ON HIERARCHIC MODELS FOR NONHOMOGENEOUS ANISOTROPIC ELASTIC PRISMATIC SHELLS [1]

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(Received: 09.06.2003; revised: 01.09.2003; accepted: 01.10.2003)

Abstract

In the present paper static boundary value problems for nonhomogeneous anisotropic elastic prismatic shells are considered and corresponding two-dimensional hierarchic models are constructed. The existence and uniqueness of solutions to the obtained boundary value problems are proved and the relation of the models to the original three-dimensional problems is studied.

Key words and phrases: Static boundary value problems for prismatic shells, linear theory of elasticity, modelling error estimation.
AMS subject classification: 74K25, 65N30, 42C10.

The lower-dimensional hierarchic models are widely used while investigating various objects in the theory of elasticity and mathematical physics, since they have simpler mathematical structure than three-dimensional models and an increase of the model order within the hierarchy always leads to a reduction of the modelling error. In the paper [1] I. Vekua suggested a new method of constructing hierarchic models for elastic plates with variable thickness (i.e., for prismatic shells). The mentioned method consists in expanding of fields of displacement vectors, strain and stress tensors of the three-dimensional model of the prismatic shell into Fourier-Legendre series with respect to the thickness variable and leaving then only the first $N+1$, $N \in \mathbb{N} \cup \{0\}$, terms of the expansion. Applying this method in the paper [2], mathematical model for thin shallow shells was obtained.

It must be pointed out, that in [1, 2] initial boundary-value problems were considered in C^k spaces and convergence of the sequence of approximate solutions to the exact solution of three-dimensional problem was not investigated. For static boundary value problem the existence and uniqueness of the solution to reduced two-dimensional problem obtained from the

[1] Dedicated to the memory of Professor Victor Kupradze on the occasion of the 100^{th} anniversary of his birth

linear system in the theory of elasticity by I. Vekua's method in Sobolev spaces first were investigated in [3]. The rate of approximation of the exact solution to the original problem by vector-functions restored from the solution of the reduced problem in classical spaces of regular functions was estimated in [4]. The hierarchic models constructed by I. Velua's method and its generalizations and various aspects of hierarchic modelling are studied in [5-15].

The present work is devoted to the construction and investigation of two-dimensional models of static boundary value problems for nonhomogeneous anisotropic linearly elastic prismatic shells. We consider boundary value problems with Dirichlet and Neumann type conditions on the upper and lower faces of the prismatic shell and generalizing I. Vekua's reduction method, we construct their two-dimensional models. For the obtained boundary value problems we prove the existence and uniqueness of their solutions, show convergence of the sequence of approximate solutions constructed by means of the solutions to reduced problems and obtain *a priori* estimates of the modelling error.

Let us consider an elastic body with initial configuration $\overline{\Omega}$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain ([16]) with boundary $\Gamma = \partial\Omega$. We assume that the body consists of arbitrary (i.e. of nonhomogeneous and anisotropic) linearly elastic material for which the stress tensor (σ_{ij}) linearly depends on the strain tensor $(e_{pq}(\boldsymbol{u}))$, $\sigma_{ij} = a_{ijpq}e_{pq}(\boldsymbol{u})$, where the indices i, j, p, q take their values in the set $\{1, 2, 3\}$ and summation convention with respect to the repeated indices is used, $e_{pq}(\boldsymbol{u}) = 1/2(\partial_p u_q + \partial_q u_p)$, $\boldsymbol{u} = (u_i)$ is a displacement vector field, ∂_p denotes the partial derivative $\partial/\partial x_p$, a_{ijpq} are elastic coefficients depending on $x = (x_1, x_2, x_3) \in \Omega$.

Assume that the elastic body Ω is clamped on $\Gamma_0 \subset \Gamma$, i.e. $\boldsymbol{u} = \mathbf{0}$ on Γ_0, and on the rest part of the boundary the following Neumann type conditions are given: $\sum\limits_{j=1}^{3}(\sigma_{ij}\nu_j + b_{ij}u_j) = g_i$ on $\Gamma_1 = \Gamma\backslash\overline{\Gamma}_0$, where b_{ij}, g_i are given functions $(i, j = \overline{1,3})$, $\boldsymbol{\nu} = (\nu_j)$ is the outward unit normal vector to Γ_1, $\Gamma = \Gamma_0 \cup \Gamma_{01} \cup \Gamma_1$ is a Lipschitz dissection of Γ and $\Gamma_0 \neq \emptyset$ ([16]). The variational formulation of the corresponding static three-dimensional problem of linearized elasticity is the following: find a vector-function $\boldsymbol{u} = (u_i) \in V(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\}$, which for all $\boldsymbol{v} = (v_i) \in V(\Omega)$ satisfies equation

$$\sum_{i,j,p,q=1}^{3} \int\limits_{\Omega} a_{ijpq}(x)e_{pq}(\boldsymbol{u})e_{ij}(\boldsymbol{v})dx + \sum_{i,j=1}^{3} \int\limits_{\Gamma_1} b_{ij}(x)u_j v_i d\Gamma_1 =$$

$$= \sum_{i=1}^{3} \int_{\Omega} f_i v_i dx + \sum_{i=1}^{3} \int_{\Gamma_1} g_i v_i d\Gamma_1, \tag{1}$$

where $\boldsymbol{f} = (f_i) \in \mathbf{L}^{6/5}(\Omega)$ is a density of the applied body forces, acting on the body Ω, $\boldsymbol{g} = (g_i) \in \mathbf{L}^{4/3}(\Gamma_1)$. In the paper we denote by $H^s(D)$ the usual Sobolev space of order $s \in \mathbb{R}$, $s \geq 0$, on Lipschitz domain D or on part of Lipschitz dissection of the boundary of Lipschitz domain, based on $L^2(D)$, $\mathbf{H}^s(D) = [H^s(D)]^3$, $\mathbf{L}^s(D) = [L^s(D)]^3$. Let us denote the bilinear form in the left part of the equation (1) by $A(\boldsymbol{u}, \boldsymbol{v})$.

The three-dimensional problem (1) has a unique solution if $a_{ijpq} \in L^\infty(\Omega)$, $b_{ij} \in L^\infty(\Gamma_1)$, $i,j,p,q = \overline{1,3}$, and the elasticity tensor (a_{ijpq}), matrix (b_{ij}) satisfy the following coercivity, positive definiteness, and symmetry conditions for positive constants $\alpha \in \mathbb{R}$, for almost all $x \in \Omega$, $y \in \Gamma_1$, and for all ε_{ij}, $\varepsilon_i \in \mathbb{R}$,

$$\begin{gathered}\sum_{i,j,p,q=1}^{3} a_{ijpq}(x)\varepsilon_{ij}\varepsilon_{pq} \geq \alpha \sum_{i,j=1}^{3} \varepsilon_{ij}\varepsilon_{ij}, \quad \sum_{i,j=1}^{3} b_{ij}(y)\varepsilon_i\varepsilon_j \geq 0, \\ a_{ijpq}(x) = a_{jipq}(x) = a_{pqij}(x), \ b_{ij}(y) = b_{ji}(y), \ \ i,j,p,q = \overline{1,3}.\end{gathered} \tag{2}$$

Moreover, the solution $\boldsymbol{u}$ of the problem (1) is also a unique solution to the following minimization problem: find $\boldsymbol{u} \in V(\Omega)$ such that $I(\boldsymbol{u}) = \inf\limits_{\boldsymbol{v} \in V(\Omega)} I(\boldsymbol{v})$,

$$I(\boldsymbol{v}) = \frac{1}{2}A(\boldsymbol{v}, \boldsymbol{v}) - \sum_{i=1}^{3} \int_{\Omega} f_i v_i dx - \sum_{i=1}^{3} \int_{\Gamma_1} g_i v_i d\Gamma_1, \quad \forall \boldsymbol{v} \in V(\Omega).$$

Let us consider now the particular cases of the three-dimensional problem (1), when Ω is a prismatic shell

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3; \ h^-(x_1, x_2) < x_3 < h^+(x_1, x_2), \ \ (x_1, x_2) \in \omega\},$$

where $\omega \subset \mathbb{R}^2$ is a two-dimensional Lipschitz domain with boundary $\gamma = \partial\omega$, $h^\pm \in C^1(\bar{\omega})$, $h^+(x_1, x_2) > h^-(x_1, x_2)$, for $(x_1, x_2) \in \bar{\omega}$. The upper and lower faces of the shell Ω, which are given by the equations $x_3 = h^+(x_1, x_2)$ and $x_3 = h^-(x_1, x_2)$, $(x_1, x_2) \in \omega$, denote by Γ^+ and Γ^-, respectively, and the lateral surface denote by $\widetilde{\Gamma} = \Gamma \backslash (\overline{\Gamma^+ \cup \Gamma^-})$.

We study the problem (1) in three cases: when the shell is clamped on both surfaces Γ^+ and Γ^-, the shell is clamped only on one of the upper or lower faces, and the shell is not clamped on Γ^+ and Γ^-. The clamped part of the lateral surface we denote by $\widetilde{\Gamma}_0 = \{(x_1, x_2, x_3) \in \widetilde{\Gamma}; \ (x_1, x_2) \in \gamma_0 \subset \gamma\}$,

γ_0 is a Lipschitz curve with positive length if $\widetilde{\Gamma}_0$ is a non-empty set. In the second case, without loss of generality, we consider the shell clamped on Γ^+ and $\widetilde{\Gamma}_0$. In order to construct the two-dimensional models of the shell let us introduce the subspaces $V^k_{\mathbf{N}}(\Omega)$ $(k = 1, 2, 3)$ of $V(\Omega)$ of polynomials with respect to the thickness variable x_3, i.e.

$$V^k_{\mathbf{N}}(\Omega) = \{\boldsymbol{v}^k_{\mathbf{N}} = (v^k_{\mathbf{N}i});\ v^k_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h}(r_i + \frac{1}{2}) \overset{r_i}{v}_i P_{r_i}(z) - \frac{3-k}{4} \Bigg[k \sum_{r_i=0}^{N_i} \frac{1}{h}$$

$$(1 + (-1)^{k(r_i+N_i+1)})(r_i + \frac{1}{2}) \overset{r_i}{v}_i P_{N_i+1}(z) - (2-k) \sum_{r_i=0}^{N_i} \frac{1}{h}(1 + (-1)^{r_i+N_i})$$

$$(r_i + \frac{1}{2}) \overset{r_i}{v}_i P_{N_i+2}(z) \Bigg], \overset{r_i}{v}_i \in H^1(\omega), \overset{r_i}{v}_i = 0 \quad \text{on } \gamma_0,\ 0 \le r_i \le N_i, i = \overline{1,3}\},$$

where $z = (x_3 - \overline{h})/h$, $h = (h^+ - h^-)/2$, $\overline{h} = (h^+ + h^-)/2$ and P_r is Legendre polynomial of degree $r \in \mathbb{N} \cup \{0\}$. In the case of $k = 1$, the vector-functions of the space $V^1_{\mathbf{N}}(\Omega)$ vanish on the surface $\Gamma^+ \cup \Gamma^- \cup \widetilde{\Gamma}_0$ and hence, $V^1_{\mathbf{N}}(\Omega)$ is a subspace of $V(\Omega)$, when the shell is clamped on both upper and lower faces. Similarly, the case of $k = 2$ corresponds to the shell clamped on the upper and on the part of the lateral surface, and the case of $k = 3$ corresponds to shell, which is not clamped on the faces Γ^+ and Γ^-.

On the subspace $V^k_{\mathbf{N}}(\Omega)$ $(k = \overline{1,3})$ from the problem (1) we obtain the following reduced problem: the unknown is the vector-function $\boldsymbol{w}^k_{\mathbf{N}} = (w^k_{\mathbf{N}i}) \in V^k_{\mathbf{N}}(\Omega)$, which for all $\boldsymbol{v}^k_{\mathbf{N}} \in V^k_{\mathbf{N}}(\Omega)$ satisfies equation

$$\sum_{i,j,p,q=1}^{3} \int_{\Omega} a_{ijpq}(x) e_{pq}(\boldsymbol{w}^k_{\mathbf{N}}) e_{ij}(\boldsymbol{v}^k_{\mathbf{N}}) dx + \sum_{i,j=1}^{3} \int_{\Gamma_1} b_{ij}(x) w^k_{\mathbf{N}j} v^k_{\mathbf{N}i} d\Gamma_1 =$$

$$= \sum_{i=1}^{3} \int_{\Omega} f_i v^k_{\mathbf{N}i} dx + \sum_{i=1}^{3} \int_{\Gamma_1} g_i v^k_{\mathbf{N}i} d\Gamma_1. \tag{3}$$

The obtained problem (3) is equivalent to the following one: find $\vec{w}^k_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega) = \{\vec{v}_{\mathbf{N}} \in [H^1(\omega)]^{N_1+N_2+N_3+3};\ \vec{v}_{\mathbf{N}} = (\overset{r_i}{v}_i), \overset{r_i}{v}_i = 0 \text{ on } \gamma_0,\ 0 \le r_i \le N_i, i = \overline{1,3}\}$, such that

$$A^k_{\mathbf{N}}(\vec{w}^k_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = L^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}}), \qquad \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)\ (k = 1, 2, 3), \tag{4}$$

where $A^k_{\mathbf{N}}(\vec{w}^k_{\mathbf{N}}, \vec{v}_{\mathbf{N}})$ is the bilinear form $A(\boldsymbol{w}^k_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}})$ rewritten in terms of $\vec{w}^k_{\mathbf{N}}$ and $\vec{v}_{\mathbf{N}}$, the linear form $L^k_{\mathbf{N}}$ is given by

$$L^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}}) = \sum_{i=1}^{3} \sum_{r_i=0}^{N_i} (r_i + \frac{1}{2}) \Bigg[\int_{\omega} \frac{1}{h} \overset{r_i}{v}_i \Bigg(\overset{r_i}{f}_i - \sum_{\alpha=1}^{2} \frac{(3-k)(2(\alpha-1) - (-1)^{\alpha} k)}{4}$$

$$\left(1+(-1)^{k^{2-\alpha}(r_i+N_i+\alpha)}\right)\overset{N_i+\alpha}{f_i}+\frac{k-1}{2}\left((k-2)\left.g_i\right|_{\Gamma^+}\lambda_+ +(4-k)\left.g_i\right|_{\Gamma^-}\lambda_-\right.$$

$$\left.\left((-1)^{r_i}+(3-k)(-1)^{N_i}\right)\right)\Big)\,d\omega+\int\limits_{\gamma_1}\frac{1}{h}\overset{r_i}{v_i}\left(\overset{r_i}{g}_i-\sum_{\alpha=1}^{2}\left(1+(-1)^{k^{2-\alpha}(r_i+N_i+\alpha)}\right)\right.$$

$$\left.\frac{(3-k)(2(\alpha-1)-(-1)^{\alpha}k)}{4}\overset{N_i+\alpha}{g_i}\right)d\gamma_1\Bigg], \qquad \gamma_1=\gamma\backslash\overline{\gamma}_0,$$

where $\left.g_i\right|_{\Gamma^\pm}(x_1,x_2)=g_i(x_1,x_2,h^\pm(x_1,x_2))$, $\lambda_\pm=\sqrt{1+(\partial_1 h^\pm)^2+(\partial_2 h^\pm)^2}$,
$\overset{r_i}{f}_i=\int\limits_{h^-}^{h^+}f_i(x)P_{r_i}(z)dx_3$, $\overset{r_i}{g}_i=\int\limits_{h^-}^{h^+}g_i(x)P_{r_i}(z)dx_3$, $r_i=\overline{0,N_i}$, $i=\overline{1,3}$.

So, for the three-dimensional static problem (1) with various boundary conditions on the upper and lower faces of elastic prismatic shell, which correspond to $k=1,2,3$, we have constructed the hierarchies of two-dimensional models. In the following theorem we prove the well-posedness of the corresponding boundary value problem (4).

Theorem 1. *If the elasticity tensor* (a_{ijpq}) *and matrix* (b_{ij}) *satisfy conditions (2),* $a_{ijpq}\in L^\infty(\Omega)$, $b_{ij}\in L^\infty(\Gamma_1)$ *and for all* $r_i=\overline{0,N_i}$, i, j, p, q, $k=1,2,3$, $\widetilde{k}=3-k$,

$$\overset{r_i}{f}_i-\frac{\widetilde{k}}{4}\left[k\left(1+(-1)^{k(r_i+N_i+1)}\right)\overset{N_i+1}{f_i}-(2-k)\left(1+(-1)^{r_i+N_i}\right)\overset{N_i+2}{f_i}\right]+$$

$$+\frac{k-1}{2}\left[(4-k)\left.g_i\right|_{\Gamma^-}\lambda_-\left((-1)^{r_i}+\widetilde{k}(-1)^{N_i}\right)+(k-2)\left.g_i\right|_{\Gamma^+}\lambda_+\right]\in L^{6/5}(\omega),$$

$$\overset{r_i}{g}_i-\sum_{\alpha=1}^{2}\frac{\widetilde{k}(2(\alpha-1)-(-1)^{\alpha}k)}{4}\left(1+(-1)^{k^{2-\alpha}(r_i+N_i+\alpha)}\right)\overset{N_i+\alpha}{g_i}\in L^{4/3}(\gamma_1),$$

then the two-dimensional problem (4) has a unique solution $\vec{w}^k_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)$, *which also is a solution to the minimization problem: find* $\vec{w}^k_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)$ *such that*

$$I^k_{\mathbf{N}}(\vec{w}^k_{\mathbf{N}})=\inf_{\vec{v}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)}I^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}}),\quad I^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}})=\frac{1}{2}A^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}},\vec{v}_{\mathbf{N}})-L^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}}).$$

Proof. Note, that from the conditions of the theorem, embedding and trace theorems for Sobolev spaces we infer that $L^k_{\mathbf{N}}$ is a linear continuous form on $\vec{V}_{\mathbf{N}}(\omega)$. Furthermore, applying the formulas for derivatives of Legendre polynomials ([1])

$$\begin{aligned} P_r'(t) &= \sum_{s=0}^{r-1}\left(s+\frac{1}{2}\right)(1-(-1)^{r+s})P_s(t), \\ tP_r'(t) &= rP_r(t)+\sum_{s=0}^{r-1}\left(s+\frac{1}{2}\right)(1+(-1)^{r+s})P_s(t), \end{aligned} \qquad r\in\mathbb{N}, \tag{5}$$

for any $\boldsymbol{v}_{\mathbf{N}}^k \in V_{\mathbf{N}}^k(\Omega)$, we obtain

$$\left\|\boldsymbol{v}_{\mathbf{N}}^k\right\|^2_{\mathbf{H}^1(\Omega)} = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i+2}(r_i+\frac{1}{2})\left[\left\|h^{-1/2}\overset{r_i *}{v_i}\right\|^2_{L^2(\omega)} + \left\|\sum_{s_i=r_i}^{N_i+2}(s_i+\frac{1}{2})(1-\right.\right.$$

$$-(-1)^{r_i+s_i})h^{-3/2}\overset{s_i *}{v_i}\Big\|^2_{L^2(\omega)} + \sum_{\alpha=1}^{2}\Big\|h^{-1/2}\partial_\alpha\overset{r_i *}{v_i} - (r_i+1)h^{-3/2}\partial_\alpha h\overset{r_i *}{v_i} -$$

$$\left.\left. - \sum_{s_i=r_i+1}^{N_i+2}(s_i+\frac{1}{2})\ (\partial_\alpha h^+ - (-1)^{r_i+s_i}\partial_\alpha h^-)h^{-3/2}\overset{s_i *}{v_i}\right\|^2_{L^2(\omega)}\right] < \infty,$$

where $\overset{r_i *}{v_i} = \overset{r_i}{v_i}$, for $0 \le r_i \le N_i$, $\overset{N_i+1 *}{v_i} = -\dfrac{k(3-k)}{2(2N_i+3)}\displaystyle\sum_{s_i=0}^{N_i}(1+(-1)^{k(s_i+N_i+1)})$ $(s_i+\dfrac{1}{2})\overset{s_i}{v_i}$, $\overset{N_i+2 *}{v_i} = -\dfrac{(3-k)(2-k)}{2(2N_i+5)}\displaystyle\sum_{s_i=0}^{N_i}(1+(-1)^{s_i+N_i})(s_i+\dfrac{1}{2})\overset{s_i}{v_i}$, $i,k=\overline{1,3}$ and we assume that the sum with lower limit greater, than the upper one is equal to zero.

Consequently, identity mappings from the space $\vec{V}_{\mathbf{N}}^0(\omega)$ onto $\vec{V}_{\mathbf{N}}^k(\omega)$, $k=1,2,3$, which coincide with $\vec{V}_{\mathbf{N}}(\omega)$ and are equipped with the norms $\|\vec{v}_{\mathbf{N}}\|_0 = \|\vec{v}_{\mathbf{N}}\|_{[H^1(\omega)]^{N_1+N_2+N_3+3}}$, $\|\vec{v}_{\mathbf{N}}\|_k = \left\|\boldsymbol{v}_{\mathbf{N}}^k\right\|_{\mathbf{H}^1(\Omega)}$, $\vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, respectively, are continuous and according to the closed graph theorem we have that these spaces are isomorphic and, hence, the norms $\|.\|_0$, $\|.\|_k$ $(k=\overline{1,3})$ are equivalent.

Since the bilinear form A is continuous and coercive on $V(\Omega)$, then it is coercive on the subspace $V_{\mathbf{N}}^k(\Omega) \subset V(\Omega)$, and, therefore, the equivalence of the norms $\|.\|_0$ and $\|.\|_k$ implies that the bilinear form $A_{\mathbf{N}}^k$ $(k=1,2,3)$ is coercive on the space $\vec{V}_{\mathbf{N}}(\omega)$,

$$\begin{aligned} A_{\mathbf{N}}^k(\vec{v}_{\mathbf{N}},\vec{v}_{\mathbf{N}}) = A(\boldsymbol{v}_{\mathbf{N}}^k,\boldsymbol{v}_{\mathbf{N}}^k) \ge \alpha\left\|\boldsymbol{v}_{\mathbf{N}}^k\right\|^2_{\mathbf{H}^1(\Omega)} \ge \\ \ge \widetilde{\alpha}\,\|\vec{v}_{\mathbf{N}}\|^2_{\vec{V}_{\mathbf{N}}(\omega)} = \widetilde{\alpha}\,\|\vec{v}_{\mathbf{N}}\|^2_{[H^1(\omega)]^{N_1+N_2+N_3+3}}, \quad \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega). \end{aligned}$$

Thus, for each $k=1,2,3$, the existence and uniqueness of the solution to the problem (4) and equivalence to the minimization problem is a consequence of Lax-Milgram lemma. □

It must be pointed out, that one of the most important questions that arise while investigating the hierarchic models is convergence of the sequence of approximate solutions to the exact solution of original problem, as the model order tends to infinity. The following theorem gives the result on the relation of the obtained hierarchies of two-dimensional models and corresponding three-dimensional problems.

Theorem 2. *Assume that components of the elasticity tensor $a_{ijpq} \in L^\infty(\Omega)$, the given functions $b_{ij} \in L^\infty(\Gamma_1)$ and conditions (2) are fulfilled $(i,\ j,\ p,\ q = \overline{1,3})$. If $f_i \in L^{6/5}(\Omega)$, $g_i \in L^{4/3}(\Gamma_1)$, then the vector-function $\boldsymbol{w}^k_{\mathbf{N}} = (w^k_{\mathbf{N}i})$, $i,k = 1,2,3$, restored from the solution $\vec{w}^k_{\mathbf{N}} = (\overset{0}{w^k_1}, ..., \overset{N_1}{w^k_1}, ..., \overset{0}{w^k_3}, ..., \overset{N_3}{w^k_3})^T$ of the two-dimensional problem (4) tends to the solution $\boldsymbol{u}$ of the three-dimensional problem (1) in the space $\mathbf{H}^1(\Omega)$, as $N_1, N_2, N_3 \to \infty$. Moreover, if $\partial_3^n \boldsymbol{u} \in \mathbf{H}^1(\Omega)$, $0 \le n \le s-1$, $s \in \mathbb{N}$, $s \ge 2$, then*

$$\|\boldsymbol{u} - \boldsymbol{w}^k_{\mathbf{N}}\|^2_{\mathbf{H}^1(\Omega)} \le \frac{1}{N^{2(s-1)}} \delta_k(\Omega, \Gamma_0, h^\pm, \mathbf{N}), \quad N = \min_{1\le i\le 3}\{N_i\},$$

where $\delta_k(\Omega,\Gamma_0,h^\pm,\mathbf{N}) \to 0$, as $N_i \to \infty$, $i,k = 1,2,3$. If additionally $\sum_{n=1}^{s-1} \|\partial_3^n \boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \le c$, c is independent from $h_{\max} = \max_{(x_1,x_2)\in\overline{\omega}} h(x_1,x_2)$, then

$$\left\|\boldsymbol{u} - \boldsymbol{w}^k_{\mathbf{N}}\right\|^2_{E(\Omega)} \le \frac{h_{\max}^{2(s-1)}}{N^{2(s-1)}} \overline{\delta}_k(\mathbf{N}), \qquad \overline{\delta}_k(\mathbf{N}) \to 0, \text{ as } N \to \infty,$$

where $k = 1,2,3$, $\|\boldsymbol{v}\|^2_{E(\Omega)} = A(\boldsymbol{v},\boldsymbol{v})$, for all $\boldsymbol{v} \in V(\Omega)$.

Proof. By our assumptions on the functions f_i, g_i $(i = \overline{1,3})$ we get that $\overset{r_i}{f_i} \in L^{6/5}(\omega)$, $\overset{r_i}{g_i} \in L^{4/3}(\gamma_1)$, $g_i|_{\Gamma^\pm} \in L^{4/3}(\omega) \subset L^{6/5}(\omega)$, and consequently, all the conditions of Theorem 1 are fulfilled. Hence, a unique solution $\vec{w}^k_{\mathbf{N}}$ of the two-dimensional problem (4) minimizes the energy functional $I^k_{\mathbf{N}}$ on the space $\vec{V}_{\mathbf{N}}(\omega)$, i.e.,

$$\frac{1}{2} A^k_{\mathbf{N}}(\vec{w}^k_{\mathbf{N}}, \vec{w}^k_{\mathbf{N}}) - L^k_{\mathbf{N}}(\vec{w}^k_{\mathbf{N}}) \le \frac{1}{2} A^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) - L^k_{\mathbf{N}}(\vec{v}_{\mathbf{N}}), \ \ \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega). \quad (6)$$

Taking into account that $A^k_{\mathbf{N}}(\vec{w}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = A(\boldsymbol{w}^k_{\mathbf{N}}, \boldsymbol{v}^k_{\mathbf{N}})$, for all $\vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, where $\boldsymbol{v}^k_{\mathbf{N}} = (v^k_{\mathbf{N}i}) \in V^k_{\mathbf{N}}(\Omega)$ corresponds to $\vec{v}_{\mathbf{N}}$, $v^k_{\mathbf{N}i}$, are defined in the definition of the space $V^k_{\mathbf{N}}(\Omega)$, $k = 1,2,3$, then from (6) and (1), (3) we obtain

$$A(\boldsymbol{u} - \boldsymbol{w}^k_{\mathbf{N}}, \boldsymbol{u} - \boldsymbol{w}^k_{\mathbf{N}}) \le A(\boldsymbol{u} - \boldsymbol{v}^k_{\mathbf{N}}, \boldsymbol{u} - \boldsymbol{v}^k_{\mathbf{N}}), \quad \forall \boldsymbol{v}^k_{\mathbf{N}} \in V^k_{\mathbf{N}}(\Omega). \qquad (7)$$

According to the trace theorems for Sobolev spaces ([16]), for any $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{v} = \mathbf{0}$ on Γ_0, there exists continuation $\tilde{\boldsymbol{v}} \in \mathbf{H}_0^1(\Omega_1)$ of $\boldsymbol{v}$, where Ω_1 is a Lipschitz domain such that $\Omega_1 \supset \Omega$, $\partial\Omega_1 \supset \Gamma_0$. From the density of the set of infinitely differentiable functions $\mathcal{D}(\Omega_1)$ with compact support in Ω_1 in $H_0^1(\Omega_1)$ we obtain, that the set of infinitely differentiable vector-functions $[C_{\Gamma_0}^\infty(\overline{\Omega})]^3$ on Ω, which are equal to zero on Γ_0, is dense in the space $V_{\mathbf{N}}^k(\Omega)$. From the estimates obtained below it follows, that the union of the spaces $[C^\infty(\overline{\Omega})]^3 \cap V_{\mathbf{N}}^k(\Omega)$, for all $N_i \geq 0$, $i = \overline{1,3}$, is dense in $V(\Omega)$. Hence, due to coerciveness of the bilinear form A and density of $[C_{\Gamma_0}^\infty(\overline{\Omega})]^3$ in $V_{\mathbf{N}}^k(\Omega)$ from (7) we get that $\boldsymbol{w}_{\mathbf{N}}^k \to \boldsymbol{u}$ in the space $\mathbf{H}^1(\Omega)$, as $N_1, N_2, N_3 \to \infty$, for each hierarchy of two-dimensional problems corresponding to $k = 1, 2, 3$.

Let us now estimate the rate of approximation of $\boldsymbol{u}$ by $\boldsymbol{w}_{\mathbf{N}}^k$, when $\boldsymbol{u}$ satisfies additional regularity conditions stated in the theorem. In order to obtain the estimates of the theorem for each $k = 1, 2, 3$, by means of the solution $\boldsymbol{u}$ we construct the vector-function $\boldsymbol{u}_{\mathbf{N}}^k$, which is an element of the space $V_{\mathbf{N}}^k(\Omega)$ and we can estimate the norm of the difference $\boldsymbol{u} - \boldsymbol{u}_{\mathbf{N}}^k$. Let us consider $\boldsymbol{u}_{\mathbf{N}}^k = (u_{\mathbf{N}i}^k)$,

$$u_{\mathbf{N}i}^k = \sum_{r_i=0}^{N_i^k} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{u_i} P_{r_i}(z) + \sum_{r_i=N_i^k}^{N_i^k+1} \frac{1}{2} \overset{r_i}{\partial_3 u_i} P_{r_i-1}(z),$$

where $N_i^k = N_i + 3 - k$, $\overset{r_i}{\partial_3^\beta u_i} = \int\limits_{h^-}^{h^+} \partial_3^\beta u_i P_{r_i}(z) dx_3$, $\beta = 0, 1$, $i, k = 1, 2, 3$.

Due to the conditions of the theorem we have that $\boldsymbol{u}_{\mathbf{N}}^k \in \mathbf{H}^1(\Omega)$. Hence, from the construction of $\boldsymbol{u}_{\mathbf{N}}^k$ it follows, that $\boldsymbol{u}_{\mathbf{N}}^k \in V_{\mathbf{N}}^k(\Omega)$ if and only if $\boldsymbol{u}_{\mathbf{N}}^k$ vanishes on both upper and lower faces of the shell in the case of $k = 1$, or only on Γ^+ for $k = 2$. Note, that the cases of $k = 1, 2$, correspond to the problem (1), when the shell is clamped either on both Γ^+, Γ^- faces or only on Γ^+, i.e. $\boldsymbol{u} = \mathbf{0}$ on $\Gamma^\pm$, for $k = 1$, and $\boldsymbol{u} = \mathbf{0}$ on Γ^+, for $k = 2$. Consequently, we have the following equalities

$$\begin{aligned} &\overset{0}{\partial_3 u_i} = 0, \quad \overset{1}{\partial_3 u_i} + \frac{1}{h} \overset{0}{u_i} = 0, && \text{for } k = 1, \\ &\overset{0}{\partial_3 u_i} + \overset{1}{\partial_3 u_i} + \frac{1}{h} \overset{0}{u_i} = 0, && \text{for } k = 2. \end{aligned} \tag{8}$$

By the property of Legendre polynomials ([17])

$$P_r(t) = \frac{1}{2r+1}(P'_{r+1}(t) - P'_{r-1}(t)), \qquad r \geq 1,$$

for almost all $(x_1, x_2) \in \omega$,

$$\overset{r}{u}_i (x_1, x_2) = \frac{h}{2r+1}(\overset{r-1}{\partial_3 u_i} (x_1, x_2) - \overset{r+1}{\partial_3 u_i} (x_1, x_2)), \qquad r \geq 1. \tag{9}$$

Taking into account (8) and applying formula (9), we infer

$$\sum_{r_i=0}^{N_i^k} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{u_i} + \sum_{r_i=N_i^k}^{N_i^k+1} \frac{1}{2} \overset{r_i}{\partial_3 u_i} = 0, \qquad k = 1, 2,$$

$$\sum_{r_i=0}^{N_i+2} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{u_i} (-1)^{r_i} - \sum_{r_i=N_i+2}^{N_i+3} \frac{1}{2} \overset{r_i}{\partial_3 u_i} (-1)^{r_i} = 0, \quad k = 1,$$

and, therefore, $\boldsymbol{u}_{\mathbf{N}}^k \in V_{\mathbf{N}}^k(\Omega)$, $k = 1, 2, 3$.

Note, that the following equality holds almost everywhere in ω, $\alpha = 1, 2$,

$$\partial_\alpha(\overset{r}{u}_i) = \overset{r}{\partial_\alpha u_i} + \frac{\partial_\alpha h}{h}(r+1) \overset{r}{u}_i + \partial_\alpha \bar{h} \overset{r}{\partial_3 u_i} + \partial_\alpha h \overset{r+1}{\partial_3 u_i}, \quad r \in \mathbb{N} \cup \{0\}.$$

By virtue of the last formula, applying (5) and (9), we obtain the expressions for derivatives of $\boldsymbol{u}_{\mathbf{N}}^k = \{u_{\mathbf{N}i}^k\}$, for $i, k = 1, 2, 3$,

$$\partial_3 u_{\mathbf{N}i}^k = \sum_{r_i=0}^{N_i^k-1} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{\partial_3 u_i} P_{r_i}(z),$$

$$\begin{aligned}\partial_\alpha u_{\mathbf{N}i}^k = &\sum_{r_i=0}^{N_i^k} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{\partial_\alpha u_i} P_{r_i}(z) + \frac{\partial_\alpha \bar{h}}{h}\left(N_i^k + \frac{1}{2}\right) \overset{N_i^k}{\partial_3 u_i} P_{N_i^k}(z) + \\ &+ \sum_{r_i=N_i^k}^{N_i^k+1} \frac{\partial_\alpha h}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{\partial_3 u_i} P_{r_i-1}(z) + \sum_{r_i=N_i^k}^{N_i^k+1} \frac{1}{2}(\partial_\alpha \overset{r_i}{\partial_3 u_i} + \\ &+ \partial_\alpha \bar{h}\, \partial_3 \overset{r_i}{\partial_3 u_i} + \partial_\alpha h\, \partial_3 \overset{r_i+1}{\partial_3 u_i}) P_{r_i-1}(z).\end{aligned}$$

Therefore, by the orthogonality property of Legendre polynomials and

Parseval equality, for residue $\boldsymbol{\varepsilon}^k_{\mathbf{N}} = (\varepsilon^k_{\mathbf{N}i})$, $\varepsilon^k_{\mathbf{N}i} = u_i - u^k_{\mathbf{N}i}$ we have

$$\|\varepsilon^k_{\mathbf{N}i}\|^2_{L^2(\Omega)} = \sum_{r_i=N^k_i+1}^{\infty} \int_\omega \frac{1}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{u}_i)^2 d\omega + \sum_{r_i=N^k_i-1}^{N^k_i} \int_\omega \frac{h(\overset{r_i+1}{\partial_3 u_i})^2}{2r_i+1} d\omega,$$

$$\left\|\partial_3\varepsilon^k_{\mathbf{N}i}\right\|^2_{L^2(\Omega)} = \sum_{r_i=N^k_i}^{\infty} \int_\omega \frac{1}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{\partial_3 u_i})^2 d\omega,$$

$$\left\|\partial_\alpha\varepsilon^k_{\mathbf{N}i}\right\|^2_{L^2(\Omega)} \le \sum_{r_i=N^k_i+1}^{\infty} \int_\omega \frac{1}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{\partial_\alpha u_i})^2 d\omega +$$

$$+\frac{9}{2}\sum_{r_i=N^k_i}^{N^k_i+1} \int_\omega \frac{h}{2r_i-1}\left((\partial_\alpha \overset{r_i}{\partial_3 u_i})^2 + (\partial_\alpha \bar{h}\ \partial_3 \overset{r_i}{\partial_3 u_i})^2 + (\partial_\alpha h\ \partial_3 \overset{r_i+1}{\partial_3 u_i})^2\right) d\omega +$$

$$+9\sum_{r_i=N^k_i}^{N^k_i+1} \int_\omega \frac{(N^k_i + 1 - r_i)(\partial_\alpha \bar{h})^2 + (\partial_\alpha h)^2}{h}\left(r_i + \frac{1}{2}\right)(\overset{r_i}{\partial_3 u_i})^2 d\omega,$$

where $\alpha = 1, 2, i, k = \overline{1,3}$. Applying (9), we obtain that

$$\|\ \overset{r}{\partial^\beta_p u_i}\ \|^2_{L^2(\omega)} \le \frac{c}{r^{2(s-\beta)}} \sum_{n=r-s+\beta}^{r+s-\beta} \|h^{s-\beta}\ (\overset{n}{\partial_3^{s-\beta}\partial^\beta_p u_i})\ \|^2_{L^2(\omega)}, \qquad (10)$$

where $r \ge s-\beta, \beta = 0, 1, i, p = \overline{1,3}$, $c = const > 0$ is independent from h^+, h^- and r. Therefore, from the last estimate, for all $i, p, k = 1, 2, 3$, we infer

$$\begin{array}{ll} \|\varepsilon^k_{\mathbf{N}i}\|^2_{L^2(\Omega)} \le \dfrac{1}{N_i^{2s}}\delta^k_i(h^+, h^-, N_i), & \\ \left\|\partial_p\varepsilon^k_{\mathbf{N}i}\right\|^2_{L^2(\Omega)} \le \dfrac{1}{N_i^{2(s-1)}}\delta^k_i(h^+, h^-, N_i), & \end{array} \quad \delta^k_i(h^+, h^-, N_i) \to 0, \text{ as } N_i \to \infty.$$

Hence, the inequality (7) and coerciveness of the bilinear form A imply

$$\|\boldsymbol{u} - \boldsymbol{w}^k_{\mathbf{N}}\|^2_{\mathbf{H}^1(\Omega)} \le \frac{1}{N^{2(s-1)}}\delta_k(\Omega, \Gamma_0, h^\pm, \mathbf{N}), \qquad N = \min_{1\le i\le 3} N_i,$$

where $\delta_k(\Omega, \Gamma_0, h^\pm, \mathbf{N}) \to 0$, as $N \to \infty$.

In addition, by virtue of the conditions of the theorem from (10) we obtain

$$\begin{array}{ll} \|\varepsilon^k_{\mathbf{N}i}\|^2_{L^2(\Omega)} \le \dfrac{h^{2s}_{\max}}{N_i^{2s}}\bar{\delta}^k_i(N_i), & \\ \left\|\partial_p\varepsilon^k_{\mathbf{N}i}\right\|^2_{L^2(\Omega)} \le \dfrac{h^{2(s-1)}_{\max}}{N_i^{2(s-1)}}\bar{\delta}^k_i(N_i), & \end{array} \quad \bar{\delta}^k_i(N_i) \to 0, \text{ as } N_i \to \infty, i, p, k = \overline{1,3},$$

which yields the second estimate of the theorem. □

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