AN ANISOTROPIC ELECTROMAGNETIC-ELASTIC ANALOGY

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Abstract

An analogy is exhibited between results for electromagnetic waves in linear media which are both electrically and magnetically anisotropic (crystals) and results for finite amplitude elastic waves in deformed Mooney-Rivlin materials. More precisely, the results for elastic waves in deformed Mooney-Rivlin materials appear formally as a special case of the results for electromagnetic waves in electrically and magnetically anisotropic crystals. The analogy is used to formulate the problem of finding the wave speeds and the polarization directions of the finite amplitude elastic waves as an eigenvalue problem.

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1. Introduction

The analogy between the propagation of electromagnetic waves in vacuum and the propagation of acoustic waves in a hypothetical linear incompressible elastic material ("aether") is well known and has played a crucial role in the origins of the electromagnetic theory of light [1]. The term "electric displacement" in electromagnetism has its origin in this analogy.

Here, however, we are interested in anisotropic properties of waves.

On the one hand, detailed results are available [2],[3] for electromagnetic waves in linear, non dissipative, non dispersive materials which are both electrically and magnetically anisotropic. The structure of these results arises from the fact that the electric permittivity and magnetic permeability tensors are placed on an equal footing.

On the other hand, a widely used model of incompressible material in finite elasticity theory is the Mooney-Rivlin model (see, for instance [4]) which is used to describe the behaviour of rubberlike materials. Mooney-Rivlin materials are isotropic, but when subjected to a finite static homogeneous deformation they behave as anisotropic for superimposed elastic waves. Detailed results for finite-amplitude waves in deformed Mooney-Rivlin materials are also available [5]-[8]. Although the theory is non linear, these waves have many features in common with anisotropic linear waves. In particular, their wave speed is a constant for any given propagation direction (but varies with the propagation direction).

In this paper, an analogy between the two sets of results is exhibited. More exactly, it is shown that the results for elastic waves in deformed Mooney-Rivlin materials may be obtained from the results for electromagnetic waves in electrically and magnetically anisotropic crystals by appropriate formal substitutions. The anisotropic properties of electromagnetic waves in these crystals are described in terms of two tensors, the electric permittivity and magnetic permeability tensors, which do not necessarily have the same principal axes. The anisotropic properties of elastic waves in deformed Mooney-Rivlin materials are described in terms of one tensor, the left Cauchy-Green strain tensor of the static homogeneous deformation of the material. The formal substitutions proposed here express both the electric permittivity and magnetic permeability tensors in terms of one tensor, the left Cauchy-Green strain tensor of the static deformation of Mooney-Rivlin materials. Hence, the results for elastic waves in deformed Mooney-Rivlin materials appear formally as a special case of the results for electromagnetic waves in crystals.

In §2, basic results for electromagnetic waves in linear crystals which are both electrically and magnetically anisotropic are recalled. Introducing the generalized optic axes, further results are collected in §3.

Similarly, in §4, basic results for finite-amplitude elastic waves in deformed Mooney-Rivlin materials are recalled, and the corresponding acoustic axes are introduced in §5.

The formal substitutions for obtaining the results of §4–5 from those of §2–3 are presented in §6. They are used to show that the wave speeds and polarization directions of finite amplitude elastic waves propagating in deformed Mooney-Rivlin materials may be obtained by solving an eigenvalue problem. The main results are summarized in §7.

2. Results for electromagnetic waves in crystals

Here, we recall results obtained for time harmonic plane waves propagating in crystals which are both electrically and magnetically anisotropic. These results are mainly taken from [2], where both homogeneous and inhomogeneous plane waves are studied. Here, we restrict our attention to homogeneous plane waves propagating in an arbitrary direction $\mathbf{n}$ ($\mathbf{n}\cdot\mathbf{n} = 1$) with phase speed v. For these, Maxwell's equations yield

$$\mathbf{n}\cdot\mathbf{D} = 0, \quad \mathbf{n}\cdot\mathbf{B} = 0, \tag{2.1}$$

$$\mathbf{n}\times\mathbf{E} = v\mathbf{B}, \quad \mathbf{n}\times\mathbf{H} = -v\mathbf{D}, \tag{2.2}$$

where $\mathbf{E}, \mathbf{H}, \mathbf{D}$ and $\mathbf{B}$ are constant amplitude vectors of the electric field, the magnetic field, the electric displacement, and the magnetic induction, respectively. The constitutive equations are assumed to be such that

$$\mathbf{D} = \boldsymbol{\kappa}\mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu}\mathbf{H}, \tag{2.3}$$

where the electric permittivity tensor $\boldsymbol{\kappa}$ and the magnetic permeability tensor $\boldsymbol{\mu}$ are constant, real, symmetric and positive definite (linear, non dissipative and non dispersive media).

Let $\boldsymbol{\Gamma}$ denotes the dual skew-symmetric tensor associated with $\mathbf{n}$, and $\boldsymbol{\Pi} = -\boldsymbol{\Gamma}^2$ the projection tensor onto the plane orthogonal to $\mathbf{n}$:

$$\Gamma_{ij} = \epsilon_{ikj} n_k, \quad \Pi_{ij} = \delta_{ij} - n_i n_j. \tag{2.4}$$

The propagation condition in terms of $\mathbf{E}$ reads [2]

$$\left(\boldsymbol{\Gamma}\boldsymbol{\mu}^{-1}\boldsymbol{\Gamma} + v^2\boldsymbol{\kappa}\right)\mathbf{E} = \mathbf{0}, \tag{2.5}$$

or, in terms of $\mathbf{H}$,

$$\left(\boldsymbol{\Gamma}\boldsymbol{\kappa}^{-1}\boldsymbol{\Gamma} + v^2\boldsymbol{\mu}\right)\mathbf{H} = \mathbf{0}. \tag{2.6}$$

Alternatively, in terms of $\mathbf{D}$, the propagation condition is [3]

$$\left\{\boldsymbol{\Pi}\boldsymbol{\kappa}^{-1}\boldsymbol{\Pi} - v^2(\det\boldsymbol{\mu})(\mathbf{n}\cdot\boldsymbol{\mu}\mathbf{n})^{-1}\boldsymbol{\Pi}\boldsymbol{\mu}^{-1}\boldsymbol{\Pi}\right\}\mathbf{D} = \mathbf{0}, \tag{2.7}$$

or, in terms of $\mathbf{B}$,

$$\left\{\boldsymbol{\Pi}\boldsymbol{\mu}^{-1}\boldsymbol{\Pi} - v^2(\det\boldsymbol{\kappa})(\mathbf{n}\cdot\boldsymbol{\kappa}\mathbf{n})^{-1}\boldsymbol{\Pi}\boldsymbol{\kappa}^{-1}\boldsymbol{\Pi}\right\}\mathbf{B} = \mathbf{0}. \tag{2.8}$$

In general, for any given propagation direction $\mathbf{n}$, two linearly polarized plane waves may propagate. The two possible directions of the amplitude

$\mathbf{D}$ (or $\mathbf{B}$) are the common conjugate directions of the ellipses in which the plane $\mathbf{n} \cdot \mathbf{x} = 0$ cuts the two ellipsoids $\mathbf{x} \cdot \boldsymbol{\kappa}^{-1}\mathbf{x} = 1$ ("index ellipsoid") and $\mathbf{x} \cdot \boldsymbol{\mu}^{-1}\mathbf{x} = 1$. The two phase speeds are obtained from the corresponding eigenvalues of $\mathbf{\Pi}\boldsymbol{\kappa}^{-1}\mathbf{\Pi}$ with respect to $\mathbf{\Pi}\boldsymbol{\mu}^{-1}\mathbf{\Pi}$ (or $\mathbf{\Pi}\boldsymbol{\mu}^{-1}\mathbf{\Pi}$ with respect to $\mathbf{\Pi}\boldsymbol{\kappa}^{-1}\mathbf{\Pi}$). They are the solutions of the secular equation, which reads [3]

$$v^4 - (\mathbf{n} \cdot \mathbf{\Phi}\mathbf{n})\, v^2 + (\mathbf{n} \cdot \boldsymbol{\kappa}\mathbf{n})\,(\mathbf{n} \cdot \boldsymbol{\mu}\mathbf{n}) \det \boldsymbol{\kappa}^{-1} \det \boldsymbol{\mu}^{-1} = 0, \tag{2.9}$$

where $\mathbf{\Phi}$ is the real symmetric tensor given by

$$\mathbf{\Phi} = \det \boldsymbol{\kappa}^{-1} \left\{ tr(\boldsymbol{\kappa}\boldsymbol{\mu}^{-1})\boldsymbol{\kappa} - \boldsymbol{\kappa}\boldsymbol{\mu}^{-1}\boldsymbol{\kappa} \right\} = \det \boldsymbol{\mu}^{-1} \left\{ tr(\boldsymbol{\mu}\boldsymbol{\kappa}^{-1})\boldsymbol{\mu} - \boldsymbol{\mu}\boldsymbol{\kappa}^{-1}\boldsymbol{\mu} \right\}. \tag{2.10}$$

Let v_1^2, v_2^2 be the two solutions of (2.9) and let $\mathbf{D}_1$, $\mathbf{D}_2$ be the corresponding amplitudes of the electric displacement. Because $\mathbf{D}_1$ and $\mathbf{D}_2$ are along the common conjugate directions of the ellipses in which the plane $\mathbf{n} \cdot \mathbf{x} = 0$ cuts the two ellipsoids $\mathbf{x}\boldsymbol{\kappa}^{-1} \cdot \mathbf{x} = 1$ and $\mathbf{x}\boldsymbol{\mu}^{-1} \cdot \mathbf{x} = 1$, their directions are characterized by

$$\mathbf{n} \cdot \mathbf{D}_1 = \mathbf{n} \cdot \mathbf{D}_2 = 0\ , \quad \mathbf{D}_1 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_2 = \mathbf{D}_1 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_2 = 0\ , \tag{2.11}$$

and the phase speeds are given by [2]

$$\begin{aligned} v_1^2 &= \det \boldsymbol{\kappa}^{-1}(\mathbf{n} \cdot \boldsymbol{\kappa}\mathbf{n}) \frac{\mathbf{D}_2 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_2}{\mathbf{D}_2 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_2} = \det \boldsymbol{\mu}^{-1}(\mathbf{n} \cdot \boldsymbol{\mu}\mathbf{n}) \frac{\mathbf{D}_1 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_1}{\mathbf{D}_1 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_1}\ , \\ v_2^2 &= \det \boldsymbol{\kappa}^{-1}(\mathbf{n} \cdot \boldsymbol{\kappa}\mathbf{n}) \frac{\mathbf{D}_1 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_1}{\mathbf{D}_1 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_1} = \det \boldsymbol{\mu}^{-1}(\mathbf{n} \cdot \boldsymbol{\mu}\mathbf{n}) \frac{\mathbf{D}_2 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_2}{\mathbf{D}_2 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_2}\ . \end{aligned} \tag{2.12}$$

Equivalently,

$$v_1^{-2} = \frac{(\mathbf{D}_1 \times \mathbf{D}_2) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{(\mathbf{D}_1 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_1)(\mathbf{D}_2 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_2)}\ , \quad v_2^{-2} = \frac{(\mathbf{D}_1 \times \mathbf{D}_2) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{(\mathbf{D}_2 \cdot \boldsymbol{\kappa}^{-1}\mathbf{D}_2)(\mathbf{D}_1 \cdot \boldsymbol{\mu}^{-1}\mathbf{D}_1)}\ . \tag{2.13}$$

This may be geometrically interpreted [2]: the slowness $1/v_1$ ($1/v_2$) of the wave propagating along $\mathbf{n}$ with amplitude $\mathbf{D}_1$ $(\mathbf{D}_2)$ is equal to the area of the parallelogram formed by the radius along $\mathbf{D}_1$ ($\mathbf{D}_2$) to the ellipsoid $\mathbf{x} \cdot \boldsymbol{\kappa}^{-1}\mathbf{x} = 1$ and the radius along $\mathbf{D}_2$ ($\mathbf{D}_1$) to the ellipsoid $\mathbf{x} \cdot \boldsymbol{\mu}^{-1}\mathbf{x} = 1$.

The energy flux velocity $\mathbf{g}$, defined as the mean energy flux vector (Poynting vector) divided by the mean energy density, is given by

$$(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})\mathbf{g} = 2\mathbf{E} \times \mathbf{H}\ . \tag{2.14}$$

For the waves with amplitudes $\mathbf{D}_1$ and $\mathbf{D}_2$, we have [2], for their respective energy flux velocities $\mathbf{g}_1$ and $\mathbf{g}_2$,

$$v_1\mathbf{g}_1 = \frac{\boldsymbol{\kappa}^{-1}\mathbf{D}_1 \times \boldsymbol{\mu}^{-1}\mathbf{D}_2}{|\mathbf{D}_1 \times \mathbf{D}_2|}\ , \quad v_2\mathbf{g}_2 = \frac{\boldsymbol{\mu}^{-1}\mathbf{D}_1 \times \boldsymbol{\kappa}^{-1}\mathbf{D}_2}{|\mathbf{D}_1 \times \mathbf{D}_2|}\ . \tag{2.15}$$

3. Optic axes

As in [2], we introduce here the eigenvectors $\mathbf{V}_i$, $(i = 1, 2, 3)$, of the tensor $\boldsymbol{\mu}^{-1}$ with respect to $\boldsymbol{\kappa}^{-1}$, corresponding to the eigenvalues λ_i :

$$(\boldsymbol{\mu}^{-1} - \lambda_i \boldsymbol{\kappa}^{-1})\mathbf{V}_i = \mathbf{0} \ . \qquad \text{(no sum)} \tag{3.1}$$

Their directions are conjugate with respect to both ellipsoids $\mathbf{x} \cdot \boldsymbol{\kappa}^{-1}\mathbf{x} = 1$ and $\mathbf{x} \cdot \boldsymbol{\mu}^{-1}\mathbf{x} = 1$, that is

$$\mathbf{V}_i \cdot \boldsymbol{\kappa}^{-1}\mathbf{V}_j = \mathbf{V}_i \cdot \boldsymbol{\mu}^{-1}\mathbf{V}_j = 0 \ . \qquad (i \neq j) \tag{3.2}$$

Let k_i and m_i be defined by

$$k_i^{-1} = \mathbf{V}_i \cdot \boldsymbol{\kappa}^{-1}\mathbf{V}_i \ , \quad m_i^{-1} = \mathbf{V}_i \cdot \boldsymbol{\mu}^{-1}\mathbf{V}_i \ , \qquad \text{(no sum)} \tag{3.3}$$

and let

$$\mathbf{V}_*^i = k_i \boldsymbol{\kappa}^{-1}\mathbf{V}_i = m_i \boldsymbol{\mu}^{-1}\mathbf{V}_i \ . \qquad \text{(no sum)} \tag{3.4}$$

The set $\mathbf{V}_*^i$ is reciprocal to the set $\mathbf{V}_i$: $\mathbf{V}_*^i \cdot \mathbf{V}_j = \delta_j^i$, and we have

$$(\boldsymbol{\kappa} - \lambda_i \boldsymbol{\mu})\mathbf{V}_*^i = \mathbf{0} \ , \qquad \text{(no sum)} \tag{3.5}$$

so that $\mathbf{V}_*^i$ are the eigenvectors of $\boldsymbol{\kappa}$ with respect to $\boldsymbol{\mu}$, corresponding to the eigenvalues λ_i. Their directions are conjugate with respect to both ellipsoids $\mathbf{x} \cdot \boldsymbol{\kappa}\mathbf{x} = 1$ and $\mathbf{x} \cdot \boldsymbol{\mu}\mathbf{x} = 1$, that is

$$\mathbf{V}_*^i \cdot \boldsymbol{\kappa}\mathbf{V}_*^j = \mathbf{V}_*^i \cdot \boldsymbol{\mu}\mathbf{V}_*^j = 0 \ . \qquad (i \neq j) \tag{3.6}$$

Also,

$$k_i = \mathbf{V}_*^i \cdot \boldsymbol{\kappa}\mathbf{V}_*^i \ , \quad m_i = \mathbf{V}_*^i \cdot \boldsymbol{\mu}\mathbf{V}_*^i \ . \qquad \text{(no sum)} \tag{3.7}$$

Here we consider the general case when the three eigenvalues $\lambda_i = k_i/m_i$ are all different, and we order them $\lambda_1 > \lambda_2 > \lambda_3$ ("biaxial" crystals). Then, there are two "generalized optic axes" [2] whose directions are given by the vectors

$$\mathbf{n}_O^{\pm} = \left\{ \frac{k_3(\lambda_1 - \lambda_2)}{k_2(\lambda_1 - \lambda_3)} \right\}^{1/2} \mathbf{V}_*^1 \pm \left\{ \frac{k_1(\lambda_2 - \lambda_3)}{k_2(\lambda_1 - \lambda_3)} \right\}^{1/2} \mathbf{V}_*^3 \ , \tag{3.8}$$

which are normal to the two planes that cut the ellipsoids $\mathbf{x} \cdot \boldsymbol{\kappa}^{-1}\mathbf{x} = 1$ and $\mathbf{x} \cdot \boldsymbol{\mu}^{-1}\mathbf{x} = 1$ in similar and similarly situated ellipses [9]. We note the identity [9]

$$\boldsymbol{\mu}^{-1} = \lambda_2 \boldsymbol{\kappa}^{-1} + \frac{k_2(\lambda_1 - \lambda_3)}{2k_1k_3}(\mathbf{n}_O^+ \otimes \mathbf{n}_O^- + \mathbf{n}_O^- \otimes \mathbf{n}_O^+) \ . \tag{3.9}$$

The generalized optic axes are the propagation directions for which the two phase speeds v_1, v_2 are equal.

Referring an arbitrary direction $\mathbf{n}$ to the basis $\mathbf{V}_*^i$, we write $\mathbf{n} = n_i \mathbf{V}_*^i$, and the secular equation (2.9) may be put in the form [2]

$$\sum_{i=1}^{3} \frac{m_i n_i^2}{\lambda_i^{-1}(\mathbf{n}\cdot\boldsymbol{\mu}\mathbf{n}) - (\det\boldsymbol{\mu})v^2} = 0, \qquad (3.10)$$

or, equivalently

$$\sum_{i=1}^{3} \frac{k_i n_i^2}{\lambda_i(\mathbf{n}\cdot\boldsymbol{\kappa}\mathbf{n}) - (\det\boldsymbol{\kappa})v^2} = 0. \qquad (3.11)$$

Note that (3.10) generalizes Fresnels's equation (see, for instance [10]) to the case of electrically and magnetically anisotropic crystals. We call (3.10) and (3.11) the "Fresnel forms" of the secular equation for electromagnetic waves in crystals. Using the results of [9], it may be seen that the amplitudes $\mathbf{D}_1, \mathbf{D}_2$ may be expressed in terms of the propagation direction $\mathbf{n}$, using the generalized optic axes $\mathbf{n}_O^{\pm}$. We have [9]

$$\mathbf{D}_{1,2} = (\mathbf{n}\times\mathbf{n}_O^+)/\sigma_+ \pm (\mathbf{n}\times\mathbf{n}_O^-)/\sigma_-, \qquad (3.12)$$

where $\sigma_\pm$ are given by

$$\sigma_\pm^2 = m_2(\mathbf{n}\times\mathbf{n}_O^\pm)\cdot\boldsymbol{\mu}^{-1}(\mathbf{n}\times\mathbf{n}_O^\pm) = k_2(\mathbf{n}\times\mathbf{n}_O^\pm)\cdot\boldsymbol{\kappa}^{-1}(\mathbf{n}\times\mathbf{n}_O^\pm). \qquad (3.13)$$

The corresponding phase speeds are given by [3]

$$\begin{aligned} v_{1,2}^2 = & \frac{1}{2}\det\boldsymbol{\mu}^{-1}(\mathbf{n}\cdot\boldsymbol{\mu}\mathbf{n})(\lambda_3^{-1}+\lambda_1^{-1}) + \\ & \frac{\lambda_3^{-1}-\lambda_1^{-1}}{2m_1m_3}\left\{m_2(\det\boldsymbol{\mu}^{-1})(\mathbf{n}\cdot\boldsymbol{\mu}\mathbf{n}_O^+)(\mathbf{n}\cdot\boldsymbol{\mu}\mathbf{n}_O^-) \pm \sigma_+\sigma_-\right\}, \end{aligned} \qquad (3.14)$$

or, equivalently,

$$\begin{aligned} v_{1,2}^2 = & \frac{1}{2}\det\boldsymbol{\kappa}^{-1}(\mathbf{n}\cdot\boldsymbol{\kappa}\mathbf{n})(\lambda_1+\lambda_3) - \\ & \frac{\lambda_1-\lambda_3}{2k_1k_3}\left\{k_2(\det\boldsymbol{\kappa}^{-1})(\mathbf{n}\cdot\boldsymbol{\kappa}\mathbf{n}_O^+)(\mathbf{n}\cdot\boldsymbol{\kappa}\mathbf{n}_O^-) \mp \sigma_+\sigma_-\right\}. \end{aligned} \qquad (3.15)$$

These formulae generalize formulae derived by Neumann for the case of electrically anisotropic but magnetically isotropic crystals (see, for instance, [11]).

4. Results for finite amplitude elastic waves in deformed Mooney-Rivlin materials

Here we recall results obtained for the propagation of finite amplitude elastic plane waves in a Mooney-Rivlin material which is maintained in a state of arbitrary static homogeneous deformation. These results are taken from [5],[6] (see also [7],[8] for collected results).

Mooney-Rivlin materials are incompressible isotropic elastic materials characterized by a strain-energy density W per unit volume given by

$$2W = C(I-3) + D(II-3), \tag{4.1}$$

where C and D are material constants (we here assume $C > 0$, $D > 0$), and

$$I = \mathrm{tr}\mathbb{B}, \quad 2II = (\mathrm{tr}\mathbb{B})^2 - \mathrm{tr}(\mathbb{B}^2). \tag{4.2}$$

Here $\mathbb{B}$ denotes the left Cauchy-Green strain tensor, whose components, in a rectangular Cartesian coordinate system, are

$$\mathbb{B}_{ij} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_j}{\partial X_A}, \tag{4.3}$$

where x_i, $(i = 1, 2, 3)$, denotes the position, after deformation, of the particle whose position is X_A, $(A = 1, 2, 3)$, before deformation. Because the material is incompressible, $III = \det\mathbb{B} = 1$. The corresponding constitutive equation for the symmetric Cauchy stress tensor $\mathbb{T}$ may be written in the form

$$\mathbb{T} = -p\mathbf{1} + C\mathbb{B} - D\mathbb{B}^{-1}\ , \tag{4.4}$$

where p is an indeterminate pressure corresponding to the incompressibility constraint. We also note that, because $III = 1$, we have $II = \mathrm{tr}\mathbb{B}^{-1}$.

Consider now a static finite homogeneous deformation $x_i = \mathbb{F}_{iA}X_A$, with $\det\mathbb{F} = 1$, of a Mooney-Rivlin material, and let $\mathbb{B}=\mathbb{F}\mathbb{F}^T$ be the corresponding constant left Cauchy-Green strain tensor. Then, superimposed on this finite homogeneous static deformation, we consider finite amplitude linearly polarized plane waves taking the particle at $\mathbf{x}$ in the static deformation to $\overline{\mathbf{x}}$, given by

$$\overline{\mathbf{x}} = \mathbf{x} + \mathbf{a}f(\eta, t)\ , \qquad \eta = \mathbf{n}\cdot\mathbf{x}\ . \tag{4.5}$$

Here, $\mathbf{n}$ is a unit vector along the propagation direction, and $\mathbf{a}$ a vector along the polarization direction. For the sake of comparison with results of §2-3, we do not assume that $\mathbf{a}$ is a *unit* vector. It has been shown [5] that along any direction $\mathbf{n}$, two such finite amplitude waves may propagate with polarization directions $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2$ characterized by

$$\mathbf{n}\cdot\mathbf{a}_1=\mathbf{n}\cdot\mathbf{a}_2=0,\quad \mathbf{a}_1\cdot\mathbf{a}_2=\mathbf{a}_1\cdot\mathbb{B}^{-1}\mathbf{a}_2=0. \tag{4.6}$$

The two possible polarization directions $\mathbf{a}_1, \mathbf{a}_2$ are along the principal axes of the elliptical section of the ellipsoid $\mathbf{x}\cdot\mathbb{B}^{-1}\mathbf{x}=1$ by the plane $\mathbf{n}\cdot\mathbf{x}=0$. Although the material is nonlinear and the waves are of finite amplitude, the function $f(\eta,t)$ is governed by the second order linear wave equation

$$\partial_t^2 f - v^2\partial_\eta^2 f = 0, \tag{4.7}$$

where $v^2 = v_1^2, v_2^2$ is given by

$$\rho v_1^2 = C\mathbf{n}\cdot\mathbb{B}\mathbf{n} + D\,\hat{\mathbf{a}}_1\cdot\mathbb{B}^{-1}\,\hat{\mathbf{a}}_1,\quad \rho v_2^2 = C\mathbf{n}\cdot\mathbb{B}\mathbf{n} + D\,\hat{\mathbf{a}}_2\cdot\mathbb{B}^{-1}\,\hat{\mathbf{a}}_2\,. \tag{4.8}$$

Here ρ denotes the constant mass density of the material, and $\hat{\mathbf{a}}_{1,2}$ are unit vectors along $\mathbf{a}_{1,2}$: $\hat{\mathbf{a}}_{1,2}=\mathbf{a}_{1,2}/\left|\mathbf{a}_{1,2}\right|$. In particular, we may choose $f=\cos\omega(v^{-1}\eta-t)$ as a solution of (4.7), so that the corresponding waves are time-harmonic plane waved (with angular frequency ω). The vector $\mathbf{a}$ then represents the amplitude vector of the displacement field $\mathbf{u}=\overline{\mathbf{x}}-\mathbf{x}$.

Also, it has been shown [6] that it is useful to introduce the tensor $\mathbb{E}$, defined by

$$\rho\mathbb{E} = C\mathbf{1} + D\mathbb{B}, \tag{4.9}$$

for the formulation of various properties of the waves. In particular, the two squared wave speeds are the solutions of the secular equation

$$v^4 - v^2\mathbf{n}\cdot\{\mathrm{tr}\mathbb{E})\mathbb{B} - \mathbb{B}\mathbb{E}\}\mathbf{n} + (\det\mathbb{E})(\mathbf{n}\cdot\mathbb{B}\mathbf{n})(\mathbf{n}\cdot\mathbb{B}\mathbb{E}^{-1}\mathbf{n}) = 0. \tag{4.10}$$

The solutions v_1^2, v_2^2 of this equation may also be expressed using the tensor $\mathbb{E}$. Indeed, we have [8]

$$\begin{aligned} v_1^2 &= (\mathbf{n}\cdot\mathbb{B}\mathbf{n})\frac{(\mathbf{a}_2\cdot\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2)}{(\mathbf{a}_2\cdot\mathbb{B}^{-1}\mathbf{a}_2)} = (\det\mathbb{E})(\mathbf{n}\cdot\mathbb{B}\mathbb{E}^{-1}\mathbf{n})\frac{(\mathbf{a}_1\cdot\mathbb{B}^{-1}\mathbf{a}_1)}{(\mathbf{a}_1\cdot\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_1)}, \\ v_2^2 &= (\mathbf{n}\cdot\mathbb{B}\mathbf{n})\frac{(\mathbf{a}_1\cdot\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_1)}{(\mathbf{a}_1\cdot\mathbb{B}^{-1}\mathbf{a}_1)} = (\det\mathbb{E})(\mathbf{n}\cdot\mathbb{B}\mathbb{E}^{-1}\mathbf{n})\frac{(\mathbf{a}_2\cdot\mathbb{B}^{-1}\mathbf{a}_2)}{(\mathbf{a}_2\cdot\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2)}. \end{aligned} \tag{4.11}$$

Alternatively,

$$v_1^{-2} = \frac{(\mathbf{a}_1\times\mathbf{a}_2)\cdot(\mathbf{a}_1\times\mathbf{a}_2)}{(\mathbf{a}_1\cdot\mathbb{B}^{-1}\mathbf{a}_1)(\mathbf{a}_2\cdot\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2)}\,,\quad v_2^{-2} = \frac{(\mathbf{a}_1\times\mathbf{a}_2)\cdot(\mathbf{a}_1\times\mathbf{a}_2)}{(\mathbf{a}_2\cdot\mathbb{B}^{-1}\mathbf{a}_2)(\mathbf{a}_1\cdot\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_1)}\,. \tag{4.12}$$

Of course, here, $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$ so that $(\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = a_1^2 a_2^2$. The expressions (4.12) may be geometrically interpreted: the slowness $1/v_1$ $(1/v_2)$ of the wave propagating along $\mathbf{n}$ with amplitude $\mathbf{a}_1(\mathbf{a}_2)$ is equal to the area of the rectangle formed by the radius along $\mathbf{a}_1(\mathbf{a}_2)$ to the ellipsoid $\mathbf{x} \cdot \mathbb{B}^{-1}\mathbf{x} = 1$ and the radius along $\mathbf{a}_2(\mathbf{a}_1)$ to the ellipsoid $\mathbf{x} \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{x} = 1$. We also note that the conditions (4.6) characterizing the two polarization directions $\mathbf{a}_1, \mathbf{a}_2$ may be written equivalently as

$$\mathbf{n} \cdot \mathbf{a}_1 = \mathbf{n} \cdot \mathbf{a}_2 = 0, \quad \mathbf{a}_1 \cdot \mathbb{B}^{-1}\mathbf{a}_2 = \mathbf{a}_2 \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_1 = 0. \tag{4.13}$$

Considering now time-periodic waves propagating with speed $+v$, thus solutions of (4.7) of the type $f = F(\eta - vt)$, where F is any periodic function, we introduce the energy flux velocity $\mathbf{g}$ defined as the mean energy flux vector divided by the mean energy density (see details in [6],[13]). For the waves with amplitudes $\mathbf{a}_1, \mathbf{a}_2$, the energy flux velocities are $\mathbf{g}_1, \mathbf{g}_2$ given by

$$v_1\mathbf{g}_1 = \frac{\mathbb{B}^{-1}\mathbf{a}_1 \times \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}, \quad v_2\mathbf{g}_2 = \frac{\mathbb{E}\mathbb{B}^{-1}\mathbf{a}_1 \times \mathbb{B}^{-1}\mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}. \tag{4.14}$$

Of course, here, $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$ so that $|\mathbf{a}_1 \times \mathbf{a}_2| = |\mathbf{a}_1||\mathbf{a}_2|$.

5. Acoustic axes

Let $\mathbf{v}_i$, $(i = 1, 2, 3)$, be unit vectors along the principal axes of the basic static deformation. Thus, in the orthonormal basis $\mathbf{v}_i$, the left Cauchy-Green strain tensor $\mathbb{B}$ of this deformation is diagonal. The diagonal elements are the squared principal stretches $b_1 = \lambda_1^2, b_2 = \lambda_2^2, b_3 = \lambda_3^2$ (the notation λ_i is usual in elasticity for the principal stretches, but it has already been used with another meaning in §3, so we shall here use the notation b_i for the squared principal stretches). Because of the incompressibility constraint they are such that $b_1 b_2 b_3 = 1$. The tensors $\mathbb{B}^{-1}$ and $\mathbb{E}\mathbb{B}^{-1}$, with $\mathbb{E}$ defined by (4.9), are also diagonal in this basis:

$$\mathbf{v}_i \cdot \mathbb{B}^{-1}\mathbf{v}_j = \mathbf{v}_i \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{v}_j = 0, \qquad (i \neq j) \tag{5.1}$$

and

$$\mathbf{v}_i \cdot \mathbb{B}^{-1}\mathbf{v}_i = b_i^{-1}, \quad \mathbf{v}_i \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{v}_i = \mathrm{E}_i b_i^{-1}, \qquad \text{(no sum)} \tag{5.2}$$

where E_i are the eigenvalues of $\mathbb{E}$, given by $\rho\mathrm{E}_i = C + Db_i$.

Here we consider the general case when the static deformation is triaxial so that the three squared principal stretches are all different, and we order them $b_1 > b_2 > b_3$. Then, there are two "acoustic axes" [5] for the propagation of elastic waves in the deformed material. The unit vectors along the acoustic axes are $\mathbf{n}_A^{\pm}$ given by

$$\mathbf{n}_A^{\pm} = \left\{ \frac{b_2^{-1} - b_1^{-1}}{b_3^{-1} - b_1^{-1}} \right\}^{1/2} \mathbf{v}_1 \pm \left\{ \frac{b_3^{-1} - b_2^{-1}}{b_3^{-1} - b_1^{-1}} \right\}^{1/2} \mathbf{v}_3 \ . \tag{5.3}$$

They are normal to the planes of central circular section of the ellipsoid $\mathbf{x} \cdot \mathbb{B}^{-1} x{=}1$. Equivalently (see [6]), we have

$$\mathbf{n}_A^{\pm} = \left\{ \frac{b_3(\mathrm{E}_1 - \mathrm{E}_2)}{b_2(\mathrm{E}_1 - \mathrm{E}_3)} \right\}^{1/2} \mathbf{v}_1 \pm \left\{ \frac{b_1(\mathrm{E}_2 - \mathrm{E}_3)}{b_2(\mathrm{E}_1 - \mathrm{E}_3)} \right\}^{1/2} \mathbf{v}_3 \ . \tag{5.4}$$

We note the identities [6]

$$\mathbb{B}^{-1} = b_2^{-1}\mathbf{1} - \frac{1}{2}(b_3^{-1} - b_1^{-1})(\mathbf{n}_A^{+}\otimes\mathbf{n}_A^{-}+\mathbf{n}_A^{-}\otimes\mathbf{n}_A^{+}) \ , \tag{5.5}$$

and

$$\mathbb{E}\mathbb{B}^{-1} = \mathrm{E}_2\mathbb{B}^{-1} + \frac{b_2(\mathrm{E}_1 - \mathrm{E}_3)}{2b_1b_3}(\mathbf{n}_A^{+}\otimes\mathbf{n}_A^{-}+\mathbf{n}_A^{-}\otimes\mathbf{n}_A^{+}) \ . \tag{5.6}$$

The acoustic axes are the only propagation directions for which the two phase speeds v_1, v_2 are equal.

Referring an arbitrary propagation direction $\mathbf{n}$ to the basis v_i, we write $\mathbf{n} = n_i v_i$ and the secular equation (4.10) may be put in the form [6]

$$\sum_{i=1}^{3} \frac{b_i n_i^2}{\mathrm{E}_i(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) - v^2} = 0. \tag{5.7}$$

We call this the "Fresnel form" of the secular equation for elastic waves in deformed Mooney-Rivlin materials. Also, the amplitudes $\mathbf{D}_1, \mathbf{D}_2$ may be expressed in terms of the propagation direction $\mathbf{n}$, using the acoustic axes $\mathbf{n}_A^{\pm}$. We have [6] [7]

$$\mathbf{a}_{1,2} = (\mathbf{n} \times \mathbf{n}_A^{+})/\delta_{+} \pm (\mathbf{n} \times \mathbf{n}_A^{-})/\delta_{-}, \tag{5.8}$$

where $\delta_{\pm}$ are given by

$$\begin{aligned}\delta_{\pm}^2 &= b_2(\mathbf{n} \times \mathbf{n}_A^{\pm}) \cdot \mathbb{B}^{-1}(\mathbf{n} \times \mathbf{n}_A^{\pm}) = \mathrm{E}_2^{-1} b_2(\mathbf{n} \times \mathbf{n}_A^{\pm}) \cdot \mathbb{E}\mathbb{B}^{-1}(\mathbf{n} \times \mathbf{n}_A^{\pm}) \\ &= (\mathbf{n} \times \mathbf{n}_A^{\pm}) \cdot (\mathbf{n} \times \mathbf{n}_A^{\pm}).\end{aligned} \tag{5.9}$$

We note that here $\delta_{\pm} = \sin\varphi_{\pm}$, where $\varphi_{\pm}$ denote the angles that the propagation direction $\mathbf{n}$ makes with the acoustic axes $\mathbf{n}_A^{\pm}$. The corresponding phase speeds are given by [6] [7]

$$v_{1,2}^2 = \frac{1}{2}(\mathrm{E}_1 + \mathrm{E}_3)(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) - \frac{\mathrm{E}_1 - \mathrm{E}_3}{2b_1 b_3} \left\{ b_2 (\mathbf{n} \cdot \mathbb{B}\mathbf{n}_A^+)(\mathbf{n} \cdot \mathbb{B}\mathbf{n}_A^-) \mp \delta_+ \delta_- \right\} . \tag{5.10}$$

These expressions yield results for the squared wave speeds as functions of the angles φ_+ and φ_- that the propagation direction $\mathbf{n}$ makes with the acoustic axes $\mathbf{n}_A^+$ and $\mathbf{n}_A^-$ (see [5] for these results).

6. An electromagnetic-elastic analogy

Here we show that elastic (finite-amplitude) waves in deformed Mooney-Rivlin materials formally behave as electromagnetic waves in crystals with appropriate electric permittivity and magnetic permeability tensors. Thus, the results of §4–5 for elastic waves in deformed Mooney-Rivlin materials formally appear as a special case of the results of §2–3 for electromagnetic waves in electrically and magnetically anisotropic crystals.

Indeed, a comparison of the results of §2–3 on the one hand and of §4–5 on the other hand suggests that the properties of elastic waves in deformed Mooney-Rivlin materials may be obtained from corresponding properties of electromagnetic waves in electrically and magnetically anisotropic media by the formal substitutions

$$\boldsymbol{\kappa}^{-1} = \mathbb{B}^{-1}, \qquad \boldsymbol{\mu}^{-1} = \mathbb{E}\mathbb{B}^{-1} \ , \tag{6.1}$$

the amplitude $\mathbf{D}$ of the *electric* displacement field being replaced by the vector $\mathbf{a}$ along the polarization direction of the elastic waves (amplitude of the *elastic* displacement field for time-harmonic plane waves). This will now be checked in detail. Prior to that, we note that $\boldsymbol{\kappa}^{-1}$ and $\boldsymbol{\mu}^{-1}$ given by (6.1) have the same principal axes, the principal axes of the basic static deformation. This was not assumed in §2–3, so that the results of §4–5 will formally appear as a specialization of the results of §2–3.

Consider first the conditions (4.6) characterizing the two polarization directions $\mathbf{a}_1, \mathbf{a}_2$ of the elastic waves for a given propagation direction $\mathbf{n}$. As we have noted, they are equivalent to the conditions (4.13). Clearly, they may be obtained from the conditions (2.11) for electromagnetic waves by the substitutions $\boldsymbol{\kappa}^{-1} = \mathbb{B}^{-1}$, $\boldsymbol{\mu}^{-1} = \mathbb{E}\mathbb{B}^{-1}$, and $\mathbf{D}_1 = \mathbf{a}_1$, $\mathbf{D}_2 = \mathbf{a}_2$. Consider next the secular equation (4.10) for elastic waves. It may be obtained from the secular equation (2.9) for electromagnetic waves by the substitutions $\boldsymbol{\kappa}^{-1} = \mathbb{B}^{-1}$, $\boldsymbol{\mu}^{-1} = \mathbb{E}\mathbb{B}^{-1}$, on using the first expression (2.10) of $\boldsymbol{\Phi}$ and recalling that $\det \mathbb{B} = 1$. The expressions (4.11) of the squared wave speeds v_1^2, v_2^2 for the elastic waves are obtained from the corresponding expressions (2.12) for the squared wave speeds of the electromagnetic waves by the substitutions $\boldsymbol{\kappa}^{-1} = \mathbb{B}^{-1}$, $\boldsymbol{\mu}^{-1} = \mathbb{E}\mathbb{B}^{-1}$, and $\mathbf{D}_1 = \mathbf{a}_1$, $\mathbf{D}_2 = \mathbf{a}_2$ (the same is valid for (4.12) which is the counterpart of (2.13)). Also, the expressions (4.14) of the energy flux velocities $\mathbf{g}_1, \mathbf{g}_2$ for the elastic waves are obtained from the corresponding expressions (2.15) for electromagnetic waves by the same substitutions.

Considering the results of §3 and §5, using the generalized optic axes and the acoustic axes, respectively, the analogy goes through. Indeed, substituting (6.1) into (3.1), we conclude that then $\mathbb{B}^{-1}\mathbf{V}_i$, $(i = 1, 2, 3)$, are along the principal axes of the tensor $\mathbb{E}$ defined by (4.9), thus along the principal axes of $\mathbb{B}$ (principal axes of the basic static deformation of the Mooney-Rivlin material). Hence, with the substitutions (6.1), $\mathbf{V}_i$, $(i = 1, 2, 3)$, may be taken to be unit vectors along the principal axes of $\mathbb{B}$, and using also (3.3) or (3.7), we have the corresponding substitutions

$$\mathbf{V}_i = \mathbf{V}_*^i = \mathbf{v}_i, \qquad \lambda_i = \mathrm{E}_i \ , \tag{6.2}$$

and

$$k_i = b_i, \quad m_i = b_i \mathrm{E}_i^{-1}. \tag{6.3}$$

Using these substitutions, the acoustic axes $\mathbf{n}_A^{\pm}$ given by (5.4) may clearly be obtained from the generalized optic axes $\mathbf{n}_O^{\pm}$ given by (3.8), and the identity (5.6) is the counterpart of the identity (3.9). Moreover, with the substitutions (6.1)–(6.3), the Fresnel form (5.7) of the secular equation for elastic waves in deformed Mooney-Rivlin materials is obtained from the Fresnel form (3.11) of the secular equation for electromagnetic waves in crystals. Also, the results (5.8) and (5.10) are the counterparts of (3.12) and (3.15) because, with the substitutions (6.1)–(6.3), we have $\mathbf{n}_O^{\pm} = \mathbf{n}_A^{\pm}$ and $\sigma_{\pm} = \delta_{\pm}$.

Considering now Maxwell's equations (2.1) (2.2) for the amplitudes of the electromagnetic field, it first seems that there is no counterpart of these equations for elastic waves in deformed Mooney-Rivlin materials, because electromagnetic waves involve four amplitudes $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ and elastic waves involve just one amplitude $\mathbf{a}$. However, a counterpart of equations (2.1)–(2.3) may be constructed for elastic waves in deformed Mooney-Rivlin materials. Indeed, consider the elastic wave propagating in the direction $\mathbf{n}$ with speed $v(=v_1$ or $v_2)$ and polarization vector $\mathbf{a}(=\mathbf{a}_1$ or $\mathbf{a}_2)$, and define the vectors $\mathbf{e}, \mathbf{c}, \mathbf{h}$ as

$$\mathbf{e} = \mathbb{B}^{-1}\mathbf{a}, \quad \mathbf{c} = v^{-1}\mathbf{n} \times \mathbf{e}, \quad \mathbf{h} = \mathbb{E}\mathbb{B}^{-1}\mathbf{c}. \tag{6.4}$$

Because $\mathbf{n} = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2 = (\mathbf{a}_1 \times \mathbf{a}_2)/\left|\mathbf{a}_1 \times \mathbf{a}_2\right|$, we have, for the elastic wave propagating with speed v_1 and polarization vector $\mathbf{a}_1$,

$$\mathbf{c}_1 = \left(\frac{\mathbf{a}_1 \times \mathbf{a}_2}{v_1\,|\mathbf{a}_1 \times \mathbf{a}_2|}\right) \times \mathbb{B}^{-1}\mathbf{a}_1 = \left(\frac{\mathbf{a}_1 \cdot \mathbb{B}^{-1}\mathbf{a}_1}{\mathbf{a}_2 \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2}\right)^{1/2} \mathbf{a}_2, \tag{6.5}$$

on using (4.12) and (4.13), and hence,

$$\mathbf{h}_1 = \left(\frac{\mathbf{a}_1 \cdot \mathbb{B}^{-1}\mathbf{a}_1}{\mathbf{a}_2 \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2}\right)^{1/2} \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2. \tag{6.6}$$

Calculating now $\mathbf{n} \times \mathbf{h}_1$, we obtain

$$\begin{aligned}\mathbf{n} \times \mathbf{h}_1 &= \left(\frac{\mathbf{a}_1 \cdot \mathbb{B}^{-1}\mathbf{a}_1}{\mathbf{a}_2 \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2}\right)^{1/2} \left(\frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}\right) \times \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2 \\ &= -\frac{\left(\mathbf{a}_1 \cdot \mathbb{B}^{-1}\mathbf{a}_1\right)^{1/2}\left(\mathbf{a}_2 \cdot \mathbb{E}\mathbb{B}^{-1}\mathbf{a}_2\right)^{1/2}}{|\mathbf{a}_1 \times \mathbf{a}_2|}\mathbf{a}_1 = -v_1\mathbf{a}_1, \end{aligned}\tag{6.7}$$

on using again (4.12) and (4.13). A similar calculation for the elastic wave propagating with speed v_2 and polarization vector $\mathbf{a}_2$ yields $\mathbf{n} \times \mathbf{h}_2 = -v_2\mathbf{a}_2$. Hence, the vectors $\mathbf{a}(=\mathbf{a}_1$ or $\mathbf{a}_2)$ and $\mathbf{h}(=\mathbf{h}_1$ or $\mathbf{h}_2)$ satisfy

$$\mathbf{n} \times \mathbf{h} = -v\mathbf{a}. \tag{6.8}$$

Collecting now the definitions (6.4) and the result (6.8), and noting that $\mathbf{a}$ and $\mathbf{c}$ are orthogonal to the propagation direction $\mathbf{n}$, we write

$$\mathbf{n} \cdot \mathbf{a} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{c} = \mathbf{0}, \tag{6.9}$$

$$\mathbf{n} \times \mathbf{e} = v\mathbf{c}, \quad \mathbf{n} \times \mathbf{h} = -v\mathbf{a}, \tag{6.10}$$

and

$$\mathbf{a} = \mathbb{B}\mathbf{e}, \quad \mathbf{c} = \mathbb{B}\mathbb{E}^{-1}\mathbf{h}. \tag{6.11}$$

These may be obtained from Maxwell's equations (2.1), (2.2) and constitutive equations (2.3) by the formal substitutions $\boldsymbol{\kappa}^{-1} = \mathbb{B}^{-1}$, $\boldsymbol{\mu}^{-1} = \mathbb{E}\mathbb{B}^{-1}$, $\mathbf{D} = \mathbf{a}, \mathbf{E} = \mathbf{e}, \mathbf{B} = \mathbf{c}, \mathbf{H} = \mathbf{h}$.

The analogy between equations (2.1)–(2.3) and equations (6.9)–(6.11) may then be used to find counterparts to the propagation condition in the form (2.5) or (2.7) for finite-amplitude elastic waves propagating in deformed Mooney-Rivlin materials. With the formal substitutions (6.1) and $\mathbf{E} = \mathbf{e} = \mathbb{B}^{-1}\mathbf{a}$, $\mathbf{D} = \mathbf{a}$, equations (2.5) and (2.7) read

$$(\boldsymbol{\Gamma}\mathbb{E}\mathbb{B}^{-1}\boldsymbol{\Gamma}\mathbb{B}^{-1} + v^2 1)\mathbf{a} = \mathbf{0}\ , \tag{6.12}$$

and

$$\{\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi} - v^2(\det E^{-1})(\mathbf{n}\cdot\mathbb{B}\mathbb{E}^{-1}\mathbf{n})^{-1}\mathbf{\Pi}\mathbb{E}\mathbb{B}^{-1}\mathbf{\Pi}\}\mathbf{a} = \mathbf{0}\ . \tag{6.13}$$

Similarly, counterparts to the propagation condition in the form (2.6) or (2.8) may be obtained using (6.1) and $\mathbf{H} = \mathbf{h} = \mathbb{E}\mathbb{B}^{-1}\mathbf{c}$, $\mathbf{B} = \mathbf{c}$. They read

$$(\mathbf{\Gamma}\mathbb{B}^{-1}\mathbf{\Gamma}\mathbb{E}\mathbb{B}^{-1} + v^2\mathbf{1})\mathbf{c} = \mathbf{0}\ , \tag{6.14}$$

and

$$\{\mathbf{\Pi}\mathbb{E}\mathbb{B}^{-1}\mathbf{\Pi} - v^2(\mathbf{n}\cdot\mathbb{B}\mathbf{n})^{-1}\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi}\}\mathbf{c} = \mathbf{0}\ , \tag{6.15}$$

where, as defined by (6.4), $\mathbf{c} = v^{-1}\mathbf{\Gamma}\mathbb{B}^{-1}\mathbf{a}$. Hence, for elastic waves propagating in deformed Mooney-Rivlin materials, the two squared phase speeds and the corresponding polarization directions may be obtained by solving any of the eigenvalue problems (6.12)–(6.15). Of course, if (6.14) or (6.15) is used, then after obtaining the directions of the eigenvectors $\mathbf{c}$, the corresponding directions of $\mathbf{a}$ must be determined. This is easily done because, from $(6.10)_2$ and $(6.11)_2$, we have $\mathbf{a} = -v^{-1}\mathbf{\Gamma}\mathbb{E}\mathbb{B}^{-1}\mathbf{c}$.

Remark. The two tensors $\boldsymbol{\kappa}^{-1}$ and $\boldsymbol{\mu}^{-1}$ given by (6.1) are both expressed in terms of one tensor, the left Cauchy-Green strain tensor $\mathbb{B}$. Thus, recalling the definition (4.9) of the tensor $\mathbb{E}$, we note that $\boldsymbol{\kappa}^{-1}$ and $\boldsymbol{\mu}^{-1}$ given by (6.1) satisfy

$$\rho\boldsymbol{\mu}^{-1} - C\boldsymbol{\kappa}^{-1} = D\mathbf{1}\ . \tag{6.16}$$

As noticed previously, the results for elastic waves in deformed Mooney-Rivlin materials appear as formally analogous to a *special case* of the results for electromagnetic waves in crystals. We note here that the condition for existence of three constants ρ, C, D (not all zero) such that the two positive definite tensors $\boldsymbol{\kappa}^{-1}$ and $\boldsymbol{\mu}^{-1}$ satisfy (6.16) is that the ellipsoids $\mathbf{x}\cdot\boldsymbol{\kappa}^{-1}\mathbf{x} = 1$ and $\mathbf{x}\cdot\boldsymbol{\mu}^{-1}\mathbf{x} = 1$ have the same planes of central circular section (see [12]). This implies in particular that the two ellipsoids have the same principal axes, with the same intermediate axis. The condition for C to be positive ($\rho > 0$ being given) is then that the two ellipsoids have the same major and minor principal axes (k_1, k_2, k_3 ordered as m_1, m_2, m_3). The condition for D to be positive is then that the ellipsoid $\mathbf{x}\cdot\boldsymbol{\kappa}\boldsymbol{\mu}^{-1}\mathbf{x} = 1$ has the same major and minor axes as the ellipsoid $\mathbf{x}\cdot\boldsymbol{\kappa}\mathbf{x} = 1$ ($\lambda_1, \lambda_2, \lambda_3$ ordered as k_1, k_2, k_3 and m_1, m_2, m_3, thus, recalling the order chosen in §3, $k_1 > k_2 > k_3$ and $m_1 > m_2 > m_3$).

7. Conclusion

The results for electromagnetic plane waves in electrically and magnetically anisotropic crystals (§2–3) have been compared with the results for finite-amplitude elastic plane waves in deformed Mooney-Rivlin materials (§4–5). Although the theory of electromagnetic waves considered here is linear whilst the theory of elastic waves in Mooney-Rivlin is nonlinear, an analogy has been exhibited. This was possible because finite-amplitude waves in Mooney-Rivlin materials behave essentially as linear waves (constant wave speeds).

More precisely, it has been shown that the results for elastic waves in deformed Mooney-Rivlin materials may be obtained as a special case of the results for electromagnetic waves in crystals by appropriate formal substitutions. We here list these substitutions (Table 1): the basic substitutions are $\boldsymbol{\kappa}^{-1} = \mathbb{B}^{-1}$, $\boldsymbol{\mu}^{-1} = \mathbb{E}\mathbb{B}^{-1}$, $\mathbf{D} = \mathbf{a}$, the others being consequences of these three.

electromagnetic waves in crystals ⇢	elastic waves in Mooney-Rivlin materials
$\boldsymbol{\kappa}^{-1}$ (inverse electric permittivity)	$\mathbb{B}^{-1}$ (inverse left Cauchy-Green strain tensor)
$\boldsymbol{\mu}^{-1}$ (inverse magnetic permeability)	$\mathbb{E}\mathbb{B}^{-1}$ (with $\rho\mathbb{E} \equiv C\mathbf{1} + D\mathbb{B}$)
$\mathbf{D}$ (electric displacement amplitude)	$\mathbf{a}$ (elastic displacement amplitude)
$\mathbf{V}_i$, $\mathbf{V}_*^i$ (eigenvectors of $\boldsymbol{\mu}^{-1}$ with respect to $\boldsymbol{\kappa}^{-1}$ and of $\boldsymbol{\kappa}$ with respect to $\boldsymbol{\mu}$)	$\mathbf{v}_i$, $\mathbf{v}_i$ (eigenvectors of $\mathbb{B}$ and $\mathbb{B}^{-1}$)
$k_i = \mathbf{V}_*^i \cdot \boldsymbol{\kappa}\mathbf{V}_*^i$, $m_i = \mathbf{V}_*^i \cdot \boldsymbol{\mu}\mathbf{V}_*^i$	$b_i = \mathbf{v}_i \cdot \mathbb{B}\mathbf{v}_i$, $b_i\mathrm{E}_i^{-1} = \mathbf{v}_i \cdot \mathbb{B}\mathbb{E}^{-1}\mathbf{v}_i$
$\lambda_i = k_i/m_i$	$\mathrm{E}_i = b_i/(b_i\mathrm{E}_i^{-1})$
$\mathbf{n}_O^{\pm}$ (generalized optic axes)	$\mathbf{n}_A^{\pm}$ (acoustic axes)
$\mathbf{E}$ (electric field amplitude)	$\mathbf{e} \equiv \mathbb{B}^{-1}\mathbf{a}$
$\mathbf{B}$ (magnetic induction amplitude)	$\mathbf{c} \equiv v^{-1}\mathbf{n} \times \mathbb{B}^{-1}\mathbf{a}$
$\mathbf{H}$ (magnetic field amplitude)	$\mathbf{h} \equiv v^{-1}\mathbb{E}\mathbb{B}^{-1}(\mathbf{n} \times \mathbb{B}^{-1}\mathbf{a})$

Table 1 Formal substitutions to be done in order to read off results for elastic waves in deformed Mooney-Rivlin materials from corresponding results for electromagnetic waves in crystals.

Of course, it should be kept in mind that the theory of finite-amplitude waves in deformed Mooney-Rivlin materials is basically nonlinear, so that, for instance, a superposition of two waves propagating in different directions is, in general, no longer a solution. However, if we consider small-amplitude waves in finitely deformed Mooney-Rivlin materials (linearized theory), the analogy extends also to the superposition of these waves.

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