ON THE INVESTIGATION OF ONE NONCLASSICAL PROBLEM FOR NAVIER-STOKES EQUATIONS

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Abstract

In the present paper initial boundary value problem with discrete-integral nonclassical initial condition for Navier-Stokes equations is investigated and is proved, that in suitable functional spaces the formulated problem is solvable.

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Investigations of the nonclassical problems are considerably stimulated by an increasing number of mathematical models for various ecological, physical and biological processes, which are reduced to problems of such type ([1-3]). A certain type of nonclassical problems first was considered by A.V. Bitsadze and A.A. Samarskii in [4]. Later, various generalizations of nonclassical boundary value problem formulated in [4] for Laplace equation were investigated in [5-8] for elliptic, parabolic and hyperbolic equations. Another type of nonclassical problems was considered in [9] for parabolic equation, where instead of classical initial condition a certain relation between the values of unknown function at initial and later times is given. Nonclassical problems for ordinary and partial differential equations were studied in [10-12].

In the present paper we consider initial boundary value problem for multidimensional Navier-Stokes equations with discrete-integral nonclassical initial conditions. For the above mentioned problems we introduce the corresponding functional spaces and prove the solvability of the initial boundary value problem under certain assumptions on the nonclassical initial operator.

Let us consider nonclassical in time problem for Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} = f - \mathrm{grad} p, \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\mathrm{div} u = 0, \qquad \text{in } \Omega \times (0,T), \tag{2}$$

with homogeneous boundary and nonclassical initial conditions

$$u(x,t) = 0, \qquad (x,t) \in \Gamma \times (0,T), \tag{3}$$

$$u(x,0) = \sum_{j=1}^{m} \beta_j u(x,T_j) + \sum_{j=1}^{m} \int_{T_j^1}^{T_j^2} \gamma_j(u,\tau) u(x,\tau) d\tau + u_0(x), \quad x \in \Omega, \tag{4}$$

where $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, $x = (x_1, ..., x_n)$, $\nu > 0$, $u = \{u_i\}_{i=1}^n$ is an unknown n-component vector-function and p is an unknown scalar function, $0 < T_j \leq T$, $0 \leq T_j^1 < T_j^2 \leq T$, $\gamma_j(u,\tau) = \sin^{r_j}(\int_{T_j^1}^{T_j^2} \int_\Omega |u(x,\tau|^2 dx d\tau)\tilde{\gamma}_j(\tau)$, $r_j \in \mathbf{N} \cup \{0\}$, $\tilde{\gamma}_j$ are given real functions $(j = \overline{1,m})$, $f(x,t) = \{f_i(x,t)\}_{i=1}^n$, $u_0(x) = \{u_{0_i}(x)\}_{i=1}^n$ are given vector-functions, $\Delta u = \sum_{i=1}^n \partial_i^2 u$, $\mathrm{div} u = \sum_{i=1}^n \partial_i u_i$, $(\mathrm{grad} p)_i = \partial_i p$, ∂_i denotes the partial derivative with respect to x_i $(i = \overline{1,n})$.

Let us now introduce the basic functional spaces, in which we investigate the problem (1)-(4). Let $\mathcal{D} = \{v \mid v \in (D(\Omega))^n, \ \mathrm{div} v = 0\}$, $D(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. Denote by H the closure of $\mathcal{D}$ in the space $(L^2(\Omega))^n$, and by V_s the closure of $\mathcal{D}$ in $(W^{s,2}(\Omega))^n$, where $W^{s,r}(\Omega)$ is the Sobolev space of order s with respect to $L^r(\Omega)$ $(s, r \in \mathbf{R}, \ s \geq 0, \ 1 \leq r \leq \infty)$. If H is identified with its dual space by scalar product in H and $s \geq 1$, then $V_s \subset V_1 \subset H \subset V_1' \subset V_s'$ with continuous and dense embeddings, V_s' denotes the dual space of V_s. Let $\mathcal{L}(X;Y)$ be a space of linear continuous operators from X to Y, where X, Y are Banach spaces. Denote by $L^r(0,T;X)$, $1 \leq r \leq \infty$, the space of measurable vector-functions $g : (0,T) \to X$, such that $\int_0^T \|g(t)\|_X^r dt < \infty$, for $1 \leq r < \infty$ and $\operatorname{supess}_{t\in(0,T)} \|g(t)\|_X < \infty$, for $r = +\infty$. It must be pointed out, that each $g \in L^r(0,T;X)$ can be identified with distribution in $(0,T)$ with values in X and its generalized derivative is denoted by $g' = dg/dt \in D'((0,T);X) = \mathcal{L}(D(0,T);X)$. Also, let us consider the following forms corresponding to the elliptic and nonlinear operators

$$a(v,w) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} dx, \quad b(v,w,w^1) = \sum_{i,k=1}^{n} \int_\Omega v_k \frac{\partial w_i}{\partial x_k} w_i^1 dx,$$

where $v, w \in V_1$, $w^1 \in V_1 \cap (L^n(\Omega))^n$. From the Sobolev embedding theorem $H_0^1(\Omega) \subset L^{q(n)}(\Omega)$ $(q(n) = 2n/(n-2)$, for $n > 2$, $q(n)$ is an arbitrary number for $n=2$) and the Hölder's inequality, it follows that the form b is continuous on $V_1 \times V_1 \times [V_1 \cap (L^n(\Omega))^n]$.

The problem (1)-(4) admits the following variational formulation: find the vector-function $u \in L^2(0,T;V_1) \cap L^\infty(0,T;H)$, such that

$$\frac{d}{dt}(u(.),v) + \nu a(u(.),v) + b(u(.),u(.),v) = \langle f(.),v\rangle_1, \tag{5}$$

for all $v \in V_1 \cap (L^n(\Omega))^n$, in the sense of distributions in $(0,T)$ and the following nonclassical initial condition

$$u(0) = \sum_{j=1}^m \beta_j u(T_j) + \sum_{j=1}^m \int_{T_j^1}^{T_j^2} \gamma_j(u,\tau)u(\tau)d\tau + u_0, \tag{6}$$

where $u_0 \in H$, $f \in L^2(0,T;V_1')$, $\gamma_j(u,\tau) = \sin^{r_j}(\|u\|^2_{L^2(0,T;H)})\tilde{\gamma}_j(\tau)$, $\tilde{\gamma}_j \in L^1(0,T)$, $j=\overline{1,m}$, $(.,.)$ denotes the scalar product in H, $\langle .,.\rangle_s$ is the duality relation between the spaces V_s' and V_s.

Note that, if u is a solution of the equation (5), then $\mathcal{P} = u' - \nu\Delta u + \sum\limits_{i=1}^n u_i\partial_i u - f \in [D'(\Omega\times(0,T))]^n$, where $D'(\Omega\times(0,T))$ denotes the space of distributions in $\Omega\times(0,T)$. Since $\langle\mathcal{P},\varphi\rangle = 0$ in $[D'(0,T)]^n$ for all $\varphi\in\mathcal{D}$, there exists $p \in D'(\Omega\times(0,T))$, such that $\mathcal{P} = -\mathrm{grad}p$. Thus, under the weak solution of the problem (1)-(4) we can mean solution u of the problem (5), (6).

In order to give a sense to the condition (6) we determine the space to which belongs the vector-function u', that allows to apply the interpolation theorem. It must be pointed out that nonlinear term $b(u,u,v)$ of the equation (5) is linear with respect to v. Applying Hölder's inequality, for any vector-functions v,w from $\mathcal{D}$ we have

$$|b(w,w,v)| = |-b(w,v,w)| \le c_1 \|w\|^2_{(L^{p(n)}(\Omega))^n} \sum_{i,j=1}^n \|\partial_i v_j\|_{L^n(\Omega)}, \tag{7}$$

where $p(n) = 2n/(n-1)$. From the embedding theorem for fractional order Sobolev spaces, it follows that $\partial_i v_j \in H^{s-1}(\Omega) \subset L^n(\Omega)$, for each $v\in V_s$, $s=n/2$. Consequently, due to density of $\mathcal{D}$ in V_s, we can pass to the limit in (7) and then for any $w\in V_1$, $v\in V_s$, we obtain that $b(w,w,v) = -b(w,v,w)$ and $|b(w,w,v)| \le c_2\|w\|^2_{(L^{p(n)}(\Omega))^n}\|v\|_{V_s}$. Thus, there exists $Bw \in V_s'$, such that $b(w,w,v) = \langle Bw,v\rangle_s$ and $\|Bw\|_{V_s'} \le c_2\|w\|^2_{(L^{p(n)}(\Omega))^n}$.

Note that $L^2(0,T;V_1)\cap L^\infty(0,T;H) \subset L^4(0,T;(L^{p(n)}(\Omega))^n)$. Indeed, for any $w \in L^2(0,T;V_1)\cap L^\infty(0,T;H)$, we have that $w_i \in L^2(0,T;L^{q(n)}(\Omega))\cap$

$L^\infty(0,T;L^2(\Omega))$, $i=\overline{1,n}$, for $n\geq 3$, and applying Hölder's inequality

$$\int_\Omega |w_i|^{p(n)}dx \leq \int_\Omega |w_i|^{\frac{p(n)}{2}}|w_i|^{\frac{p(n)}{2}}dx \leq \left(\int_\Omega |w_i|^{q(n)}dx\right)^{\frac{p(n)}{2q(n)}}\left(\int_\Omega |w_i|^2dx\right)^{\frac{p(n)}{4}}, \quad (8)$$

for almost all $t\in(0,T)$, since $p(n)/2q(n)+p(n)/4=1$. Hence, from Sobolev embedding theorem $H_0^1(\Omega)\subset L^{q(n)}(\Omega)$ we infer, that $\|w\|_{L^4(0,T;(L^{p(n)}(\Omega))^n)}\leq c_3(\|w\|_{L^2(0,T;V_1)}+\|w\|_{L^\infty(0,T;H)})$.

If $n=2$, then $p=4$ and in order to prove that $w\in L^4(0,T;(L^4(\Omega))^n)$ note, that for any infinitely differentiable function v defined on $\mathbf{R}^2$ with compact support in Ω the following estimates are valid

$$\int_\Omega v^4(x)dx = \int_{\mathbf{R}^2} v^4(x)dx = 4\int_{\mathbf{R}^2}\left(\int_{-\infty}^{x_1} v\partial_1 v d\xi_1\right)\left(\int_{-\infty}^{x_2} v\partial_2 v d\xi_2\right)dx_1dx_2 \leq$$

$$\leq 4\int_{\mathbf{R}^2}|v||\partial_1 v|dx_1dx_2\int_{\mathbf{R}^2}|v||\partial_2 v|dx_1dx_2 \leq 4\|v\|^2_{L^2(\Omega)}\|\partial_1 v\|_{L^2(\Omega)}\|\partial_2 v\|_{L^2(\Omega)}.$$

Since $D(\Omega)$ is dense in $L^4(\Omega)$ and in $H_0^1(\Omega)$, from the latter inequalities for any $v\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, we have

$$\|v(t)\|^2_{L^4(\Omega)} \leq \sqrt{2}\,\|v(t)\|_{L^2(\Omega)}\,\|v(t)\|_{H_0^1(\Omega)}, \quad \text{for almost all } t\in(0,T), \quad (9)$$

that implies $\|w\|_{L^4(0,T;(L^4(\Omega))^n)}\leq c_4(\|w\|_{L^2(0,T;V_1)}+\|w\|_{L^\infty(0,T;H)})$.

So, if u is a solution of the problem (5), (6), then $b(u,u,v)=\langle Bu,v\rangle_s$ and $Bu\in L^2(0,T;V_s')$. Also $a(u,v)=-\langle\Delta u,v\rangle_1$, $\Delta\in\mathcal{L}(V_1,V_1')$, $f\in L^2(0,T;V_1')$ and $V_1'\subset V_s'$, that implies $u'=\nu\Delta u-Bu+f\in L^2(0,T;V_s')$. Hence, from regularity theorem we obtain that $u\in C^0([0,T];V_s')$ ([13]), u is weakly continuous from $[0,T]$ to H and, hence, the condition (6), can be interpreted as the equality in the space H.

For the formulated nonclassical problem (5), (6) the following theorem is true.

Theorem. *If $\Omega\subset\mathbf{R}^n$, $n\geq 2$, is a bounded domain with Lipschitz boundary and there exists $0<\varkappa<2\nu\inf\limits_{v\in V_1,v\neq 0}\frac{a(v,v)}{(v,v)}$ such, that*

$$\sum_{j=1}^m|\beta_j|\exp\left(-\frac{\varkappa T_j}{2}\right)+\sum_{j=1}^m\int_{T_j^1}^{T_j^2}|\tilde{\gamma}_j(\tau)|\exp\left(-\frac{\varkappa\tau}{2}\right)d\tau<1, \quad (10)$$

then for $f\in L^2(0,T;V_1')$, $u_0\in H$, the nonclassical problem (5), (6) has a solution $u\in L^2(0,T;V_1)\cap L^\infty(0,T;H)$, $u'\in L^2(0,T;V_s')$, $s=n/2$.

Proof. Note, that the embedding of V_s in H is continuous and compact, since $V_s \subset V_1 \subset [H_0^1(\Omega)]^n$ and the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact. Hence, in the space V_s there exists a complete system of orthonormal in H vectors $\{v^k\}_{k=1}^{\infty}$, which are solutions of the spectral problem $(v^k, v)_{V_s} = \lambda_k(v^k, v)$, for all $v \in V_s$ and $0 < \lambda_1 \leq \lambda_2 \leq ..., \lambda_k \to \infty$, $k \to \infty$ ([14]).

In order to prove that the problem (5), (6) has a solution, let us consider the sequence of approximate solutions $u_N(t) = \sum\limits_{k=1}^{N} w_k^N(t)v^k$, where u_N is a solution of the following problem

$$\frac{d}{dt}\left(u_N(.), v_N\right) + \nu a\left(u_N(.), v_N\right) + b\left(u_N(.), u_N(.), v_N\right) = \langle f(.), v_N\rangle_1, \quad (11)$$

$$u_N(0) = \sum_{j=1}^{m} \beta_j u_N(T_j) + \sum_{j=1}^{m} \int_{T_j^1}^{T_j^2} \gamma_j(u_N, \tau) u_N(\tau) d\tau + u_{0N}, \quad (12)$$

where $v_N \in V_{s,N} = \{v_N \in V_s \mid v_N = \sum\limits_{i=1}^{N} \lambda_i v^i, \lambda_i \in \mathbf{R},\ i = \overline{1, N}\}$, $u_{0N} = \sum\limits_{k=1}^{N}(u_0, v^k)v^k$. Since the system $\{v^k\}_{k=1}^{\infty}$ is orthonormal in H, we infer that (11) is the system of nonlinear ordinary differential equations with respect to the vector-function $\vec{w}^N = \{w_k^N\}_{k=1}^N$,

$$\frac{d\vec{w}^N}{dt} + A_N \vec{w}^N + B_N \vec{w}^N = \vec{f}^N, \quad (13)$$

where $A_N = (A_{N_{ij}}), A_{N_{ij}} = \nu a(v^i, v^j)$, $(B_N \vec{w}^N)_k = \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} b(v^i, v^j, v^k) w_i^N w_j^N$, $\vec{f}^N = \{f_k^N\}_{k=1}^N$, $f_k^N = \langle f, v^k\rangle_1$, $i, j, k = \overline{1, N}$. The nonlinear term in the system (13) satisfies Lipschitz's condition and, applying the method of successive approximations, we get that the Cauchy problem with initial condition $\vec{w}^N(0) = \{\vec{w}_k^N(0)\}_{k=1}^N$ has solution $\vec{w}^N \in C^0([0, t_N]; \mathbf{R}^N)$, $\vec{w}^{N'} \in L^2(0, t_N; \mathbf{R}^N)$ on some subinterval $[0, t_N]$. Let us prove that $\vec{w}^N$ is defined on the whole interval $[0, T]$, i.e. $t_N = T$. Indeed, scalarly multiplying the both sides of the system (13) on $\vec{w}^N$ in the space $\mathbf{R}^N$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\vec{w}^N\|_{\mathbf{R}^N}^2 + \left(A_N \vec{w}^N, \vec{w}^N\right)_{\mathbf{R}^N} + \left(B_N \vec{w}^N, \vec{w}^N\right)_{\mathbf{R}^N} = (\vec{f}^N, \vec{w}^N)_{\mathbf{R}^N},$$

or

$$\frac{1}{2}\frac{d}{dt}\|u_N(t)\|_H^2 + \nu a\left(u_N(t), u_N(t)\right) = \langle f(t), u_N(t)\rangle_1, \quad (14)$$

since $b(u_N, u_N, u_N) = 0$. Integrating the both sides of the latter equation from 0 to t and applying the Cauchy-Schwartz inequality we infer, that

$$\frac{1}{2}\|u_N(t)\|_H^2 + \nu\int_0^t a(u_N(\tau), u_N(\tau))\,d\tau \le \frac{1}{2}\|u_N(0)\|_H^2 +$$

$$+\frac{1}{2\varepsilon}\int_0^t \|f(\tau)\|_{V_1'}^2\,d\tau + \frac{\varepsilon}{2}\int_0^t \|u_N(\tau)\|_{V_1}^2\,d\tau, \qquad \forall\varepsilon > 0.$$

As well-known, $a(v,v) \ge c_a \|v\|_H^2$ for all $v \in V_1$, $c_a = \inf\limits_{v\in V_1, v\neq 0} \frac{a(v,v)}{(v,v)}$, whence $\|v\|_{V_1}^2 \le \left(1 + \frac{1}{c_a}\right) a(v,v)$, and from the latter inequality for sufficiently small $\varepsilon > 0$, we obtain

$$\|u_N(t)\|_H^2 + \int_0^t \|u_N(\tau)\|_{V_1}^2\,d\tau \le \tilde{c}\left(\|u_N(0)\|_H^2 + \int_0^t \|f(\tau)\|_{V_1'}^2\,d\tau\right). \qquad (15)$$

Hence, $\left\|\vec{w}^N(t_N)\right\|_{\mathbf{R}^N}^2 = \|u_N(t_N)\|_H^2 < \infty$ and, consequently, $t_N = T$.

So, for any $\varphi_N \in V_{s,N}$, there exists a solution $u_N \in C^0([0,T]; V_{s,N})$, $u_N' \in L^2(0,T; V_{s,N})$ of Cauchy problem for the system (11) with initial condition $u_N(0) = \varphi_N$. Let us show that u_N is unique and continuously depends on the initial condition φ_N. Assume that u_N^1 is a solution of Cauchy problem for the system (11) with initial condition φ_N^1. Then the difference $\delta_N = u_N - u_N^1$ is a solution of the following Cauchy problem

$$\left(\delta_N'(.), v_N\right) + \nu a(\delta_N(.), v_N) + b(\delta_N(.), u_N(.), v_N) +$$

$$+b(u_N(.), \delta_N(.), v_N) - b(\delta_N(.), \delta_N(.), v_N) = 0, \quad \forall v_N \in V_{s,N}, \qquad (16)$$

$$\delta_N(0) = \varphi_N - \varphi_N^1. \qquad (17)$$

Note, that for any $v, w \in \mathcal{D}$, $b(v,w,w) = b(w,w,w) = 0$, and, consequently, from the density of $\mathcal{D}$ in V_s and continuous embedding $V_s \subset L^r(\Omega)$, for any $r \ge 1$, we obtain $b(u_N, \delta_N, \delta_N) = b(\delta_N, \delta_N, \delta_N) = 0$. Therefore, if we substitute v_N by δ_N in the equation (16) and integrate from 0 to t, we get

$$\frac{1}{2}\|\delta_N(t)\|_H^2 + \nu\int_0^t a(\delta_N(\tau), \delta_N(\tau))\,d\tau =$$

$$= \frac{1}{2}\|\delta_N(0)\|_H^2 - \int_0^t b(\delta_N(\tau), u_N(\tau), \delta_N(\tau))\,d\tau. \qquad (18)$$

Applying the inequalities (8), (9), Sobolev embedding theorem $H_0^1(\Omega) \subset L^{q(n)}(\Omega)$ and Hölder's inequality we obtain

$$\int_0^t b\left(\delta_N(\tau), u_N(\tau), \delta_N(\tau)\right) d\tau \leq \tilde{c}_1 \int_0^t \|\delta_N(\tau)\|^2_{(L^{p(n)}(\Omega))^n} \|u_N(\tau)\|_{V_s} d\tau \leq$$
$$\leq \tilde{c}_2 \varepsilon \int_0^t \|\delta_N(\tau)\|^2_{V_1} d\tau + \frac{\tilde{c}_3}{\varepsilon} \int_0^t \|\delta_N(\tau)\|^2_H \|u_N(\tau)\|^2_{V_s} d\tau.$$

From the latter inequality and (18), taking $\varepsilon > 0$ sufficiently small, we infer

$$\|\delta_N(t)\|^2_H \leq \|\delta_N(0)\|^2_H + \tilde{c}_4 \int_0^t \|\delta_N(\tau)\|^2_H \|u_N(\tau)\|^2_{V_s} d\tau, \quad 0 \leq t \leq T,$$

whence, applying Gronwall's lemma ([13]), it follows that

$$\|\delta_N(t)\|^2_H \leq \|\delta_N(0)\|^2_H \exp\left(\tilde{c}_4 \int_0^t \|u_N(\tau)\|^2_{V_s} d\tau\right), \quad 0 \leq t \leq T.$$

Therefore, the operator $S_N : H_N \to C^0([0,T]; H_N)$, $S_N(u_N(0)) = u_N$ is continuous, where H_N is the linear subspace of H defined by the vectors $v^1, v^2, ..., v^N$.

In order to prove the existence of the solution to the problem (11), (12) we have to find $\varphi_N \in H_N$, such that $u_N(0) = \varphi_N$ and $\varphi_N = \tilde{S}_N(\varphi_N)$,

$$\tilde{S}_N(\varphi_N) = \sum_{j=1}^m \beta_j S_N(\varphi_N)(T_j) + \sum_{j=1}^m \int_{T_j^1}^{T_j^2} \gamma_j(S_N(\varphi_N), \tau) S_N(\varphi_N)(\tau) d\tau + u_{0N}.$$

From (14), for any $0 < \varepsilon < 1$, we have

$$\frac{d}{dt} \|u_N(t)\|^2_H + (2\nu c_a - \varepsilon(c_a + 1)) \|u_N(t)\|^2_H \leq \frac{1}{\varepsilon} \|f(t)\|^2_{V_1'},$$

and, consequently, by virtue of the condition of the theorem, we infer that

$$\|u_N(t)\|^2_H \leq e^{-\varkappa t} \left(\|\varphi_N\|^2_H + \frac{c_a + 1}{2\nu c_a - \varkappa} \int_0^t e^{\varkappa\tau} \|f(\tau)\|^2_{V_1'} d\tau\right), \quad \forall t \in [0,T].$$

Applying the latter inequality we can estimate the norm of $\tilde{S}_N(\varphi_N)$

$$\left\|\tilde{S}_N(\varphi_N)\right\|_H \le \sum_{j=1}^{m} |\beta_j| \|u_N(T_j)\|_H + \sum_{j=1}^{m} \int_{T_j^1}^{T_j^2} |\tilde{\gamma}_j(\tau)| \|u_N(\tau)\|_H d\tau + \|u_{0N}\|_H \le$$

$$\le \sqrt{\|\varphi_N\|_H^2 + c_f} \left(\sum_{j=1}^{m} |\beta_j| \, \zeta(T_j) + \sum_{j=1}^{m} \int_{T_j^1}^{T_j^2} |\tilde{\gamma}_j(\tau)| \, \zeta(\tau) d\tau \right) + \|u_{0N}\|_H \,,$$

where $c_f = \dfrac{c_a + 1}{2\nu c_a - \varkappa} \displaystyle\int_0^T e^{\varkappa\tau} \|f(\tau)\|_{V_1'}^2 d\tau$, $\zeta(\tau) = \exp\left(-\dfrac{\varkappa\tau}{2}\right)$, $\tau \in [0, T]$.

Since the system $\{v^k\}_{k=1}^{\infty}$ is orthonormal in H, we have $\|u_{0N}\|_H \le \|u_0\|_H$, and, hence, by virtue of (10), $\|\tilde{S}_N(\varphi_N)\|_H \le R$, if $\|\varphi_N\|_H \le R$, where R is sufficiently large

$$R \ge \sqrt{\frac{Q}{1-Q} c_f + \frac{\|u_0\|_H^2}{(1-Q)^2}}, \quad Q = \sum_{j=1}^{m} |\beta_j| \, \zeta(T_j) + \sum_{j=1}^{m} \int_{T_j^1}^{T_j^2} |\tilde{\gamma}_j(\tau)| \, \zeta(\tau) d\tau.$$

From the continuity of the operator S_N it follows, that $\tilde{S}_N : H_N \to H_N$ is a continuous operator too, which maps the ball with radius R and centre 0 to itself. Consequently, applying the Brower's fixed point theorem we obtain that there exists $\varphi_N^0 \in H_N$, such that $\tilde{S}_N(\varphi_N^0) = \varphi_N^0$, $\left\|\varphi_N^0\right\|_H \le R$.

Thus, if we let $u_N(0) = \varphi_N^0$, then the solution $u_N(t)$ of the system (11) satisfies condition (12) and from the inequality (15) we obtain that the sequence $\{u_N\}_{N\ge1}$ is bounded in $L^2(0,T;V_1) \cap L^\infty(0,T;H)$. Let us prove that the sequence $\{u_N'\}_{N\ge1}$ is also bounded in corresponding space. Note that for any $g \in L^2(0,T;V_s')$,

$$\left\| \sum_{k=1}^{N} \left\langle g(t), v^k \right\rangle_s v^k \right\|_{V_s'} = \sup_{\|v\|_{V_s} \le 1} \left| \left\langle g(t), \sum_{k=1}^{N} (v, v^k) v^k \right\rangle_s \right| \le$$

$$\le \|g(t)\|_{V_s'} \sup_{\|v\|_{V_s} \le 1} \left| \sum_{k=1}^{N} \lambda_k (v, v^k)^2 \right|^{1/2} \le \|g(t)\|_{V_s'} \,, \tag{19}$$

for almost all $t \in (0,T)$. Since u_N is bounded in $L^2(0,T;V_1)$, then Δu_N is bounded in $L^2(0,T;V_1') \subset L^2(0,T;V_s')$. Also, from the inequalities (8), (9) for any vector-function $w \in L^2(0,T;V_1) \cap L^\infty(0,T;H)$ we infer that $\|Bw\|_{L^2(0,T;V_s')} \le \tilde{c}_5 \|w\|_{L^\infty(0,T;H)} \|w\|_{L^2(0,T;V_1)}$. Hence the sequence

$\{Bu_N\}_{N\geq 1}$ is bounded in the space $L^2(0,T;V_s')$. The system (11) can be written in the following equivalent form

$$u_N' - \nu \sum_{k=1}^{N} \left\langle \Delta u_N, v^k \right\rangle_1 v^k + \sum_{k=1}^{N} \left\langle Bu_N, v^k \right\rangle_s v^k = \sum_{k=1}^{N} \left\langle f, v^k \right\rangle_1 v^k,$$

in the sense of the space $L^2(0,T;V_{s,N})$, whence, due to the inequality (19), it follows, that the sequence $\{u_N'\}_{N\geq 1}$ is bounded in $L^2(0,T;V_s')$.

So, the sequences $\{u_N\}_{N\geq 1}$ and $\{u_N'\}_{N\geq 1}$ are bounded in corresponding spaces and, consequently, there exists subsequence $\{u_{N_k}\}_{k=1}^{\infty}$, which converges to the vector-function u weakly in the space $L^2(0,T;V_1)$, weakly-$*$ in $L^\infty(0,T;H)$ and $\{u_{N_k}'\}_{k=1}^{\infty}$ converges weakly to u' in $L^2(0,T;V_s')$. Due to the compactness theorem ([15]) the bounded set $\{u_N\}_{N\geq 1}$ is relatively compact in $L^2(0,T;H)$. Therefore, we can choose the subsequence $\{u_{N_k}\}_{k\geq 1}$ in such a way that $u_{N_k} \to u$ strongly in $L^2(0,T;H)$. Also, according to the inequalities (8), (9), $(u_N)_i(u_N)_j$ is bounded in $L^2(0,T;L^{p(n)/2}(\Omega))$ and we can assume that $\{(u_{N_k})_i(u_{N_k})_j\}_{k\geq 1}$ weakly converges in the space $L^2(0,T;L^{\frac{p(n)}{2}}(\Omega))$ $(i,j=\overline{1,n})$.

By virtue of weak convergence of the sequences $\{u_{N_k}\}_{k\geq 1}$, $\{u_{N_k}'\}_{k\geq 1}$ in the spaces $L^2(0,T;V_1)$ and $L^2(0,T;V_s')$ respectively, they weakly converge in the spaces $L^2(0,t_0;V_1)$ and $L^2(0,t_0;V_s')$, where $t_0 \in (0,T]$. For any $v \in V_s$ and ψ_1, $\psi_2 \in C^\infty([0,t_0])$, $\psi_1(0)\neq 0$, $\psi_1(t_0)=0$, $\psi_2(t_0)\neq 0$, $\psi_2(0)=0$, the following equality is valid

$$\int_0^{t_0} \left\langle u'(\tau) - u_{N_k}'(\tau), v\right\rangle_s \psi_\alpha(\tau)d\tau = -\int_0^{t_0} \left\langle u(\tau) - u_{N_k}(\tau), v\right\rangle_s \psi_\alpha'(\tau)d\tau +$$
$$+(-1)^\alpha \left\langle u\left((\alpha-1)t_0\right) - u_{N_k}\left((\alpha-1)t_0\right), v\right\rangle_s \psi_\alpha\left((\alpha-1)t_0\right), \quad \alpha=1,2.$$

Hence, due to weak convergence of the sequences $\{u_{N_k}\}_{k\geq 1}$, $\{u_{N_k}'\}_{k\geq 1}$ we have that $u_{N_k}(t_0) \to u(t_0)$ weakly in V_s', for all $t_0 \in [0,T]$. Also, since $\{u_{N_k}\}_{k\geq 1}$ weakly-$*$ converges in $L^\infty(0,T;H)$, it weakly-$*$ converges to $u(t)$ in $L^\infty(t_0,t_1;H)$, for any $t_0 < t_1$, $t_0,t_1 \in [0,T]$. Therefore,

$$\left(\int_{T_j^1}^{T_j^2} \tilde\gamma_j(\tau)(u(\tau) - u_{N_k}(\tau))d\tau, v \right) = \int_{T_j^1}^{T_j^2} (u(\tau) - u_{N_k}(\tau), \tilde\gamma_j(\tau)v)d\tau \to 0,$$

as $k\to\infty$, for all $v\in H$, $j=\overline{1,m}$. Thus, $u_{N_k}(0) \to u(0)$ weakly in V_s' and from strong convergence of $\{u_{N_k}\}_{k\geq 1}$ in $L^2(0,T;H)$, we obtain

$$\tilde{S}_{N_k}(u_{N_k}(0)) \to \sum_{j=1}^{m} \beta_j u(T_j) + \sum_{j=1}^{m} \int_{T_j^1}^{T_j^2} \gamma_j(u,\tau) u(\tau) d\tau + u_0 \quad \text{weakly in } V_s'.$$

Consequently, the limit $u \in L^2(0,T;V_1) \cap L^\infty(0,T;H)$ of the sequence $\{u_{N_k}\}_{k\geq 1}$ satisfies nonclassical initial condition (6).

Let us prove now that u is a solution of the equation (5). By virtue of density of V_s in $V_1 \cap (L^n(\Omega))^n$, for any $v \in V_1 \cap (L^n(\Omega))^n$, there exists sequence $\{v_r\}_{r\geq 1}$, which strongly converges to v, where $v_r \in V_{s,r}$, $r \geq 1$. From the equation (11) we obtain

$$-\int_0^T (u_{N_k}(\tau), v_r)\, \psi'(\tau) d\tau + \nu \int_0^T a(u_{N_k}(\tau), v_r)\, \psi(\tau) d\tau +$$

$$+\int_0^T b(u_{N_k}(\tau), u_{N_k}(\tau), v_r)\, \psi(\tau) d\tau = \int_0^T \langle f(\tau), v_r\rangle_1\, \psi(\tau) d\tau, \tag{20}$$

for all $\psi \in D(0,T)$ and $N_k \geq r$.

Since $\{u_{N_k}\}_{k\geq 1}$ weakly converges to u in $L^2(0,T;V_1) \subset L^2(0,T;(L^2(\Omega))^n)$ and $u_{N_k}\varphi$ strongly converges to $u\varphi$ in $L^2(0,T;(L^2(\Omega))^n)$ for any $\varphi \in D(\Omega \times (0,T))$, we infer that $(u_{N_k})_i (u_{N_k})_j$ converges to $u_i u_j$ in $D'(\Omega \times (0,T))$ and, consequently, $\{(u_{N_k})_i (u_{N_k})_j\}_{k\geq 1}$ weakly converges to $u_i u_j$ in the space $L^2(0,T;L^{p(n)/2}(\Omega))$ $(i,j = \overline{1,n})$. Hence, from the embedding $H^{s-1}(\Omega) \subset L^n(\Omega)$, we have

$$\int_0^T b(u_{N_k}(\tau), u_{N_k}(\tau), v_r)\, \psi(\tau) d\tau = -\int_0^T b(u_{N_k}(\tau), v_r, u_{N_k}(\tau))\, \psi(\tau) d\tau =$$

$$= -\sum_{i,j=1}^{n} \int_0^T \int_\Omega (u_{N_k}(\tau))_i \frac{\partial (v_r)_j}{\partial x_i} (u_{N_k}(\tau))_j \psi(\tau) dx d\tau \to$$

$$\to -\sum_{i,j=1}^{n} \int_0^T \int_\Omega u_i(\tau) \frac{\partial (v_r)_j}{\partial x_i} u_j(\tau) \psi(\tau) dx d\tau = \int_0^T b(u(\tau), u(\tau), v_r)\, \psi(\tau) d\tau,$$

as $k \to \infty$, for all $v_r \in V_{s,r}$. Therefore, tending $k \to \infty$ in (20) we infer, that

$$-\int_0^T (u(\tau), v_r)\,\psi'(\tau)d\tau + \nu\int_0^T a\,(u(\tau), v_r)\,\psi(\tau)d\tau + \int_0^T b\,(u, u, v_r)\,\psi(\tau)d\tau =$$

$$= \int_0^T \langle f(\tau), v_r\rangle_1\,\psi(\tau)d\tau, \qquad \forall\psi \in D(0,T),$$

whence, tending $r \to \infty$, we deduce that u satisfies the equation (5). □

Corollary. If in the nonclassical initial condition (6) $\tilde{\gamma}_2 \equiv ... \equiv \tilde{\gamma}_m \equiv 0$, $T_1^1 = 0$, $T_1^2 = T$, all β_j, $\tilde{\gamma}_1$ are nonnegative or nonpositive, $\tilde{\gamma}_1(t) = \alpha\rho(t)$, $\rho \in L^1(0,T)$, $0 < \rho(t) \leq 1$ for almost all $t \in (0,T)$ and the following inequality is valid $\left|\sum\limits_{j=1}^{m} \beta_j + \alpha T\right| \leq 1$, then the problem (5), (6) is solvable.

Proof. To prove the assertion it is sufficient to show that the condition (10) is fulfilled. Indeed,

$$\sum_{j=1}^{m} |\beta_j| \exp\left(-\frac{\varkappa T_j}{2}\right) d\tau + |\alpha| \int_0^T |\rho(\tau)| \exp\left(-\frac{\varkappa\tau}{2}\right) d\tau \leq$$

$$\leq \sum_{j=1}^{m} |\beta_j| + |\alpha| \int_0^T \exp\left(-\frac{\varkappa\tau}{2}\right) d\tau = \sum_{j=1}^{m} |\beta_j| + |\alpha|\, T \frac{1-\exp\left(-\frac{\varkappa T}{2}\right)}{\frac{\varkappa T}{2}} \leq 1,$$

where one of the inequalities is strict, since $1 - \exp(-y) < y$, for $y > 0$. □

It must be pointed out, that from the formulated theorem we can also obtain the results on the solvability of the problem (5), (6), for $T_j^1 > 0$, $j = \overline{1,m}$, and arbitrary coefficients β_j and functions $\tilde{\gamma}_j$, if the moments of time T_j, T_j^1, T_j^2 or the coefficient of viscosity ν are sufficiently large.

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