x_1$, $x_2$, i.e.

$$u_{\mathbf{N}_1\mathbf{N}_2 i} = \sum_{k_1^i=0}^{N_1^i}\sum_{k_2^i=0}^{N_2^i} \frac{1}{h_1h_2}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right) \overset{k_1^ik_2^i}{u_i}\, P_{k_1^i}(\omega_1)P_{k_2^i}(\omega_2),$$

where $\overset{k_1^ik_2^i}{u_i} = \int\limits_{h_1^-}^{h_1^+}\int\limits_{h_2^-}^{h_2^+} u_i P_{k_1^i}(\omega_1)P_{k_2^i}(\omega_2)dx_1dx_2$, $k_\alpha^i = \overline{0, N_\alpha^i}$, $i=\overline{1,3}$, $\alpha = 1,2$.

Note, that the vector-function $\boldsymbol{u}_{\mathbf{N}_1\mathbf{N}_2} \in V_{\mathbf{N}_1\mathbf{N}_2}(\Omega)$. Indeed, since $\boldsymbol{u} \in V(\Omega)$, we have that $\boldsymbol{u}_{\mathbf{N}_1\mathbf{N}_2} = \mathbf{0}$ on $\Gamma_2$. So, it suffices to prove that $\boldsymbol{u}_{\mathbf{N}_1\mathbf{N}_2} \in \mathbf{H}^1(\Omega)$. Applying properties of the Legendre polynomials ([14])

$$\begin{aligned} P_k(t) &= \frac{1}{2k+1}(P'_{k+1}(t) - P'_{k-1}(t)), && \forall k \in \mathbb{N}, \\ tP'_k(t) &= P'_{k+1}(t) - (k+1)P_k(t), && \forall k \in \mathbb{N}\cup\{0\}, \end{aligned}$$

we have that for all $k_1, k_2 \in \mathbb{N}$ and $i = 1,2,3$,

$$\overset{k_1k_2}{u_i} = \frac{h_1}{2k_1+1}(\overset{k_1-1,k_2}{\partial_1 u_i} - \overset{k_1+1,k_2}{\partial_1 u_i}) = \frac{h_2}{2k_2+1}(\overset{k_1,k_2-1}{\partial_2 u_i} - \overset{k_1,k_2+1}{\partial_2 u_i}), \tag{7}$$

$$\begin{aligned} h_1h_2\partial_3\left(\frac{1}{h_1h_2}\overset{k_1k_2}{u_i}\right) = \overset{k_1k_2}{\partial_3 u_i} + \bar{h}'_1\overset{k_1k_2}{\partial_1 u_i} + h'_1\left(\frac{1}{h_1}k_1\overset{k_1k_2}{u_i} + \overset{k_1+1,k_2}{\partial_1 u_i}\right) + \\ + \bar{h}'_2\overset{k_1k_2}{\partial_2 u_i} + h'_2\left(\frac{1}{h_2}k_2\overset{k_1k_2}{u_i} + \overset{k_1,k_2+1}{\partial_2 u_i}\right). \end{aligned}$$

Taking into account the latter formulas and expressions for derivatives of Legendre polynomials

$$\left.\begin{aligned} P'_k(t) &= \sum_{s=0}^{k-1}\left(s+\frac{1}{2}\right)(1-(-1)^{k+s})P_s(t), \\ tP'_k(t) &= kP_k(t) + \sum_{s=0}^{k-1}\left(s+\frac{1}{2}\right)(1+(-1)^{k+s})P_s(t), \end{aligned}\right. \qquad \forall k \in \mathbb{N},$$

we obtain, for $\alpha = 1, 2$,

$$\frac{\partial u_{\mathbf{N}_1\mathbf{N}_2 i}}{\partial x_\alpha} = \sum_{k_\alpha^i=0}^{N_\alpha^i-1} \sum_{k_{3-\alpha}^i=0}^{N_{3-\alpha}^i} \frac{1}{h_1 h_2}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right) \overset{k_1^i k_2^i}{\partial_\alpha u_i}\, P_{k_1^i}(\omega_1) P_{k_2^i}(\omega_2) -$$

$$-\sum_{k_\alpha^i=N_\alpha^i}^{N_\alpha^i+1} \sum_{k_{3-\alpha}^i=0}^{N_{3-\alpha}^i} \frac{1}{2h_1 h_2}\left(k_{3-\alpha}^i+\frac{1}{2}\right) \overset{k_1^i k_2^i}{\partial_\alpha u_i}\, P_{k_{3-\alpha}^i}(\omega_{3-\alpha})$$

$$\sum_{s_\alpha^i=0}^{k_\alpha^i-2}\left(s_\alpha^i+\frac{1}{2}\right)(1+(-1)^{k_\alpha^i+s_\alpha^i}) P_{s_\alpha^i}(\omega_\alpha),$$

$$\frac{\partial u_{\mathbf{N}_1\mathbf{N}_2 i}}{\partial x_3} = \sum_{k_1^i=0}^{N_1^i} \sum_{k_2^i=0}^{N_2^i} \frac{1}{h_1 h_2}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right) \overset{k_1^i k_2^i}{\partial_3 u_i}\, P_{k_1^i}(\omega_1) P_{k_2^i}(\omega_2) +$$

$$+\sum_{\alpha=1}^{2} \sum_{k_{3-\alpha}^i=0}^{N_{3-\alpha}^i} \frac{1}{h_1 h_2}\left(k_{3-\alpha}^i+\frac{1}{2}\right)\Bigg[\bar{h}'_\alpha\left(N_\alpha^i+\frac{1}{2}\right)\Bigg((\alpha-1)\overset{k_1^i N_2^i}{\partial_2 u_i} +$$

$$+(2-\alpha)\overset{N_1^i k_2^i}{\partial_1 u_i}\Bigg) P_{N_\alpha^i}(\omega_\alpha) + \sum_{k_\alpha^i=N_\alpha^i}^{N_\alpha^i+1} \sum_{s_\alpha^i=0}^{k_\alpha^i-1}\left(s_\alpha^i+\frac{1}{2}\right)\overset{k_1^i k_2^i}{\partial_\alpha u_i}\left(\frac{(h_\alpha^+)'}{2} +\right.$$

$$\left.+\frac{(h_\alpha^-)'(-1)^{k_\alpha^i+s_\alpha^i}}{2}\right) P_{s_\alpha^i}(\omega_\alpha)\Bigg] P_{k_{3-\alpha}^i}(\omega_{3-\alpha}).$$

From the latter expressions for $\partial_j u_{\mathbf{N}_1\mathbf{N}_2 i}$, taking into account conditions of the theorem $u_i \in H^1(\Omega)$, $h'_\alpha \partial_\alpha u_i$, $\bar{h}'_\alpha \partial_\alpha u_i \in L^2(\Omega)$, we have that $u_{\mathbf{N}_1\mathbf{N}_2 i} \in H^1(\Omega)$, $i, j = \overline{1,3}$, $\alpha = 1, 2$.

In order to obtain the estimates of the theorem, let us consider the remainder term $\boldsymbol{\varepsilon}_{\mathbf{N}_1\mathbf{N}_2} = (\varepsilon_{\mathbf{N}_1\mathbf{N}_2 i})$,

$$\varepsilon_{\mathbf{N}_1\mathbf{N}_2 i} = \sum_{k_1^i=N_1^i}^{\infty} \sum_{k_2^i=N_2^i}^{\infty}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right)\frac{\overset{k_1^i k_2^i}{u_i}}{h_1 h_2} P_{k_1^i}(\omega_1) P_{k_2^i}(\omega_2),\ i=\overline{1,3}.$$

Taking into account the formulas for derivatives of the components of $\boldsymbol{u}_{\mathbf{N}_1\mathbf{N}_2}$, properties of Legendre polynomials and applying Parseval equality, we infer that

$$\|\varepsilon_{\mathbf{N}_1\mathbf{N}_2 i}\|^2_{L^2(\Omega)} = \sum_{k_1^i=N_1^i+1}^{\infty} \vee \sum_{k_2^i=N_2^i+1}^{\infty} \int_I \frac{1}{h_1 h_2}\left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right)(\overset{k_1^i k_2^i}{u_i})^2 dx_3,$$

$$\left\|\frac{\partial \varepsilon_{\mathbf{N}_1\mathbf{N}_2 i}}{\partial x_\alpha}\right\|^2_{L^2(\Omega)} = \sum_{k_\alpha^i=N_\alpha^i}^{\infty} \vee \sum_{k_{3-\alpha}^i=N_{3-\alpha}^i+1}^{\infty} \int_I \left(k_1^i+\frac{1}{2}\right)\left(k_2^i+\frac{1}{2}\right)\frac{(\overset{k_1^i k_2^i}{\partial_\alpha u_i})^2}{h_1 h_2} dx_3 +$$

$$+\sum_{k^i_\alpha=N^i_\alpha}^{N^i_\alpha+1}\sum_{k^i_{3-\alpha}=0}^{N^i_{3-\alpha}}\int_I \frac{1}{h_1h_2}\left(k^i_{3-\alpha}+\frac{1}{2}\right)\frac{k^i_\alpha(k^i_\alpha-1)}{4}(\overset{k^i_1k^i_2}{\partial_\alpha u_i})^2dx_3,$$

$$\left\|\frac{\partial\varepsilon_{\mathbf{N}_1\mathbf{N}_2i}}{\partial x_3}\right\|^2_{L^2(\Omega)}\leq 5\left(\sum_{k^i_1=N^i_1+1}^{\infty}\vee\sum_{k^i_2=N^i_2+1}^{\infty}\int_I\left(k^i_1+\frac{1}{2}\right)\left(k^i_2+\frac{1}{2}\right)\frac{(\overset{k^i_1k^i_2}{\partial_3 u_i})^2}{h_1h_2}dx_3+\right.$$

$$+\sum_{\alpha=1}^{2}\sum_{k^i_\alpha=N^i_\alpha}^{N^i_\alpha+1}\sum_{k^i_{3-\alpha}=0}^{N^i_{3-\alpha}}\int_I\left(k^i_{3-\alpha}+\frac{1}{2}\right)\frac{N^i_\alpha+1}{4h_1h_2}(\overset{k^i_1k^i_2}{\partial_\alpha u_i})^2\left((2k^i_\alpha-N^i_\alpha)(h'_\alpha)^2+\right.$$

$$\left.+(3N^i_\alpha-2k^i_\alpha+2)(\bar{h}'_\alpha)^2\right)dx_3$$

where $\alpha=1,2$, $\sum\limits_{k_1=\widehat{N}_1}^{\infty}\vee\sum\limits_{k_2=\widehat{N}_2}^{\infty}$ denotes the sum with respect to the indices $k_1$ and $k_2$ for all pairs $(k_1,k_2)$, $k_1\geq\widehat{N}_1$ or $k_2\geq\widehat{N}_2$. From (7) we have that

$$\begin{aligned}
\|\overset{k_1k_2}{\partial^\beta_j u_i}\|^2_{L^2(I)}&\leq\frac{c_1}{k_1^{2(s-\beta)}}\sum_{\tilde{k}_1=k_1-s+\beta}^{k_1+s-\beta}\|h_1^{s-\beta}\,(\overset{\tilde{k}_1k_2}{\partial_1^{s-\beta}\partial_j u_i})\,\|^2_{L^2(I)},\\
\|\overset{k_1k_2}{\partial^\beta_j u_i}\|^2_{L^2(I)}&\leq\frac{c_2}{k_2^{2(s-\beta)}}\sum_{\tilde{k}_2=k_2-s+\beta}^{k_2+s-\beta}\|h_2^{s-\beta}\,(\overset{k_1\tilde{k}_2}{\partial_2^{s-\beta}\partial_j u_i})\,\|^2_{L^2(I)},
\end{aligned}\tag{8}$$

where $\min\{k_1,k_2\}\geq s$, $\beta=0,1$, $i,j=\overline{1,3}$, $c_1$, $c_2$ are positive constants independent from $h^\pm_1$, $h^\pm_2$ and $k_1$, $k_2$. Therefore, for $\boldsymbol{\varepsilon}_{\mathbf{N}_1\mathbf{N}_2}$ we obtain the following estimates

$$\begin{aligned}
\|\varepsilon_{\mathbf{N}_1\mathbf{N}_2i}\|^2_{L^2(\Omega)}&\leq\left(\frac{1}{(N^i_1)^{2s}}+\frac{1}{(N^i_2)^{2s}}\right)\theta_i(h^\pm_1,h^\pm_2,N^i_1,N^i_2),\\
\|\partial_j\varepsilon_{\mathbf{N}_1\mathbf{N}_2i}\|^2_{L^2(\Omega)}&\leq\left(\frac{1}{(N^i_1)^{2s-3}}+\frac{1}{(N^i_2)^{2s-3}}\right)\theta_i(h^\pm_1,h^\pm_2,N^i_1,N^i_2),
\end{aligned}$$

where $\theta_i(h^\pm_1,h^\pm_2,N^i_1,N^i_2)\to 0$, as $N^i_1,N^i_2\to\infty$, $i,j=1,2,3$. The latter estimates together with coerciveness of the bilinear form $B$ and inequality (6) imply

$$\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{\mathbf{H}^1(\Omega)}\leq\left(\frac{1}{N_1^{2s-3}}+\frac{1}{N_2^{2s-3}}\right)\theta(\Omega,\Gamma_2,h^\pm_1,h^\pm_2,\mathbf{N}_1,\mathbf{N}_2),$$

where $N_\alpha=\min\limits_{1\leq i\leq 3}\{N^i_\alpha\}$, $\theta(\Omega,\Gamma_2,h^\pm_1,h^\pm_2,\mathbf{N}_1,\mathbf{N}_2)\to 0$, as $N_1,N_2\to\infty$.

Moreover, if the norm $\|\boldsymbol{u}\|_{\mathbf{H}^{s,s,1}_{h^\pm_{1,2}}(\Omega)}$ is independent from $h^{\max}_1$, $h^{\max}_2$, then from (8) we obtain

$$\|\varepsilon_{\mathbf{N}_1\mathbf{N}_2 i}\|^2_{L^2(\Omega)} \leq \left( \frac{(h_1^{\max})^{2s}}{(N_1^i)^{2s}} + \frac{(h_2^{\max})^{2s}}{(N_2^i)^{2s}} \right) \bar{\theta}_i(N_1^i, N_2^i),$$

$$\|\partial_j \varepsilon_{\mathbf{N}_1\mathbf{N}_2 i}\|^2_{L^2(\Omega)} \leq \left( \frac{(h_1^{\max})^{2(s-1)}}{(N_1^i)^{2s-3}} + \frac{(h_2^{\max})^{2(s-1)}}{(N_2^i)^{2s-3}} \right) \bar{\theta}_i(N_1^i, N_2^i),$$

where $\bar{\theta}_i(N_1^i, N_2^i) \to 0$, as $N_1^i, N_2^i \to \infty$, $i, j = \overline{1,3}$. From the latter inequalities, applying (6), we infer the second estimate of the theorem

$$\|\boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}_1\mathbf{N}_2}\|^2_{E(\Omega)} \leq \left( \frac{(h_1^{\max})^{2(s-1)}}{N_1^{2s-3}} + \frac{(h_2^{\max})^{2(s-1)}}{N_2^{2s-3}} \right) \bar{\theta}(\mathbf{N}_1, \mathbf{N}_2),$$

where $\bar{\theta}(\mathbf{N}_1, \mathbf{N}_2) \to 0$, as $N_1, N_2 \to \infty$. □

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