INVESTIGATION OF NONLINEAR MODELS IN THE THEORY OF ELASTIC MIXTURES

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Abstract

In this paper a statical nonlinear model for multicomponent mixture is constructed and general expressions for response functions of the stress tensors of the constituents for isotropic elastic mixtures are given. For one nonlinear model of two-component elastic mixture a theorem on existence and uniqueness of local solution to corresponding boundary value problem is obtained. In the case of multicomponent hyperelastic mixture the Dirichlet boundary value problem is considered and the existence of global solution in suitable spaces is proved.

Key words and phrases: nonlinear models of elastic mixtures, boundary value problems.

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1. *Introduction*

In continuum mechanics under mixture is assumed coexistence of different ingredients mutually diffused through each other. Most bodies, astrophysical, geological, biological or otherwise are mixtures in which two or more constituents coexist. Therefore, investigation of such type materials is important not only from theoretical, but also from practical point of view. If in the mixture one of the constituents is preponderant and the other constituents essentially insignificant, the body is usually assumed to be of the predominant single constituent. However, there are innumerable situations, where none of the constituents presented in the body can be ignored with respect to the others. Particularly, plasmas and gaseous mixtures that surround stars, biological tissues and muscles, suspensions, porous rocks and soil infused with water or oil. Sometimes, the constituents in a mixture undergo chemical reactions, usually resulting in formation of new compounds. In the present paper we consider mixtures whose constituents do not react chemically.

The first works, where were studied the diffusion of one constituent of the mixture through another, were published in the fifties of XIX century. Later, interesting papers were devoted to the investigation of flow and diffusion of fluids through solid media ([1-4]).

The theoretical investigations of various mathematical models of mixtures were stimulated by monograph of C. Truesdell and R. Toupin ([5]). In this were formulated mechanical principles for constructing the new mathematical models of continuum with complicated internal structure, which later were generalized by Green, Naghdi, Adkins and others in [6-12].

It must be pointed out, that two-component elastic mixture first was considered by Green and Steel in [9]. Various mechanisms of interaction of the components in the mixture were proposed in [11, 12] and were obtained improvements of the models given in [9]. Later, propagation of waves and initial-boundary value problems for various models in the theory of mixtures were studied in [13-19].

In the present paper we consider nonlinear models of elastic mixtures and study corresponding boundary value problems. More precisely, in section 2 on the basis of fundamental assumptions we construct statical nonlinear model for mixtures. In the same section we consider a class of mixtures, the so called elastic mixtures, introduce the notions of isotropy, strong isotropy and for these type of mixtures obtain general expressions for response functions of the stress tensors for the constituents. In the section 3 boundary value problems for nonlinear models of elastic mixtures are studied. We consider one nonlinear model of two-component elastic mixture and prove the existence and uniqueness of local solution in corresponding space for Dirichlet boundary value problem. In order to investigate existence of global solution the notion of hyperelasticity for mixtures is introduced. For multicomponent hyperelastic mixture the Dirichlet boundary value problem is reduced to problem of minimization of energy functional and proved that in suitable spaces the problem has a solution.

2. *Statical nonlinear models of elastic mixtures*

The basic assumption of the mixture theory is that the space occupied by a mixture can be considered as being occupied cojointly by the various constituents of the mixture, each considered as a continuum in its own right. Thus, at each point in the domain occupied by the mixture, there is a particle belonging to each of the constituents. This presupposes that each constituent is sufficiently dense in the mixture and can be homogenized over the region of the mixture as a continuum. In this section we study static equilibrium of multicomponent mixture and, generalizing the basic principles in the mechanics of a single continuum ([20]), obtain corresponding balance equations.

Throughout the paper we refer the motion of the mixture to a Cartesian frame in three dimensional Euclidean space $\mathbf{R}^3$ with orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. In order to simplify notations we assume that indices j, k, l, m range over the integers $\{1, 2, 3\}$, summation over repeated indices is implied and partial derivative $\partial/\partial x_j$ is denoted by ∂_j. The scalar product of the vectors $\boldsymbol{a} = (a_j)$, $\boldsymbol{b} = (b_j) \in \mathbf{R}^3$ is denoted by $\boldsymbol{a} \cdot \boldsymbol{b} = a_j b_j$, the norm in $\mathbf{R}^3$ with $|a| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the exterior product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \wedge \boldsymbol{b} = \varepsilon_{jkl} a_k b_l \boldsymbol{e}_j$, where $\varepsilon_{jkl} = 1$ if $\{j, k, l\}$ is an even permutation of $\{1, 2, 3\}$, $\varepsilon_{jkl} = -1$ if $\{j, k, l\}$ is an odd permutation of $\{1, 2, 3\}$, and $\varepsilon_{jkl} = 0$ otherwise. Let $\mathbf{R}^{3\times 3}$ be a space of third order square matrices equipped with the norm $\|\boldsymbol{F}\| = (\boldsymbol{F} : \boldsymbol{F})^{1/2}$, where $\boldsymbol{F} : \boldsymbol{G} = tr(\boldsymbol{F}^T \boldsymbol{G})$ is scalar product of matrices $\boldsymbol{F}$, $\boldsymbol{G} \in \mathbf{R}^{3\times 3}$, $tr(\boldsymbol{F}) = F_{kk}$, $\boldsymbol{F}^T$ denotes the transposed matrix of $\boldsymbol{F}$. The cofactor matrix of $\boldsymbol{F}$ is denoted by $\mathbf{Cof}\boldsymbol{F}$ and $\mathbf{Cof}\boldsymbol{F} = (\det \boldsymbol{F})\boldsymbol{F}^{-T}$, if $\det \boldsymbol{F} \neq 0$. The set of matrices $\boldsymbol{F} \in \mathbf{R}^{3\times 3}$ with positive determinants $\det \boldsymbol{F} > 0$ is denoted by $\mathbf{R}^{3\times 3}_+$, $\boldsymbol{O}^3_+$ is the subset of $\mathbf{R}^{3\times 3}_+$ of orthogonal matrices. $\boldsymbol{S}^3$ denotes the set of symmetric matrices and $\boldsymbol{S}^3_>$ is the subset of $\boldsymbol{S}^3$ of all positive definite matrices. The set of second order tensors is identified with the set of third order square matrices and for any tensor $\boldsymbol{T} = (T_{kl})$ and vector $\boldsymbol{a} = (a_l)$ we denote by $\boldsymbol{T}\boldsymbol{a} = T_{kl} a_l \boldsymbol{e}_k$.

Let us consider body with initial configuration $\overline{\Omega} \subset \mathbf{R}^3$, which consists of n-component mixture, where Ω is a Lipschitz domain in $\mathbf{R}^3$, i.e. open, bounded, connected set with a Lipschitz-continuous boundary $\partial\Omega$, the set Ω is located locally on one side of $\partial\Omega$. As the mixture $\overline{\Omega}$ deforms, its constituents deform with respect to each other. The displacement and deformation of i-th constituent is denoted by $\boldsymbol{u}^i, \boldsymbol{\varphi}^i : \overline{\Omega} \to \mathbf{R}^3$, $\boldsymbol{\varphi}^i = \boldsymbol{id} + \boldsymbol{u}^i$, where $\boldsymbol{\varphi}^i$ is smooth enough, injective on $\overline{\Omega}$ mapping, which satisfies orientation preserving condition $\det(\nabla\boldsymbol{\varphi}^i) > 0$, $(\nabla\varphi^i)_{kl} = \partial_l \varphi^i_k$, $i = \overline{1, n}$. Therefore, for each constituent we have $\overline{\Omega}^{\varphi^i} = \boldsymbol{\varphi}^i(\overline{\Omega})$ deformed configuration and $\overline{\Omega}^{\varphi^1} = \overline{\Omega}^{\varphi^2} = ... = \overline{\Omega}^{\varphi^n}$.

Assume, that the components in the mixture are subjected to three types of forces: external body forces, applied surface forces and internal body forces, microforces caused by interaction between constituents. The applied external body forces and surface forces are given by their densities $\boldsymbol{f}^{\varphi^i} : \Omega^{\varphi^i} \to \mathbf{R}^3$ and $\boldsymbol{g}^{\varphi^i} : \Gamma_1^{\varphi^i} \to \mathbf{R}^3$, $\Gamma_1^{\varphi^i} \subset \Gamma^{\varphi^i}$, $i = \overline{1, n}$. Since the mixture consists of several components, there exist interaction forces, given by the densities $\boldsymbol{h}^{\varphi^i} : \Omega^{\varphi^i} \to \mathbf{R}^3$, where $\boldsymbol{h}^{\varphi^i} dx^{\varphi^i}$ is the sum of internal body forces acting on the element dx^{φ^i}, and microforces, with zero resultant force, but possibly non-zero moment given by the density $\boldsymbol{m}^{\varphi^i} : \Omega^{\varphi^i} \to \mathbf{R}^3$, where $\boldsymbol{m}^{\varphi^i}dx^{\varphi^i}$ is the moment of internal forces acting on the element dx^{φ^i}, $i=\overline{1,n}$.

The mathematical model of static equilibrium of a single continuum is constructed on the basis of Euler-Cauchy stress principle. In order to obtain model of multicomponent mixture we assume, that for each constituent the analog to Euler-Cauchy principle is valid.

Principle I. Let $\overline{\Omega}$ is initial configuration of n-component mixture and $\boldsymbol{\varphi}^1,...,\boldsymbol{\varphi}^n$ are deformations of its constituents. Assume, that on each constituent act external body forces with density $\boldsymbol{f}^{\varphi^i}:\Omega^{\varphi^i}\to\mathbf{R}^3$, surface forces with density $\boldsymbol{g}^{\varphi^i}:\Gamma_1^{\varphi^i}\to\mathbf{R}^3$ and interactive internal body forces with density $\boldsymbol{h}^{\varphi^i}:\Omega^{\varphi^i}\to\mathbf{R}^3$. There exist stress vector-fields

$$\boldsymbol{t}^{\varphi^i}:\overline{\Omega}^{\varphi^i}\times S_1\to\mathbf{R}^3,\qquad S_1=\{\boldsymbol{v}\in\mathbf{R}^3,\ |\boldsymbol{v}|=1\},\ i=1,...,n,$$

which satisfy the following conditions:

a) for any subdomain $D^{\varphi^i}\subset\Omega^{\varphi^i}$ and any point $x^{\varphi^i}\in\Gamma_1^{\varphi^i}\cap\partial D^{\varphi^i}$, where exists unit outer normal $\boldsymbol{\nu}^{\varphi^i}$, the following conditions are valid

$$\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu}^{\varphi^i})=\boldsymbol{g}^{\varphi^i}(x^{\varphi^i}),\qquad i=1,...,n;$$

b) for any subdomain $D^{\varphi^i}\subset\Omega^{\varphi^i}$ the balance equations for forces hold

$$\int\limits_{D^{\varphi^i}}\boldsymbol{f}^{\varphi^i}(x^{\varphi^i})dx^{\varphi^i}+\int\limits_{D^{\varphi^i}}\boldsymbol{h}^{\varphi^i}(x^{\varphi^i})dx^{\varphi^i}+\int\limits_{\partial D^{\varphi^i}}\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu}^{\varphi^i})d\sigma^{\varphi^i}=\mathbf{0},\tag{2.1}$$

where $\boldsymbol{\nu}^{\varphi^i}$ is a unit outer normal to ∂D^{φ^i}, $i=1,...,n$;

c) for any subdomain $D^{\varphi^i}\subset\Omega^{\varphi^i}$ the balance equations for moments are valid

$$\begin{gathered}\int\limits_{D^{\varphi^i}}\boldsymbol{x}^{\varphi^i}\wedge\boldsymbol{f}^{\varphi^i}(x^{\varphi^i})dx^{\varphi^i}+\int\limits_{D^{\varphi^i}}\boldsymbol{x}^{\varphi^i}\wedge\boldsymbol{h}^{\varphi^i}(x^{\varphi^i})dx^{\varphi^i}+\\+\int\limits_{D^{\varphi^i}}\boldsymbol{m}^{\varphi^i}(x^{\varphi^i})dx^{\varphi^i}+\int\limits_{\partial D^{\varphi^i}}\boldsymbol{x}^{\varphi^i}\wedge\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu}^{\varphi^i})d\sigma^{\varphi^i}=\mathbf{0},\ \ i=\overline{1,n}.\end{gathered}\tag{2.2}$$

On the basis of the formulated principle we obtain the analog to Cauchy theorem.

Theorem 2.1. *Let the densities of external and internal body forces* $\boldsymbol{f}^{\varphi^i}:\overline{\Omega}^{\varphi^i}\to\mathbf{R}^3$, $\boldsymbol{h}^{\varphi^i}:\overline{\Omega}^{\varphi^i}\to\mathbf{R}^3$ *and densities of the moments of microforces* $\boldsymbol{m}^{\varphi^i}:\overline{\Omega}^{\varphi^i}\to\mathbf{R}^3$ *are continuous, stress vector-fields* $\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu})$ *are continuous with respect to the variable* $\boldsymbol{\nu}\in S_1$, *for each* $x^{\varphi^i}\in\overline{\Omega}^{\varphi^i}$, *and continuously differentiable with respect to* $x^{\varphi^i}\in\overline{\Omega}^{\varphi^i}$, *for each* $\boldsymbol{\nu}\in S_1$. *Then* *there exist continuously differentiable tensor-fields* $\boldsymbol{T}^{\varphi^i} : \overline{\Omega}^{\varphi^i} \to \mathbf{R}^{3\times 3}$ *such that*

$$\boldsymbol{t}^{\varphi^i}(x^{\varphi^i}, \boldsymbol{\nu}) = \boldsymbol{T}^{\varphi^i}(x^{\varphi^i})\boldsymbol{\nu}, \qquad \forall x^{\varphi^i} \in \overline{\Omega}^{\varphi^i},\ \boldsymbol{\nu} \in S_1,\ i = \overline{1,n}. \tag{2.3}$$

and $\boldsymbol{T}^{\varphi^i}$ *satisfies the following equations:*

$$-\mathbf{div}^{\varphi^i}\boldsymbol{T}^{\varphi^i}(x^{\varphi^i}) = \boldsymbol{f}^{\varphi^i}(x^{\varphi^i}) + \boldsymbol{h}^{\varphi^i}(x^{\varphi^i}), \qquad \forall x^{\varphi^i} \in \Omega^{\varphi^i}, \tag{2.4}$$

$$\boldsymbol{T}^{\varphi^i}(x^{\varphi^i}) - \left[\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})\right]^T = \boldsymbol{M}^{\varphi^i}(x^{\varphi^i}), \qquad \forall x^{\varphi^i} \in \overline{\Omega}^{\varphi^i}, \tag{2.5}$$

$$\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})\boldsymbol{\nu}^{\varphi^i} = \boldsymbol{g}^{\varphi^i}(x^{\varphi^i}), \qquad \forall x^{\varphi^i} \in \Gamma_1^{\varphi^i}, \tag{2.6}$$

where $\mathbf{div}^{\varphi^i}\boldsymbol{T}^{\varphi^i} = (\partial_l^{\varphi^i} T_{kl}^{\varphi^i})$, $\boldsymbol{M}^{\varphi^i}(x^{\varphi^i}) = (\varepsilon_{jkl} m_j^{\varphi^i})$, $\boldsymbol{\nu}^{\varphi^i}$ *is a unit outer normal to* $\Gamma_1^{\varphi^i}$ $(i = \overline{1,n})$.

Proof. Since the set Ω^{φ^i} is open, for any point $y^{\varphi^i} \in \Omega^{\varphi^i}$, there exists tetrahedron Q with vertex in y^{φ^i}, which is located in Ω^{φ^i}, its faces, passing through y^{φ^i} are parallel to the coordinate planes and the normal of the fourth face G is $\boldsymbol{\nu} = \sum\limits_{k=1}^{3} \nu_k \boldsymbol{e}_k$. G_k denotes the face orthogonal to $\boldsymbol{e}_k$. From the balance equation (2.1), letting $D^{\varphi^i} = Q$, we have

$$\int\limits_Q \left(f_k^{\varphi^i}(x^{\varphi^i}) + h_k^{\varphi^i}(x^{\varphi^i})\right) dx^{\varphi^i} + \int\limits_G t_k^{\varphi^i}(x^{\varphi^i}, \boldsymbol{\nu}) d\sigma^{\varphi^i} +$$
$$+ \sum_{l=1}^{3} \int\limits_{G_l} t_k^{\varphi^i}(x^{\varphi^i}, -\mathrm{sign}(\nu_l)\boldsymbol{e}_l) d\sigma^{\varphi^i} = 0,$$

where $k = 1,2,3$, $\mathrm{sign}(z) = 1$, for $z \geq 0$; $\mathrm{sign}(z) = 0$, for $z = 0$; $\mathrm{sign}(z) = -1$, for $z < 0$.

From the latter equality, applying mean-value theorem for integrals and taking into account $\mathrm{meas}(G_k) = |\nu_k| \,\mathrm{meas}(G)$ $(k = 1,2,3)$ we obtain:

$$\left| t_k^{\varphi^i}(z_k, \boldsymbol{\nu}) + \sum_{l=1}^{3} t_k^{\varphi^i}(z_{kl}, -\mathrm{sign}(\nu_l)\boldsymbol{e}_l)\, |\nu_l| \right| \mathrm{meas}(G) \leq$$
$$\leq \sup_{z\in Q} \left| f_k^{\varphi^i}(z) + h_k^{\varphi^i}(z) \right| \mathrm{meas}(Q), \tag{2.7}$$

where $z_k \in G$, $z_{kl} \in G_l$, $k,l = 1,2,3$, $\mathrm{meas}(G)$ denotes the area of G, $\mathrm{meas}(Q)$ is the volume of Q.

Let us tend the vertices of the face G to y^{φ^i}. Since $f_k^{\varphi^i}$, $h_k^{\varphi^i}$ are bounded and components of the stress-vectors $t_k^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu})$ are continuous with respect to the first argument, then from the inequality (2.7) we have

$$\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},\boldsymbol{\nu}) = -\sum_{l=1}^{3}\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},-\mathrm{sign}(\nu_l)\boldsymbol{e}_l)\mathrm{sign}(\nu_l)\nu_l. \qquad (2.8)$$

If in the latter equality we tend $\boldsymbol{\nu}$ to $\mathrm{sign}(\nu_k)\boldsymbol{e}_k$ $(k=1,2,3)$ and take into account continuity of $\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu})$ with respect to the second argument we infer that

$$\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},\mathrm{sign}(\nu_k)\boldsymbol{e}_k) = -\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},-\mathrm{sign}(\nu_k)\boldsymbol{e}_k), \quad k=1,2,3,$$

and, due to (2.8),

$$\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},\boldsymbol{\nu}) = \sum_{l=1}^{3}\nu_l\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},\boldsymbol{e}_l), \qquad \forall y^{\varphi^i}\in\Omega^{\varphi^i},\ \boldsymbol{\nu}\in S_1.$$

Thus, there exist tensor-fields $\boldsymbol{T}^{\varphi^i}$, $\boldsymbol{T}^{\varphi^i}(y^{\varphi^i}) = \left\{T_{kl}^{\varphi^i}(y^{\varphi^i})\right\}$, $i=1,..,n$, where the functions $T_{kl}^{\varphi^i}:\overline{\Omega}^{\varphi^i}\to\mathbf{R}$, $T_{kl}^{\varphi^i}(y^{\varphi^i}) = t_k^{\varphi^i}(y^{\varphi^i},\boldsymbol{e}_l)$, $k,l=1,2,3$, are such that

$$\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},\boldsymbol{\nu}) = \sum_{k,l=1}^{3}\nu_l T_{kl}^{\varphi^i}(y^{\varphi^i})\boldsymbol{e}_k = \boldsymbol{T}^{\varphi^i}(y^{\varphi^i})\boldsymbol{\nu}, \qquad \forall y^{\varphi^i}\in\Omega^{\varphi^i},\ \boldsymbol{\nu}\in S_1.$$

Note that we prove equality (2.3) in the domain Ω^{φ^i}, but since $\boldsymbol{t}^{\varphi^i}$ is continuous on $\overline{\Omega}^{\varphi^i}\times S_1$, it is valid for all $x^{\varphi^i}\in\overline{\Omega}^{\varphi^i}$ and $\boldsymbol{\nu}\in S_1$. Moreover, continuously differentiability of $\boldsymbol{t}^{\varphi^i}(y^{\varphi^i},\boldsymbol{\nu})$ with respect to y^{φ^i} insures continuously differentiability of $\boldsymbol{T}^{\varphi^i}:\overline{\Omega}^{\varphi^i}\to\mathbf{R}^{3\times3}$.

Applying Green's formula, the last term in the balance equation (2.1) can be written in the following form

$$\int\limits_{\partial D^{\varphi^i}}\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu}^{\varphi^i})d\sigma^{\varphi^i} = \int\limits_{\partial D^{\varphi^i}}\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})\boldsymbol{\nu}^{\varphi^i}d\sigma^{\varphi^i} = \int\limits_{D^{\varphi^i}}\mathbf{div}^{\varphi^i}\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})dx^{\varphi^i},$$

whence $\boldsymbol{T}^{\varphi^i}$ satisfies the equation (2.4) $(i=\overline{1,n})$.

In order to establish (2.5), let us transform surface integral in the balance equation (2.2),

$$\int\limits_{\partial D^{\varphi^i}}\boldsymbol{x}^{\varphi^i}\wedge\boldsymbol{t}^{\varphi^i}(x^{\varphi^i},\boldsymbol{\nu}^{\varphi^i})d\sigma^{\varphi^i} = \int\limits_{D^{\varphi^i}}\varepsilon_{jkl}\partial_m^{\varphi^i}\left\{x_k^{\varphi^i}T_{lm}^{\varphi^i}(x^{\varphi^i})\right\}\boldsymbol{e}_j dx^{\varphi^i} =$$

$$= \int\limits_{D^{\varphi^i}} \varepsilon_{jkl}\delta_{km}T^{\varphi^i}_{lm}(x^{\varphi^i})\boldsymbol{e}_j dx^{\varphi^i} + \int\limits_{D^{\varphi^i}} \varepsilon_{jkl}x^{\varphi^i}_k \partial^{\varphi^i}_m T^{\varphi^i}_{lm}(x^{\varphi^i})\boldsymbol{e}_j dx^{\varphi^i} =$$

$$= \int\limits_{D^{\varphi^i}} \varepsilon_{jkl}T^{\varphi^i}_{lk}(x^{\varphi^i})\boldsymbol{e}_j dx^{\varphi^i} - \int\limits_{D^{\varphi^i}} \varepsilon_{jkl}x^{\varphi^i}_k (f^{\varphi^i}_l + h^{\varphi^i}_l)(x^{\varphi^i})\boldsymbol{e}_j dx^{\varphi^i},$$

where δ_{km} is Kronecker's symbol. Therefore, from the balance equation for moments (2.2) we obtain

$$\int\limits_{D^{\varphi^i}} \varepsilon_{jkl}T^{\varphi^i}_{lk}(x^{\varphi^i})dx^{\varphi^i} = -\int\limits_{D^{\varphi^i}} m^{\varphi^i}_j(x^{\varphi^i})dx^{\varphi^i}, \qquad j = 1,2,3,\ i = \overline{1,n},$$

hence,

$$\varepsilon_{jk_1l_1}T^{\varphi^i}_{l_1k_1}(x^{\varphi^i}) + \varepsilon_{jl_1k_1}T^{\varphi^i}_{k_1l_1}(x^{\varphi^i}) = -m^{\varphi^i}_j(x^{\varphi^i}), \qquad j = 1,2,3,\ i = \overline{1,n},$$

where $k_1, l_1 \neq j,\ k_1, l_1 = 1,2,3$, and summation over repeated indices is not implied. From the latter equality it follows, that

$$T^{\varphi^i}_{k_1l_1}(x^{\varphi^i}) - T^{\varphi^i}_{l_1k_1}(x^{\varphi^i}) = \varepsilon_{jk_1l_1}m^{\varphi^i}_j(x^{\varphi^i}), \qquad \forall x^{\varphi_i} \in \overline{\Omega}^{\varphi_i}, i = 1,...,n,$$

and the equality (2.5) is proved. The equation (2.6) directly follows from the point a) of the Principle I and definition of the tensor $\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})$. $\square$

So, we obtain the equations (2.4)-(2.6) for static equilibrium of multicomponent mixture from an Eulerian point of view. In order to express these equations with respect to the initial configuration let us consider Piola transform of the tensors $\boldsymbol{T}^{\varphi^i}$, $\boldsymbol{M}^{\varphi^i}$ and vectors $\boldsymbol{f}^{\varphi^i}$, $\boldsymbol{h}^{\varphi^i}$, $\boldsymbol{g}^{\varphi^i}$. Consequently, we get the tensors $\boldsymbol{T}^i(x)$, $\boldsymbol{M}^i(x)$,

$$\begin{aligned} \boldsymbol{T}^i(x) &= (\det \nabla\boldsymbol{\varphi}^i(x))\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})[\nabla\boldsymbol{\varphi}^i(x)]^{-T}, \\ \boldsymbol{M}^i(x) &= (\det \nabla\boldsymbol{\varphi}^i(x))\boldsymbol{M}^{\varphi^i}(x^{\varphi^i})[\nabla\boldsymbol{\varphi}^i(x)]^{-T}, \end{aligned} \qquad x^{\varphi^i} = \boldsymbol{\varphi}^i(x),$$

which we call the first Piola-Kirchhoff stress tensor and the first moment tensor for i-th constituent respectively. Also, let us introduce the following tensors

$$\boldsymbol{\Sigma}^i(x) = [\nabla\boldsymbol{\varphi}^i(x)]^{-1}\boldsymbol{T}^i(x), \quad \boldsymbol{\theta}^i(x) = [\nabla\boldsymbol{\varphi}^i(x)]^{-1}\boldsymbol{M}^i(x), \quad x^{\varphi^i} = \boldsymbol{\varphi}^i(x),$$

and call them the second Piola-Kirchhoff stress tensor and the second moment tensor of i-th constituent with respect to the initial configuration.

The vector-fields corresponding to the densities of the body forces $\boldsymbol{f}^{\varphi^i}$, $\boldsymbol{h}^{\varphi^i} : \Omega^{\varphi^i} \to \mathbf{R}^3$ transform into $\boldsymbol{f}^i, \boldsymbol{h}^i : \Omega \to \mathbf{R}^3$, where

$$\boldsymbol{f}^i(x) = (\det \nabla\boldsymbol{\varphi}^i(x))\boldsymbol{f}^{\varphi^i}(x^{\varphi^i}), \quad \boldsymbol{h}^i(x) = (\det \nabla\boldsymbol{\varphi}^i(x))\boldsymbol{h}^{\varphi^i}(x^{\varphi^i}),$$

$x^{\varphi^i} = \boldsymbol{\varphi}^i(x)$, and density of the surface forces $\boldsymbol{g}^{\varphi^i} : \Gamma_1^{\varphi^i} \to \mathbf{R}^3$ transform into $\boldsymbol{g}^i : \Gamma_1^i = [\varphi^i]^{-1}(\Gamma_1^{\varphi^i}) \to \mathbf{R}^3$,

$$\boldsymbol{g}^i(x) = (\det \nabla\boldsymbol{\varphi}^i(x)) \left|[\nabla\boldsymbol{\varphi}^i(x)]^{-T}\boldsymbol{\nu}\right| \boldsymbol{g}^{\varphi^i}(x^{\varphi^i}), \quad x^{\varphi^i} = \boldsymbol{\varphi}^i(x) \in \Gamma_1^{\varphi^i},$$

where $\boldsymbol{\nu}$ denotes the unit outer normal of Γ_1^i at point x $(i = \overline{1,n})$. Note, that the correspondence between $\boldsymbol{f}^{\varphi^i}$, $\boldsymbol{h}^{\varphi^i}$, $\boldsymbol{g}^{\varphi^i}$ and $\boldsymbol{f}^i$, $\boldsymbol{h}^i$, $\boldsymbol{g}^i$ is such that $\boldsymbol{f}^i dx = \boldsymbol{f}^{\varphi^i} dx^{\varphi^i}$, $\boldsymbol{h}^i dx = \boldsymbol{h}^{\varphi^i} dx^{\varphi^i}$, $\boldsymbol{g}^i d\sigma = \boldsymbol{g}^{\varphi^i} d\sigma^{\varphi^i}$ $(i = \overline{1,n})$.

Applying properties of Piola transform, we can obtain equations for static equilibrium of the mixture with respect to the initial configuration.

Theorem 2.2. *The equations of static equilibrium for multicomponent elastic mixture with respect to the initial configuration are of the following form:*

$$-\mathbf{div}\boldsymbol{T}^i(x) = \boldsymbol{f}^i(x) + \boldsymbol{h}^i(x), \qquad \forall x \in \Omega, \tag{2.9}$$

$$\boldsymbol{T}^i(x)[\nabla\boldsymbol{\varphi}^i(x)]^T - \nabla\boldsymbol{\varphi}^i(x)[\boldsymbol{T}^i(x)]^T = \boldsymbol{M}^i(x)[\nabla\boldsymbol{\varphi}^i(x)]^T, \ \forall x \in \overline{\Omega}, \tag{2.10}$$

$$\boldsymbol{T}^i(x)\boldsymbol{\nu} = \boldsymbol{g}^i(x), \qquad \forall x \in \Gamma_1^i, \ i = 1, ..., n, \tag{2.11}$$

where $\boldsymbol{\nu}$ is a unit outer normal to Γ_1^i. The system of equations (2.9), (2.11) is formally equivalent to the following variational equations:

$$\int\limits_\Omega \boldsymbol{T}^i : \nabla\boldsymbol{\xi}^i dx = \int\limits_\Omega \boldsymbol{f}^i \cdot \boldsymbol{\xi}^i dx + \int\limits_\Omega \boldsymbol{h}^i \cdot \boldsymbol{\xi}^i dx + \int\limits_{\Gamma_1^i} \boldsymbol{g}^i \cdot \boldsymbol{\xi}^i d\sigma, \quad i = 1, ..., n, \tag{2.12}$$

for sufficiently smooth vector-fields $\boldsymbol{\xi}^i : \overline{\Omega} \to \mathbf{R}^3$, which vanish on $\Gamma_0^i = \Gamma \backslash \Gamma_1^i$ $(i = \overline{1,n})$.

Proof. As well-known Piola transform satisfies

$$\mathbf{div}\boldsymbol{T}^i(x) = (\det \nabla\boldsymbol{\varphi}^i(x))\mathbf{div}^{\varphi^i}\boldsymbol{T}^{\varphi^i}(x^{\varphi^i}), \quad \forall x^{\varphi^i} = \varphi^i(x), \ x \in \overline{\Omega},$$

and from the equations (2.4), (2.5) and definition of $\boldsymbol{T}^i$, $\boldsymbol{M}^i$, $\boldsymbol{f}^i$, $\boldsymbol{h}^i$ we obtain (2.9), (2.10). Taking into account definition of $\boldsymbol{g}^i$ and the property of Piola transform $\boldsymbol{T}^{\varphi^i}\boldsymbol{\nu}^{\varphi^i} d\sigma^{\varphi^i} = \boldsymbol{T}^i\boldsymbol{\nu} d\sigma$, from (2.6) we get (2.11).

The equivalence stated in the theorem follows from Green's formula

$$\int\limits_\Omega \mathbf{div}\boldsymbol{T}^i \cdot \boldsymbol{\xi}^i dx = \int\limits_{\Gamma_1^i} \boldsymbol{T}^i\boldsymbol{\nu} \cdot \boldsymbol{\xi}^i d\sigma - \int\limits_\Omega \boldsymbol{T}^i : \nabla\boldsymbol{\xi}^i dx, \quad i = 1, ..., n. \tag{2.13}$$

Indeed, scalarly multiplying the both sides of the i-th equation (2.9) by $\boldsymbol{\xi}^i$ $(i = \overline{1,n})$, which is equal to zero on Γ_0^i, from Green's formula (2.13) we obtain (2.12). Conversely, if $\boldsymbol{\xi}^i = \mathbf{0}$ on Γ, then from the variational equations (2.12), applying (2.13), we have

$$-\int\limits_{\Omega} \mathbf{div}\boldsymbol{T}^i \cdot \boldsymbol{\xi}^i dx = \int\limits_{\Omega} \boldsymbol{f}^i \cdot \boldsymbol{\xi}^i dx + \int\limits_{\Omega} \boldsymbol{h}^i \cdot \boldsymbol{\xi}^i dx, \qquad i = 1, ..., n,$$

whence (2.9) is proved. Furthermore, from the equations (2.9) and Green's formula we infer that

$$\int\limits_{\Gamma_1^i} \boldsymbol{T}^i \boldsymbol{\nu} \cdot \boldsymbol{\xi}^i d\sigma = \int\limits_{\Gamma_1^i} \boldsymbol{g}^i \cdot \boldsymbol{\xi}^i d\sigma, \quad \boldsymbol{\xi}^i \neq \mathbf{0} \text{ on } \Gamma_1^i,\ i = 1, ..., n,$$

and, consequently, the equality (2.11) is proved. $\square$

Thus, we have constructed general mathematical model of n-component mixture in static equilibrium without any assumption on physical properties of the continuum. In order to determine the stress-strain state of the mixture it is necessary to know the constitutive equations for the stress tensors $\boldsymbol{T}^{\varphi^i}$, moment tensors $\boldsymbol{M}^{\varphi^i}$ and for the interaction forces $\boldsymbol{h}^{\varphi^i}$ between the components, which characterize the physical properties of the material.

Further we consider the so-called elastic mixtures, the stress tensors of which depend on gradients of deformations. More precisely, there exist response functions for the stress tensors of the constituents with respect to the deformed configurations, such that

$$\boldsymbol{T}^{\varphi^i}(x^{\varphi^i}) = \widetilde{\boldsymbol{T}}^i_d\left(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)\right), \qquad x^{\varphi^i} = \boldsymbol{\varphi}^i(x),\ i = \overline{1,n}.$$

$\widetilde{\boldsymbol{T}}^i_d$ defines the mappings $\widetilde{\boldsymbol{T}}^i$, $\widetilde{\boldsymbol{\Sigma}}^i : \overline{\Omega} \times \mathbf{R}^{3\times3}_+ \to \mathbf{R}^{3\times3}$,

$$\begin{aligned} \widetilde{\boldsymbol{T}}^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) &= (\det \boldsymbol{F}_i)\widetilde{\boldsymbol{T}}^i_d(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n)\boldsymbol{F}_i^{-T}, \\ \widetilde{\boldsymbol{\Sigma}}^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) &= (\det \boldsymbol{F}_i)\boldsymbol{F}_i^{-1}\widetilde{\boldsymbol{T}}^i_d(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n)\boldsymbol{F}_i^{-T}, \end{aligned} \qquad \forall x \in \overline{\Omega},$$

$\boldsymbol{F}_i \in \mathbf{R}^{3\times3}_+$, $i = \overline{1,n}$, which satisfy

$$\begin{aligned} \boldsymbol{T}^i(x) &= \widetilde{\boldsymbol{T}}^i(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)), \\ \boldsymbol{\Sigma}^i(x) &= \widetilde{\boldsymbol{\Sigma}}^i(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)). \end{aligned} \qquad \forall x \in \overline{\Omega},\ i = \overline{1,n},$$

$\widetilde{\boldsymbol{T}}^i$, $\widetilde{\boldsymbol{\Sigma}}^i$ are called the response functions of the first and second Piola-Kirchhoff stress tensors of i-th constituent respectively.

Therefore, stress tensors of the components of elastic mixture are uniquely determined by the mappings $\widetilde{\boldsymbol{T}}^i_d$ and, if they do not depend on $x \in \overline{\Omega}$, then the mixture is called homogeneous.

It must be pointed out that quantities, which characterize physical processes are independent of the frame of reference. Now we formulate the frame-independence principle for mixtures, but before note that rotation of the deformed configuration and orthogonal transformation of the coordinates are equivalent procedures.

Principle II. Let the deformed configuration $\overline{\Omega}^{\psi^i}$ be obtained from the configuration $\overline{\Omega}^{\varphi^i}$ by rotation, which is given by the matrix $\boldsymbol{Q}$, i.e. $\boldsymbol{\psi}^i = \boldsymbol{Q}\boldsymbol{\varphi}^i$, $i = 1, ..., n$, $\boldsymbol{Q} \in \boldsymbol{O}^3_+$. This rotation transforms the stress vectors of the mixture in the first configuration to corresponding stress vectors in the second configuration:

$$\boldsymbol{t}^{\psi^i}(x^{\psi^i}, \boldsymbol{Q}\boldsymbol{\nu}) = \boldsymbol{Q}\boldsymbol{t}^{\varphi^i}(x^{\varphi^i}, \boldsymbol{\nu}), \quad x^{\psi^i} = \boldsymbol{\psi}^i(x),\ x^{\varphi^i} = \boldsymbol{\varphi}^i(x), \tag{2.14}$$

where $x \in \overline{\Omega}$, $\boldsymbol{\nu} \in S_1$, $\boldsymbol{t}^{\psi^i} : \overline{\Omega}^{\psi^i} \times S_1 \to \mathbf{R}^3$, $\boldsymbol{t}^{\varphi^i} : \overline{\Omega}^{\varphi^i} \times S_1 \to \mathbf{R}^3$ are stress vectors in the deformed configurations $\overline{\Omega}^{\psi^i}$, $\overline{\Omega}^{\varphi^i}$ respectively ($i = \overline{1, n}$).

In the following theorem we formulate the necessary and sufficient conditions, when (2.14) is fulfilled.

Theorem 2.3. *An elastic mixture satisfies the Principle II if the response functions* $\widetilde{\boldsymbol{T}}^i_d : \overline{\Omega} \times \left[\mathbf{R}^{3\times3}_+\right]^n \to \mathbf{R}^{3\times3}$ *of the stress tensors satisfy the following conditions for all* $\boldsymbol{F}_1, ..., \boldsymbol{F}_n \in \mathbf{R}^{3\times3}_+$, $\boldsymbol{Q} \in \boldsymbol{O}^3_+$,

$$\widetilde{\boldsymbol{T}}^i_d(x, \boldsymbol{Q}\boldsymbol{F}_1, ..., \boldsymbol{Q}\boldsymbol{F}_n) = \boldsymbol{Q}\widetilde{\boldsymbol{T}}^i_d(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n)\boldsymbol{Q}^T, \qquad i = \overline{1, n}.$$

Proof. From the condition (2.14) of the Principle II, we obtain

$$\boldsymbol{t}^{\psi^i}(x^{\psi^i}, \boldsymbol{Q}\boldsymbol{\nu}) = \boldsymbol{T}^{\psi^i}(x^{\psi^i})\boldsymbol{Q}\boldsymbol{\nu} = \boldsymbol{Q}\boldsymbol{t}^{\varphi^i}(x^{\varphi^i}, \boldsymbol{\nu}) = \boldsymbol{Q}\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})\boldsymbol{\nu},$$

for all $\boldsymbol{Q} \in \boldsymbol{O}^3_+$, where $\boldsymbol{\nu} \in S_1$, $\boldsymbol{T}^{\varphi^i}$, $\boldsymbol{T}^{\psi^i}$ are stress tensors of i-th constituent with respect to the deformed configurations $\overline{\Omega}^{\varphi^i}$, $\overline{\Omega}^{\psi^i}$ respectively. Therefore,

$$\boldsymbol{T}^{\psi^i}(x^{\psi^i}) = \boldsymbol{Q}\boldsymbol{T}^{\varphi^i}(x^{\varphi^i})\boldsymbol{Q}^T, \qquad \forall \boldsymbol{Q} \in \boldsymbol{O}^3_+,\ i = 1, ..., n.$$

Since $\boldsymbol{\psi}^i(x) = \boldsymbol{Q}\boldsymbol{\varphi}^i(x)$, then $\nabla\boldsymbol{\psi}^i(x) = \boldsymbol{Q}\nabla\boldsymbol{\varphi}^i(x)$, $i = \overline{1, n}$, and taking into account definition of the response function of the stress tensor we deduce, that the mixture satisfies the Principle II if and only if

$$\widetilde{\boldsymbol{T}}^i_d\left(x, \nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x)\right) = \widetilde{\boldsymbol{T}}^i_d\left(x, \boldsymbol{Q}\nabla\boldsymbol{\varphi}^1(x), ..., \boldsymbol{Q}\nabla\boldsymbol{\varphi}^n(x)\right) =$$

$$= \boldsymbol{Q}\widetilde{\boldsymbol{T}}^i_d\left(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)\right)\boldsymbol{Q}^T, \qquad \forall \boldsymbol{Q} \in \boldsymbol{O}^3_+,\ i = 1, ..., n.$$

Note that for any matrix $\boldsymbol{F}_i \in \mathbf{R}_+^{3\times3}$, there exists deformation $\boldsymbol{\varphi}^i$, such that $\nabla\boldsymbol{\varphi}^i = \boldsymbol{F}_i$, whence the equivalence stated in the theorem is proved. $\square$

From the latter theorem it follows that condition (2.14) of the Principle II can be expressed in terms of the response functions of the first and second Piola-Kirchhoff stress tensors. More precisely, the condition (2.14) is equivalent to

$$\widetilde{\boldsymbol{T}}^i(x, \boldsymbol{QF}_1, ..., \boldsymbol{QF}_n) = \boldsymbol{Q}\widetilde{\boldsymbol{T}}^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n), \quad \forall \boldsymbol{F}_1, ..., \boldsymbol{F}_n \in \mathbf{R}_+^{3\times3}, \ \boldsymbol{Q} \in \boldsymbol{O}_+^3,$$

$i = 1, ..., n$, or

$$\widetilde{\boldsymbol{\Sigma}}^i(x, \boldsymbol{QF}_1, ..., \boldsymbol{QF}_n) = \widetilde{\boldsymbol{\Sigma}}^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n), \quad \forall \boldsymbol{F}_1, ..., \boldsymbol{F}_n \in \mathbf{R}_+^{3\times3}, \ \boldsymbol{Q} \in \boldsymbol{O}_+^3.$$

Now let us introduce the subclass of elastic mixtures, which are called isotropic elastic mixtures. Let $\boldsymbol{\varphi}$ be a rotation of the initial configuration $\overline{\Omega}$ around the point $x \in \Omega$, which is given by the matrix $\boldsymbol{Q}^T$, i.e.

$$\boldsymbol{\varphi}(y) = x + \boldsymbol{Q}^T \boldsymbol{xy}, \qquad y \in \overline{\Omega},$$

$\boldsymbol{xy}$ is a vector with the origin in x and the end in y. Note, that the deformed configurations $\boldsymbol{\varphi}^i(\overline{\Omega})$ can be considered with respect to new initial configuration $\boldsymbol{\varphi}(\overline{\Omega})$. Then, instead of the deformation $\boldsymbol{\varphi}^i$ we have

$$\widetilde{\boldsymbol{\varphi}}^i = \boldsymbol{\varphi}^i \circ \boldsymbol{\varphi}^{-1} : z \in \boldsymbol{\varphi}(\overline{\Omega}) \to \boldsymbol{\varphi}^i(x + \boldsymbol{Qxz}), \qquad i = 1, ..., n,$$

which are deformations too. In point $x^{\widetilde{\varphi}^i}$ stress tensor is equal to

$$\boldsymbol{T}^{\widetilde{\varphi}^i}(x^{\widetilde{\varphi}^i}) = \widetilde{\boldsymbol{T}}_d^i\left(x, \nabla\widetilde{\boldsymbol{\varphi}}^1(x), ..., \nabla\widetilde{\boldsymbol{\varphi}}^n(x)\right) = \widetilde{\boldsymbol{T}}_d^i\left(x, \nabla\boldsymbol{\varphi}^1(x)\boldsymbol{Q}, ..., \nabla\boldsymbol{\varphi}^n(x)\boldsymbol{Q}\right).$$

Under the isotropy of the material usually mean that the reactions in all directions are the same, i.e. $\boldsymbol{T}^{\widetilde{\varphi}^i}(x^{\widetilde{\varphi}^i}) = \boldsymbol{T}^{\varphi^i}(x^{\varphi^i})$, $i = 1, ..., n$. Thus, an elastic mixture is called isotropic in point x of the initial configuration $\overline{\Omega}$ if the response functions of the stress tensors satisfy the following identities for all $\boldsymbol{F}_1, ..., \boldsymbol{F}_n \in \mathbf{R}_+^{3\times3}$, $\boldsymbol{Q} \in \boldsymbol{O}_+^3$,

$$\widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1\boldsymbol{Q}, ..., \boldsymbol{F}_n\boldsymbol{Q}) = \widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n), \qquad i = 1, ..., n.$$

An elastic mixture is called isotropic if it is isotropic in all points of the initial configuration. Note that the condition of isotropy can be expressed in terms of the first and second Piola-Kirchhoff stress tensors. Particularly, an elastic mixture is isotropic in point $x \in \overline{\Omega}$ if one of the following conditions holds:

$$\widetilde{\boldsymbol{T}}^i(x, \boldsymbol{F}_1\boldsymbol{Q}, ..., \boldsymbol{F}_n\boldsymbol{Q}) = \widetilde{\boldsymbol{T}}^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n)\boldsymbol{Q}, \ \forall \boldsymbol{F}_i \in \mathbf{R}_+^{3\times3}, \boldsymbol{Q} \in \boldsymbol{O}_+^3, i = \overline{1, n},$$

$$\widetilde{\boldsymbol{\Sigma}}^i(x,\boldsymbol{F}_1\boldsymbol{Q},...,\boldsymbol{F}_n\boldsymbol{Q}) = \boldsymbol{Q}^T\widetilde{\boldsymbol{\Sigma}}^i(x,\boldsymbol{F}_1,...,\boldsymbol{F}_n)\boldsymbol{Q}, \forall \boldsymbol{F}_i \in \mathbf{R}_+^{3\times 3}, \boldsymbol{Q}\in \boldsymbol{O}_+^3, i=\overline{1,n}.$$

It must be pointed out that the response functions of the stress tensors of isotropic mixture depend only on special products of gradients of the deformations. More precisely, the following theorem is valid.

Theorem 2.4. *An elastic mixture is isotropic in point $x \in \overline{\Omega}$ if and only if there exist mappings*

$$\widehat{\boldsymbol{T}}_d^i(x,.) : \mathcal{F} \to \mathbf{R}^{3\times 3},\ \mathcal{F} = \Big\{\{\boldsymbol{F}_{ip}\} \,|\, \boldsymbol{F}_{ip} = \boldsymbol{F}_i\boldsymbol{F}_p^T, \boldsymbol{F}_i, \boldsymbol{F}_p \in \mathbf{R}_+^{3\times 3}, i,p=\overline{1,n}\Big\},$$

such that

$$\widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1,...,\boldsymbol{F}_n) = \widehat{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1\boldsymbol{F}_1^T,...,\boldsymbol{F}_1\boldsymbol{F}_n^T,...,\boldsymbol{F}_n\boldsymbol{F}_1^T,...,\boldsymbol{F}_n\boldsymbol{F}_n^T),$$

for all $\boldsymbol{F}_1,...,\boldsymbol{F}_n \in \mathbf{R}_+^{3\times 3},\ i=1,...,n.$

Proof. In order to prove the existence of the mapping $\widehat{\boldsymbol{T}}_d^i$ it suffices to show that $\widetilde{\boldsymbol{T}}_d^i$ depends only on $\boldsymbol{F}_i\boldsymbol{F}_j^T,\ i,j=\overline{1,n}$. Let $\{\boldsymbol{F}_1,...,\boldsymbol{F}_n\}$, $\{\boldsymbol{G}_1,...,\boldsymbol{G}_n\}$ be such that $\boldsymbol{F}_i\boldsymbol{F}_j^T = \boldsymbol{G}_i\boldsymbol{G}_j^T,\ i,j=\overline{1,n}$. Hence $\boldsymbol{F}_j^T\boldsymbol{G}_j^{-T} = \boldsymbol{F}_i^{-1}\boldsymbol{G}_i,\ i,j=\overline{1,n}$.

Since $\boldsymbol{F}_i\boldsymbol{F}_i^T = \boldsymbol{G}_i\boldsymbol{G}_i^T$, then $(\boldsymbol{F}_i^{-1}\boldsymbol{G}_i)(\boldsymbol{F}_i^{-1}\boldsymbol{G}_i)^T = \boldsymbol{I}$ and, therefore, the matrix $\boldsymbol{F}_i^{-1}\boldsymbol{G}_i$ is orthogonal for all $i=\overline{1,n}$. Furthermore, $\boldsymbol{F}_1^T\boldsymbol{G}_1^{-T} = \boldsymbol{F}_1^{-1}\boldsymbol{G}_1 = \boldsymbol{F}_2^{-1}\boldsymbol{G}_2 = ... = \boldsymbol{F}_n^{-1}\boldsymbol{G}_n$, whence, for all $i=\overline{1,n}$, we obtain:

$$\widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1,...,\boldsymbol{F}_n) = \widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1(\boldsymbol{F}_1^{-1}\boldsymbol{G}_1),...,\boldsymbol{F}_n(\boldsymbol{F}_n^{-1}\boldsymbol{G}_n)) = \widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{G}_1,...,\boldsymbol{G}_n).$$

Now assume that there exist mappings $\widehat{\boldsymbol{T}}_d^i$ with the properties stated in the theorem, then

$$\widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1\boldsymbol{Q},...,\boldsymbol{F}_n\boldsymbol{Q}) = \widehat{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{F}_1^T,\boldsymbol{F}_1\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{F}_2^T,...,\boldsymbol{F}_n\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{F}_n^T) =$$

$$= \widehat{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1\boldsymbol{F}_1^T,\boldsymbol{F}_1\boldsymbol{F}_2^T,...,\boldsymbol{F}_n\boldsymbol{F}_n^T) = \widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1,...,\boldsymbol{F}_n). \ \square$$

Note that if the response functions of the stress tensors of the constituents satisfy more strict conditions

$$\widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1\boldsymbol{Q}_1,...,\boldsymbol{F}_n\boldsymbol{Q}_n) = \widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1,...,\boldsymbol{F}_n), \quad \forall \boldsymbol{F}_i \in \mathbf{R}_+^{3\times 3},\ \boldsymbol{Q}_i \in \boldsymbol{O}_+^3,$$

$i=\overline{1,n}$, then the mixture is called strongly isotropic and the following theorem is true.

Theorem 2.5. *An elastic mixture is strongly isotropic in $x \in \overline{\Omega}$ if there exist mappings $\overline{\boldsymbol{T}}_d^i(x,.) : \left[\boldsymbol{S}_>^3\right]^n \to \mathbf{R}^{3\times 3}$ such that*

$$\widetilde{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1,...,\boldsymbol{F}_n) = \overline{\boldsymbol{T}}_d^i(x,\boldsymbol{F}_1\boldsymbol{F}_1^T,...,\boldsymbol{F}_n\boldsymbol{F}_n^T), \qquad \forall \boldsymbol{F}_i \in \mathbf{R}_+^{3\times 3},\ i=\overline{1,n}.$$

Proof. Let $\boldsymbol{F}_i, \boldsymbol{G}_i \in \mathbf{R}_+^{3\times 3}$ be such that $\boldsymbol{F}_i\boldsymbol{F}_i^T = \boldsymbol{G}_i\boldsymbol{G}_i^T$ $(i = \overline{1,n})$. Then $(\boldsymbol{F}_i^{-1}\boldsymbol{G}_i)(\boldsymbol{F}_i^{-1}\boldsymbol{G}_i)^T = \boldsymbol{I}$, and, consequently, $\boldsymbol{F}_i^{-1}\boldsymbol{G}_i$ is orthogonal matrix. Since $\det(\boldsymbol{F}_i^{-1}\boldsymbol{G}_i) > 0$, from the definition of strong isotropy, for all $i = \overline{1,n}$, we obtain

$$\widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) = \widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1(\boldsymbol{F}_1^{-1}\boldsymbol{G}_1), ..., \boldsymbol{F}_n(\boldsymbol{F}_n^{-1}\boldsymbol{G}_n)) = \widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{G}_1, ..., \boldsymbol{G}_n).$$

Also, if $\widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) = \overline{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1\boldsymbol{F}_1^T, \boldsymbol{F}_2\boldsymbol{F}_2^T, ..., \boldsymbol{F}_n\boldsymbol{F}_n^T)$, then

$$\widetilde{\boldsymbol{T}}_d^i(\boldsymbol{F}_1\boldsymbol{Q}_1, ..., \boldsymbol{F}_n\boldsymbol{Q}_n) = \overline{\boldsymbol{T}}_d^i\left(x, \boldsymbol{F}_1\boldsymbol{Q}_1\boldsymbol{Q}_1^T\boldsymbol{F}_1^T, ..., \boldsymbol{F}_n\boldsymbol{Q}_n\boldsymbol{Q}_n^T\boldsymbol{F}_n^T\right) =$$
$$= \overline{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1\boldsymbol{F}_1^T, \boldsymbol{F}_2\boldsymbol{F}_2^T, ..., \boldsymbol{F}_n\boldsymbol{F}_n^T) = \widetilde{\boldsymbol{T}}_d^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n). \square$$

3. *Boundary value problems for nonlinear elastic mixtures*

In the present section we use notations of the section 2 and also apply some known properties of the Sobolev spaces. Denote by $W^{p,q}(D)$, $p, q \geq 1$, the usual Sobolev space of order p with respect to $L^q(D)$, where $D \subset \mathbf{R}^3$ is a Lipschitz domain. In the case of $q = 2$ the space $W^{p,q}(D)$ is denoted by $H^p(D)$ $(H^0(D) = L^2(D))$ and let $H_0^p(D)$ be the closure of the set $C_0^\infty(D)$ of infinitely differentiable functions with compact support in D in the space $H^p(D)$. For the spaces of vector-functions we use the following notations $\mathbf{H}^p(D) = [H^p(D)]^3$, $\mathbf{W}^{p,q}(D) = [W^{p,q}(D)]^3$, $\mathbf{H}_0^p(D) = [H_0^p(D)]^3$, $\mathbf{L}^p(D) = [L^p(D)]^3$, $p, q \geq 1$. Also $\boldsymbol{L}_{3\times 3}^q(D)$ denotes the set of tensor-valued functions $\boldsymbol{F} : D \to \mathbf{R}^{3\times 3}$ such that each element F_{kl} belongs to $L^q(D)$ and $\|\boldsymbol{F}\|_{\boldsymbol{L}^q(D)} = (\sum\limits_{k,l=1}^{3} \|F_{kl}\|_{L^q(D)}^q)^{1/q}$.

Let us consider two-component homogeneous isotropic elastic mixture with the initial configuration $\overline{\Omega} \subset \mathbf{R}^3$, the boundary Γ of which is clamped. According to the Theorem 2.2 the stress-strain state of the mixture in static equilibrium is determined from the solution of the following boundary value problem

$$\begin{aligned} -\mathbf{div}\left(\nabla\boldsymbol{\varphi}^1(x)\boldsymbol{\Sigma}^1(x)\right) &= \boldsymbol{f}^1(x) + \boldsymbol{h}^1(x), \\ -\mathbf{div}\left(\nabla\boldsymbol{\varphi}^2(x)\boldsymbol{\Sigma}^2(x)\right) &= \boldsymbol{f}^2(x) + \boldsymbol{h}^2(x), \end{aligned} \qquad x \in \Omega, \qquad (3.1)$$

$$\boldsymbol{u}^1(x) = \boldsymbol{u}^2(x) = \boldsymbol{0}, \qquad x \in \Gamma, \qquad (3.2)$$

written in terms of the second Piola-Kirchhoff stress tensors, where we assume that the densities of the external body forces $\boldsymbol{f}^1, \boldsymbol{f}^2$ with respect to the initial configuration do not depend on the deformations $\boldsymbol{\varphi}^1$, $\boldsymbol{\varphi}^2$ of the constituents.

Since we consider the homogeneous isotropic elastic mixture, its Piola-Kirchhoff stress tensors depend only on gradients $\nabla\boldsymbol{\varphi}^1$, $\nabla\boldsymbol{\varphi}^2$ of the deformations of the constituents:

$$\boldsymbol{\Sigma}^1(x) = \widetilde{\boldsymbol{\Sigma}}^1(\nabla\boldsymbol{\varphi}^1(x), \nabla\boldsymbol{\varphi}^2(x)),\ \boldsymbol{\Sigma}^2(x) = \widetilde{\boldsymbol{\Sigma}}^2(\nabla\boldsymbol{\varphi}^1(x), \nabla\boldsymbol{\varphi}^2(x)), \quad x \in \overline{\Omega},$$

where $\widetilde{\boldsymbol{\Sigma}}^1$, $\widetilde{\boldsymbol{\Sigma}}^2$ satisfy conditions caused by frame-independence principle and isotropy of the mixture

$$\begin{array}{l} \widetilde{\boldsymbol{\Sigma}}^\alpha(\boldsymbol{Q}\boldsymbol{F}_1, \boldsymbol{Q}\boldsymbol{F}_2) = \widetilde{\boldsymbol{\Sigma}}^\alpha(\boldsymbol{F}_1, \boldsymbol{F}_2), \\ \widetilde{\boldsymbol{\Sigma}}^\alpha(\boldsymbol{F}_1\boldsymbol{Q}, \boldsymbol{F}_2\boldsymbol{Q}) = \boldsymbol{Q}^T\widetilde{\boldsymbol{\Sigma}}^\alpha(\boldsymbol{F}_1, \boldsymbol{F}_2)\boldsymbol{Q}. \end{array} \quad \forall \boldsymbol{F}_\alpha \in \mathbf{R}_+^{3\times 3},\ \boldsymbol{Q} \in \boldsymbol{O}_+^3,\ \alpha = 1, 2,$$

Let us introduce a denotation $\boldsymbol{E}^\alpha \equiv \dfrac{1}{2}\left([\nabla\boldsymbol{\varphi}^\alpha]^T\nabla\boldsymbol{\varphi}^\alpha - \boldsymbol{I}\right)$, $\alpha = 1, 2$, and assume, that the second Piola-Kirchhoff stress tensors $\widetilde{\boldsymbol{\Sigma}}^1$, $\widetilde{\boldsymbol{\Sigma}}^2$ are of the following form:

$$\widetilde{\boldsymbol{\Sigma}}^1(\nabla\boldsymbol{\varphi}^1, \nabla\boldsymbol{\varphi}^2) = \left\{\lambda_1 tr\boldsymbol{E}^1 + \lambda_3 tr\boldsymbol{E}^2\right\}\boldsymbol{I} + 2\mu_1\boldsymbol{E}^1 + 2\mu_3\boldsymbol{E}^2 + \lambda_5\widetilde{\boldsymbol{\theta}},$$

$$\widetilde{\boldsymbol{\Sigma}}^2(\nabla\boldsymbol{\varphi}^1, \nabla\boldsymbol{\varphi}^2) = \left\{\lambda_4 tr\boldsymbol{E}^1 + \lambda_2 tr\boldsymbol{E}^2\right\}\boldsymbol{I} + 2\mu_3\boldsymbol{E}^1 + 2\mu_2\boldsymbol{E}^2 - \lambda_5\widetilde{\boldsymbol{\theta}},$$

where $\widetilde{\boldsymbol{\theta}} = \left[\nabla\boldsymbol{\varphi}^1\right]^T\nabla\boldsymbol{\varphi}^1 + \left[\nabla\boldsymbol{\varphi}^2\right]^T\nabla\boldsymbol{\varphi}^2 - 2\left[\nabla\boldsymbol{\varphi}^2\right]^T\nabla\boldsymbol{\varphi}^1$. The interaction force between the constituents of the mixture is given by

$$-\boldsymbol{h}^1 = \boldsymbol{h}^2 = \boldsymbol{\pi} = \frac{\alpha_2\rho_2}{\rho}\mathbf{grad}(tr\boldsymbol{E}^1) + \frac{\alpha_2\rho_1}{\rho}\mathbf{grad}(tr\boldsymbol{E}^1), \quad \rho = \rho_1 + \rho_2,$$

where $\mathbf{grad}$ denotes the gradient of the function, the parameters $\lambda_s(s = \overline{1,5})$, $\mu_k(k = \overline{1,3})$, α_2 characterize mechanical properties of the mixture, ρ_1, ρ_2 are densities of the constituents.

Let $\mathbf{u} = \left(\boldsymbol{u}^1, \boldsymbol{u}^2\right)^T$, $\mathbf{f} = \left(\boldsymbol{f}^1, \boldsymbol{f}^2\right)^T$ and

$$A\mathbf{u} = \begin{pmatrix} -\mathbf{div}\left\{\nabla\boldsymbol{\varphi}^1(x)\boldsymbol{\Sigma}^1(x)\right\} + \boldsymbol{\pi} \\ -\mathbf{div}\left\{\nabla\boldsymbol{\varphi}^2(x)\boldsymbol{\Sigma}^2(x)\right\} - \boldsymbol{\pi} \end{pmatrix}.$$

For the Dirichlet boundary value problem (3.1), (3.2) formulated for nonlinear model of elastic mixture the following existence and uniqueness theorem is valid.

Theorem 3.1. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with boundary $\Gamma = \partial\Omega$ of class $\mathbf{C}^s$, $s \geq 3$. If the following conditions are fulfilled:*

$$\mu_1 > 0, \mu_3^2 < \mu_1\mu_2, \lambda_5 \leq 0, \lambda_3 - \lambda_4 = \alpha_2,\ \lambda_1 + \frac{2\mu_1}{3} - \frac{\alpha_2\rho_2}{\rho} > 0,$$
$$\left(\lambda_3 + \frac{2\mu_3}{3} - \frac{\rho_1\alpha_2}{\rho}\right)^2 < \left(\lambda_1 + \frac{2\mu_1}{3} - \frac{\rho_2\alpha_2}{\rho}\right)\left(\lambda_2 + \frac{2\mu_2}{3} + \frac{\rho_1\alpha_2}{\rho}\right),$$

then there exists a neighbourhood W^s of $\mathbf{0}$ in $[\mathbf{H}^{s-2}(\Omega)]^2$ and neighbourhood U^s of $\mathbf{0}$ in the space

$$V(\Omega) = \{\mathbf{u}|\mathbf{u} \in [\mathbf{H}^s(\Omega)]^2, \mathbf{u}|_\Gamma = 0\}$$

such that, for each $\mathbf{f} \in W^s$, the boundary value problem (3.1), (3.2)

$$A\mathbf{u} = \mathbf{f}, \tag{3.3}$$

has a unique solution $\mathbf{u} \in U^s$.

Proof. As well-known, the Sobolev space $\mathbf{H}^s(\Omega)$ is a Banach algebra if $s \geq 2$. Therefore, the nonlinear operator A maps $[\mathbf{H}^s(\Omega)]^2$ to the space $[\mathbf{H}^{s-2}(\Omega)]^2$ and is differentiable in Fréchet sense, since is a polynomial of the third degree with respect to the partial derivatives of $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$.

Note that $\mathbf{u} = \mathbf{0}$ is a solution to the problem (3.3), for $\mathbf{f} = \mathbf{0}$ and, hence, to prove the theorem it suffices to show that the operator A is locally invertible in the neighbourhood of $\mathbf{0}$. Let $A'(\mathbf{0})$ be derivative of A in $\mathbf{0}$. Then

$$A'(\mathbf{0})\mathbf{u} = \overline{\mathbf{f}}, \qquad \overline{\mathbf{f}} \in [\mathbf{H}^{s-2}(\Omega)]^2, \tag{3.4}$$

is the Dirichlet problem for the linear model of two-component elastic mixture

$$-\frac{\partial\sigma_{lj}^\alpha}{\partial x_l}(\boldsymbol{u}^1, \boldsymbol{u}^2) - (-1)^\alpha\overline{\pi}_j(\boldsymbol{u}^1, \boldsymbol{u}^2) = \overline{f}_j^\alpha(x), \qquad x \in \Omega,$$

$$\boldsymbol{u}^1(x) = \boldsymbol{u}^2(x) = \mathbf{0}, \qquad x \in \Gamma,$$

where $\alpha = 1, 2$,

$$\sigma_{lj}^1 = (\lambda_1 e_{kk}^1 + \lambda_3 e_{mm}^2)\delta_{lj} + 2\mu_1 e_{lj}^1 + 2\mu_3 e_{lj}^2 - \lambda_5\overline{\theta}_{lj},$$
$$\sigma_{lj}^2 = (\lambda_4 e_{kk}^1 + \lambda_2 e_{mm}^2)\delta_{lj} + 2\mu_3 e_{lj}^1 + 2\mu_2 e_{lj}^2 + \lambda_5\overline{\theta}_{lj},$$
$$\overline{\pi}_j = \frac{\alpha_2\rho_2}{\rho}\frac{\partial e_{kk}^1}{\partial x_j} + \frac{\alpha_2\rho_1}{\rho}\frac{\partial e_{mm}^2}{\partial x_j},\ e_{kl}^\alpha = \frac{1}{2}\left(\frac{\partial u_k^\alpha}{\partial x_l} + \frac{\partial u_l^\alpha}{\partial x_k}\right),$$
$$\overline{\theta}_{lj} = \frac{\partial u_j^1}{\partial x_l} - \frac{\partial u_l^1}{\partial x_j} + \frac{\partial u_l^2}{\partial x_j} - \frac{\partial u_j^2}{\partial x_l},\ \ j, k, l, m = \overline{1, 3}.$$

This problem, under the conditions of theorem, has a unique solution $\mathbf{u} \in \left[\mathbf{H}_0^1 \cap \mathbf{H}^2(\Omega)\right]^2$, for $\overline{\mathbf{f}} \in \left[\mathbf{L}^2(\Omega)\right]^2$, and if $\overline{\mathbf{f}} \in \left[\mathbf{H}^{s-2}(\Omega)\right]^2$, then $\mathbf{u} \in \left[\mathbf{H}_0^1 \cap \mathbf{H}^s(\Omega)\right]^2$.

So, the linear continuous operator $A'(\mathbf{0}) : V(\Omega) \to \left[\mathbf{H}^{s-2}(\Omega)\right]^2$ is a bijective mapping. Due to open mapping theorem ([21]) linear continuous bijective mapping from one Banach space to another is an isomorphism and, therefore, A is locally invertible. □

Let us prove now that $\boldsymbol{\varphi}^\alpha = \boldsymbol{id} + \boldsymbol{u}^\alpha : \overline{\Omega} \to \mathbf{R}^3$, $\alpha = 1, 2$, where $\mathbf{u} = \left(\boldsymbol{u}^1, \boldsymbol{u}^2\right)^T$ is a solution to the nonlinear Dirichlet problem (3.1), (3.2), are deformations, i.e. $\det \nabla \boldsymbol{\varphi}^\alpha > 0$, $\alpha = 1, 2$, and are injective mappings.

Theorem 3.2. *If all the conditions of Theorem 3.1 are fulfilled, then there exists $\varepsilon_s > 0$ such that for each $\mathbf{f} \in W^s$, $\|\mathbf{f}\|_{[\mathbf{H}^{s-2}(\Omega)]^2} < \varepsilon_s$, the corresponding $\boldsymbol{\varphi}^\alpha = \boldsymbol{id} + \boldsymbol{u}^\alpha$, $\mathbf{u} = (\boldsymbol{u}^\alpha) \in U^s$ satisfy the following conditions:*

$$\det \nabla \boldsymbol{\varphi}^1(x) > 0, \ \det \nabla \boldsymbol{\varphi}^2(x) > 0, \qquad \forall x \in \overline{\Omega},$$

$\boldsymbol{\varphi}^\alpha : \overline{\Omega} \to \mathbf{R}^3$ *is an injective mapping,* $\boldsymbol{\varphi}^\alpha(\Omega) = \Omega$, $\boldsymbol{\varphi}^\alpha(\overline{\Omega}) = \overline{\Omega}$, $\alpha = 1, 2$.

Proof. According to the proof of the Theorem 3.1 the mapping $W^s \to U^s$ is continuous and since the space $[\mathbf{H}^s(\Omega)]^2$, $s \geq 3$, is continuously embedded in $\left[\mathbf{C}^1(\Omega)\right]^2$, there exists $\varepsilon_s > 0$ such that for any $\mathbf{f} \in W^s$, $\|\mathbf{f}\|_{[\mathbf{H}^{s-2}(\Omega)]^2} < \varepsilon_s$, we have $\sup\limits_{x \in \overline{\Omega}} \left(\|\nabla \boldsymbol{u}^1(x)\| + \|\nabla \boldsymbol{u}^2(x)\|\right) < 1$, and, consequently, $\det \nabla \boldsymbol{\varphi}^1(x) > 0$, $\det \nabla \boldsymbol{\varphi}^2(x) > 0$, for all $x \in \overline{\Omega}$. Note that $\boldsymbol{\varphi}^\alpha(x) = \boldsymbol{id}(x)$, $x \in \Gamma = \partial\Omega$, then from the latter inequalities it follows ([20]), that $\boldsymbol{\varphi}^\alpha(\Omega) = \Omega$, $\boldsymbol{\varphi}^\alpha(\overline{\Omega}) = \overline{\Omega}$ and $\boldsymbol{\varphi}^\alpha$ is an injective mapping ($\alpha = 1, 2$). □

So, the homogeneous Dirichlet boundary value problem for nonlinear model (3.1) of elastic mixture locally in the neighbourhood of $\mathbf{0}$ in the space $[\mathbf{H}^s(\Omega)]^2$, $s \geq 3$, has a unique solution, when the density $\mathbf{f}$ of the applied body forces belongs to neighbourhood of $\mathbf{0}$ in the space $\left[\mathbf{H}^{s-2}(\Omega)\right]^2$.

Further we investigate the existence of the global solution to the Dirichlet problem for nonlinear model of multicomponent elastic mixture, but before let us introduce a new class of the so-called hyperelastic mixtures.

An elastic mixture is called hyperelastic, if there exist tensor-fields $\boldsymbol{H}^i : \overline{\Omega} \to \mathbf{R}^{3\times 3}$ and the function $\widetilde{W} : \overline{\Omega} \times \left[\mathbf{R}_+^{3\times 3}\right]^n \to \mathbf{R}$, such that $\mathbf{div}\boldsymbol{H}^i(x) = \boldsymbol{h}^i(x)$, for all $x \in \overline{\Omega}$, $i = \overline{1,n}$, $\widetilde{W}(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n)$ is differentiable with respect to $\boldsymbol{F}_i$ for all $x \in \overline{\Omega}$ and

$$\widetilde{\boldsymbol{T}}^i(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) + \boldsymbol{H}^i(x) = \frac{\partial \widetilde{W}}{\partial \boldsymbol{F}_i}(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n), \qquad \forall \boldsymbol{F}_i \in \mathbf{R}_+^{3\times 3}, \ i = \overline{1,n},$$

or

$$\widetilde{T}^i_{jk}(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) + H^i_{jk}(x) = \frac{\partial \widetilde{W}}{\partial (\boldsymbol{F}_i)_{jk}}(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n), \quad \forall \boldsymbol{F}_i \in \mathbf{R}^{3\times 3}_+, i = \overline{1,n}.$$

Consequently, the Dirichlet problem for hyperelastic mixtures can be written as follows:

$$\begin{aligned} -\mathbf{div}\frac{\partial \widetilde{W}}{\partial \boldsymbol{F}_i}\left(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)\right) = \boldsymbol{f}^i(x), \qquad & x \in \Omega,\ i = \overline{1,n}, \\ \boldsymbol{u}^1(x) = ... = \boldsymbol{u}^n(x) = \mathbf{0}, \qquad & x \in \Gamma = \partial\Omega. \end{aligned} \tag{3.5}$$

We say, that the external body forces acting on the mixture are conservative, if $\int\limits_\Omega \sum\limits_{i=1}^n \boldsymbol{f}^i(x)\boldsymbol{\xi}^i(x)dx = \int\limits_\Omega \sum\limits_{i=1}^n \widetilde{\boldsymbol{f}}^i\left(x, \boldsymbol{\varphi}^1(x), ..., \boldsymbol{\varphi}^n(x)\right)\boldsymbol{\xi}^i(x)dx$ is Gâteaux derivative of the functional $F(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n)$,

$$F'(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n)(\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^n) = \int\limits_\Omega \sum_{i=1}^n \widetilde{\boldsymbol{f}}^i\left(x, \boldsymbol{\psi}^1(x), ..., \boldsymbol{\psi}^n(x)\right)\boldsymbol{\xi}^i(x)dx,$$

where $\boldsymbol{\xi}^i, \boldsymbol{\psi}^i : \Omega \to \mathbf{R}^3$, $i = \overline{1,n}$, are sufficiently smooth vector-fields.

For hyperelastic mixtures the following equivalence theorem is valid.

Theorem 3.3. *If the mixture is hyperelastic and applied body forces are conservative, then the system (3.5) is formally equivalent to the equation* $I'(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n)(\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^n) = 0$*, for any sufficiently smooth mappings* $\boldsymbol{\xi}^i : \overline{\Omega} \to \mathbf{R}^3$*,* $i = \overline{1,n}$*, which vanish on the boundary* Γ*, where*

$$I(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n) = \int\limits_\Omega \widetilde{W}\left(x, \nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x)\right)dx - F(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n),$$

for any smooth enough mappings $\boldsymbol{\psi}^i : \overline{\Omega} \to \mathbf{R}^3$*,* $i = \overline{1,n}$*.*

Proof. Since the applied body force densities $\boldsymbol{f}^1, ..., \boldsymbol{f}^n$ are conservative, it suffices to find Gâteaux derivative of the first addend (which we denote by $\widetilde{I}(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n)$) of the functional $I(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n)$. For any sufficiently smooth vector-fields $\boldsymbol{\xi}^i : \overline{\Omega} \to \mathbf{R}^3$ $(i = \overline{1,n})$, we have

$$\widetilde{I}(\boldsymbol{\psi}^1 + \boldsymbol{\xi}^1, ..., \boldsymbol{\psi}^n + \boldsymbol{\xi}^n) - \widetilde{I}(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n) = \int\limits_\Omega \Big[\widetilde{W}\left(x, \nabla\boldsymbol{\psi}^1(x) + \nabla\boldsymbol{\xi}^1(x), ...,\right.$$

$$\left.\nabla\boldsymbol{\psi}^n(x) + \nabla\boldsymbol{\xi}^n(x)\right) - \widetilde{W}\left(x, \nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x)\right)\Big]dx = \int\limits_\Omega \sum_{i=1}^n \frac{\partial \widetilde{W}}{\partial \boldsymbol{F}_i}\left(x,\right.$$

$$\left.\nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x)\right) : \nabla\boldsymbol{\xi}^i(x)dx + \int\limits_\Omega o\left(x, \nabla\boldsymbol{\xi}^1(x), ..., \nabla\boldsymbol{\xi}^n(x)\right)dx.$$

From the latter equality it follows:

$$\widetilde{I}'(\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^n)(\boldsymbol{\xi}^1,...,\boldsymbol{\xi}^n)=\int\limits_{\Omega}\sum_{i=1}^{n}\frac{\partial\widetilde{W}}{\partial\boldsymbol{F}_i}\left(x,\nabla\boldsymbol{\psi}^1(x),...,\nabla\boldsymbol{\psi}^n(x)\right):\nabla\boldsymbol{\xi}^i(x)dx,$$

if in the space of mappings $(\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^n):\overline{\Omega}\to\left[\mathbf{R}^3\right]^n$ is introduced the norm $\|.\|^*$, such that the linear form

$$(\boldsymbol{\xi}^1,...,\boldsymbol{\xi}^n)\to\int\limits_{\Omega}\sum_{i=1}^{n}\frac{\partial\widetilde{W}}{\partial\boldsymbol{F}_i}\left(x,\nabla\boldsymbol{\psi}^1(x),...,\nabla\boldsymbol{\psi}^n(x)\right):\nabla\boldsymbol{\xi}^i(x)dx$$

is continuous and $\int\limits_{\Omega} o\left(x,\nabla\boldsymbol{\xi}^1(x),...,\nabla\boldsymbol{\xi}^n(x)\right)dx=o\left(\|(\boldsymbol{\xi}^1,...,\boldsymbol{\xi}^n)\|^*\right)$.

Thus,

$$I'(\boldsymbol{\varphi}^1,...,\boldsymbol{\varphi}^n)(\boldsymbol{\xi}^1,...,\boldsymbol{\xi}^n)=\int\limits_{\Omega}\sum_{i=1}^{n}\left(\widetilde{\boldsymbol{T}}^i\left(x,\nabla\boldsymbol{\varphi}^1(x),...,\nabla\boldsymbol{\varphi}^n(x)\right)+\right.$$

$$\left.+\boldsymbol{H}^i(x)\right):\nabla\boldsymbol{\xi}^i(x)dx-\int\limits_{\Omega}\sum_{i=1}^{n}\widetilde{\boldsymbol{f}}^i\left(x,\boldsymbol{\varphi}^1(x),...,\boldsymbol{\varphi}^n(x)\right)\boldsymbol{\xi}^i(x)dx,$$

for any sufficiently smooth vector-fields $\boldsymbol{\xi}^i:\overline{\Omega}\to\mathbf{R}^3$ $(i=\overline{1,n})$ and applying Theorem 2.2 we obtain the assertion. □

Note, that the point of local extremum is a stationary point too and, consequently, from Theorem 3.3, we deduce that sufficiently smooth mapping, which is a solution to the following minimization problem

$$I(\boldsymbol{\varphi}^1,...,\boldsymbol{\varphi}^n)=\inf_{(\psi^1,...,\psi^n)\in\Phi}I(\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^n),$$
$$\Phi=\left\{(\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^n)\mid\boldsymbol{\psi}^i:\overline{\Omega}\to\mathbf{R}^3,\ \boldsymbol{\psi}^i=\boldsymbol{id}\quad\text{on }\Gamma,\ i=1,...,n\right\},$$

is a solution to Dirichlet boundary value problem (3.5).

So, the boundary value problem (3.5) for nonlinear model of hyperelastic mixture we have reduced to minimization of the functional I. In the case of a single continuum the method of investigation for such type problems was proposed by J. Ball ([22]). On the basis of the results obtained in [22] we generalize the methodology of J. Ball for multicomponent hyperelastic mixtures and obtain the existence of the solution of corresponding minimization problem, but before we formulate auxiliary theorem, which follows from Lemma 6.1 and Theorem 6.2 of [22].

Theorem 3.4. *Let $\Omega \subset \mathbf{R}^3$ be a bounded Lipschitz domain. If $\boldsymbol{\psi}^i \in \mathbf{W}^{1,p}(\Omega)$, $\mathbf{Cof}\nabla\boldsymbol{\psi}^i \in \boldsymbol{L}^q_{3\times3}(\Omega)$, then $\det\nabla\boldsymbol{\psi}^i \in L^s(\Omega)$, where $p \geq 2$, $s^{-1} = p^{-1} + q^{-1} \leq 1$, $i = \overline{1,n}$, and the mapping*

$$\left(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n, \mathbf{Cof}\nabla\boldsymbol{\psi}^1, ..., \mathbf{Cof}\nabla\boldsymbol{\psi}^n\right) \to \left(\det\nabla\boldsymbol{\psi}^1, ..., \det\nabla\boldsymbol{\psi}^n\right)$$

is continuous. Moreover, if

$$\begin{array}{ll}
\left(\boldsymbol{\varphi}^1_t, ..., \boldsymbol{\varphi}^n_t\right) \to \left(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n\right) & \textit{weakly in } \left[\mathbf{W}^{1,p}(\Omega)\right]^n, p \geq 2, \\
\left(\mathbf{Cof}\nabla\boldsymbol{\varphi}^1_t, ..., \mathbf{Cof}\nabla\boldsymbol{\varphi}^n_t\right) \to \left(\boldsymbol{G}_1, ..., \boldsymbol{G}_n\right) & \textit{weakly in } \left[\boldsymbol{L}^q_{3\times3}(\Omega)\right]^n, s \geq 1, \\
\left(\det\nabla\boldsymbol{\varphi}^1_t, ..., \det\nabla\boldsymbol{\varphi}^n_t\right) \to \left(\delta_1, ..., \delta_n\right) & \textit{weakly in } \left[L^r(\Omega)\right]^n, r \geq 1,
\end{array}$$

as $t \to \infty$, then $\boldsymbol{G}_i = \mathbf{Cof}\nabla\boldsymbol{\varphi}^i$, $\delta_i = \det\nabla\boldsymbol{\varphi}^i$, $i = \overline{1,n}$.

We now establish a result on the existence of global solution to nonlinear problem (3.5).

Theorem 3.5. *Let $\Omega \subset \mathbf{R}^3$ be a bounded Lipschitz domain and function $\widetilde{W} : \Omega \times \left[\mathbf{R}^{3\times3}_+\right]^n \to \mathbf{R}$ satisfies the following conditions:*

a) for almost all $x \in \Omega$, there exists the convex function $W(x,.) : \left[\mathbf{R}^{3\times3}\right]^n \times \left[\mathbf{R}^{3\times3}\right]^n \times (0, +\infty)^n \to \mathbf{R}$ such that for all $\boldsymbol{F}_1, ..., \boldsymbol{F}_n \in \mathbf{R}^{3\times3}_+$,

$$\widetilde{W}(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) = W(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n, \mathbf{Cof}\boldsymbol{F}_1, ..., \mathbf{Cof}\boldsymbol{F}_n, \det\boldsymbol{F}_1, ..., \det\boldsymbol{F}_n).$$

The function $W\left(., \boldsymbol{F}_1, ..., \boldsymbol{F}_n, \boldsymbol{G}_1, ..., \boldsymbol{G}_n, \delta_1, ..., \delta_n\right) : \Omega \to \mathbf{R}$ is measurable for any $\boldsymbol{F}_i, \boldsymbol{G}_i \in \mathbf{R}^{3\times3}$, $\delta_i \in (0, +\infty)$, $i = \overline{1,n}$;

b) for almost all $x \in \Omega$,

$$\widetilde{W}\left(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n\right) \to +\infty, \qquad \textit{if } \det\boldsymbol{F}_i \to 0^+, \ \textit{for some } i = \overline{1,n};$$

c) $\widetilde{W}$ is coercive, i.e. there exist constants $\alpha > 0$, $\beta \in \mathbf{R}$, $p \geq 2$, $q \geq p/(p-1)$, $r > 1$ such that for almost all $x \in \Omega$ and for all $\boldsymbol{F}_1, ..., \boldsymbol{F}_n \in \mathbf{R}^{3\times3}_+$ the following inequality is valid

$$\widetilde{W}\left(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n\right) \geq \alpha \sum_{i=1}^{n} \left(\|\boldsymbol{F}_i\|^p + \|\mathbf{Cof}\boldsymbol{F}_i\|^q + (\det\boldsymbol{F}_i)^r\right) + \beta.$$

The set of admissible deformations denote by

$$\mathcal{D} = \Big\{\Psi = (\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n) \in \left[\mathbf{W}^{1,p}(\Omega)\right]^n | \mathbf{Cof}\nabla\boldsymbol{\psi}^i \in \left[\boldsymbol{L}^q_{3\times3}(\Omega)\right]^n, \det\nabla\boldsymbol{\psi}^i \in \left[L^r(\Omega)\right]^n,$$

$$\boldsymbol{\psi}^1 = ... = \boldsymbol{\psi}^n = \boldsymbol{id} \ \textit{a. e. on } \Gamma, \ \det\nabla\boldsymbol{\psi}^i \in (0, +\infty) \ \textit{a. e. in } \Omega\Big\}.$$

Assume that the vector-functions $\boldsymbol{f}^i : \Omega \to \mathbf{R}^3$ $(i = \overline{1,n})$ *are such that the linear form* $L : \Psi \in \left[\mathbf{W}^{1,p}(\Omega)\right]^n \to L(\Psi) = \int\limits_\Omega \sum\limits_{i=1}^n \boldsymbol{f}^i \boldsymbol{\psi}^i dx$ *is continuous. If* $\inf\limits_{\Psi\in\mathcal{D}} I(\Psi) < +\infty$, $I(\Psi) = \int\limits_\Omega \widetilde{W}(x, \nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x))dx - L(\Psi)$, *then there exists* $\Phi = (\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n) \in \mathcal{D}$, *which minimizes the functional* I *on the set of admissible deformations* $I(\Phi) = \inf\limits_{\Psi\in\mathcal{D}} I(\Psi)$.

Proof. First let us prove that the function $\widetilde{W}(x, \nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x))$ is integrable in Ω for any $(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n) \in \mathcal{D}$. Indeed, from the point a) of the theorem it follows that the function $W(x,.) : \left[\mathbf{R}^{3\times3}\right]^n \times \left[\mathbf{R}^{3\times3}\right]^n \times (0,+\infty)^n \to \mathbf{R}$ is continuous, since it is convex and is defined on the open set of the finite dimensional space. For any $(\boldsymbol{F}_1, ..., \boldsymbol{F}_n, \boldsymbol{G}_1, ..., \boldsymbol{G}_n, \delta_1, ..., \delta_n) \in \left[\mathbf{R}^{3\times3}\right]^n \times \left[\mathbf{R}^{3\times3}\right]^n \times (0,+\infty)^n$ the function $W(., \boldsymbol{F}_1, .., \boldsymbol{F}_n, \boldsymbol{G}_1, .., \boldsymbol{G}_n, \delta_1, .., \delta_n) : \Omega \to \mathbf{R}$ is measurable and $\left[\mathbf{R}^{3\times3}\right]^n \times \left[\mathbf{R}^{3\times3}\right]^n \times (0,+\infty)^n$ is Borel set, hence the function $W : \Omega \times \left[\mathbf{R}^{3\times3}\right]^n \times \left[\mathbf{R}^{3\times3}\right]^n \times (0,+\infty)^n \to \mathbf{R}$ is Carathéodory function and, consequently, the function $W(x, \nabla\boldsymbol{\psi}^1(x), ..., \nabla\boldsymbol{\psi}^n(x), \mathbf{Cof}\nabla\boldsymbol{\psi}^1(x), ..., \mathbf{Cof}\nabla\boldsymbol{\psi}^n(x), \det\nabla\boldsymbol{\psi}^1(x), ..., \det\nabla\boldsymbol{\psi}^n(x))$ is measurable for any $(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n) \in \mathcal{D}$. From the coerciveness inequality we infer that $\widetilde{W}$ is bounded below, whence $I(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n)$ exists for all $(\boldsymbol{\psi}^1, ..., \boldsymbol{\psi}^n) \in \mathcal{D}$.

Since the function $\widetilde{W}$ is coercive and form L is continuous, we have:

$$I(\Psi) \geq \alpha \sum_{i=1}^n \int_\Omega \left(\|\nabla\boldsymbol{\psi}^i\|^p + \|\mathbf{Cof}\nabla\boldsymbol{\psi}^i\|^q + (\det\nabla\boldsymbol{\psi}^i)^r\right) dx +$$
$$+\beta \mathrm{meas}(\Omega) - c_L \|\Psi\|_{[\mathbf{W}^{1,p}(\Omega)]^n}, \qquad \forall\Psi \in \mathcal{D}.$$

Therefore, from the condition $\boldsymbol{\psi}^1 = ... = \boldsymbol{\psi}^n = \boldsymbol{id}$ on Γ, it follows, that there exist $\alpha_1 > 0$ and $c_1 \in \mathbf{R}$ such that for any $\Psi \in \mathcal{D}$,

$$I(\Psi) \geq \alpha_1 \sum_{i=1}^n \left(\|\boldsymbol{\psi}^i\|^p_{\mathbf{W}^{1,p}(\Omega)} + \|\mathbf{Cof}\nabla\boldsymbol{\psi}^i\|^q_{\boldsymbol{L}^q_{3\times3}(\Omega)} + (\det\nabla\boldsymbol{\psi}^i)^r_{L^r(\Omega)}\right) + c_1. \quad (3.6)$$

In order to prove the theorem, let us consider the sequence $\{\Phi_t\} \subset \mathcal{D}$, which minimizes the functional I, i.e. $\lim\limits_{t\to\infty} I(\Phi_t) = \inf\limits_{\Psi\in\mathcal{D}} I(\Psi)$.

According to the condition of the theorem $\inf\limits_{\Psi\in\mathcal{D}} I(\Psi) < +\infty$ from (3.6) we have, that $\left(\boldsymbol{\varphi}^1_t, ..., \boldsymbol{\varphi}^n_t, \mathbf{Cof}\nabla\boldsymbol{\varphi}^1_t, ..., \mathbf{Cof}\nabla\boldsymbol{\varphi}^n_t, \det\nabla\boldsymbol{\varphi}^1_t, ..., \det\nabla\boldsymbol{\varphi}^n_t\right)$ is a bounded sequence in the reflexive Banach space $\left[\mathbf{W}^{1,p}(\Omega)\right]^n \times \left[\boldsymbol{L}^q_{3\times3}(\Omega)\right]^n \times \left[L^r(\Omega)\right]^n$. Consequently, there exists the subsequence $(\boldsymbol{\varphi}^1_{t_1}, ..., \boldsymbol{\varphi}^n_{t_1}, \mathbf{Cof}\nabla\boldsymbol{\varphi}^1_{t_1}, ..., \mathbf{Cof}\nabla\boldsymbol{\varphi}^n_{t_1}, \det\nabla\boldsymbol{\varphi}^1_{t_1}, ..., \det\nabla\boldsymbol{\varphi}^n_{t_1})$, which weakly converges to $(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n,$

$\boldsymbol{G}_1$, ..., $\boldsymbol{G}_n$, δ_1, ..., δ_n) in the space $\left[\mathbf{W}^{1,p}(\Omega)\right]^n \times [\boldsymbol{L}^q_{3\times3}(\Omega)]^n \times [L^r(\Omega)]^n$, and, due to Theorem 3.4, $\boldsymbol{G}_i = \mathbf{Cof}\nabla\boldsymbol{\varphi}^i$, $\delta_i = \det\nabla\boldsymbol{\varphi}^i$, $i = \overline{1,n}$. Thus, there exists the sequence $\{\Phi_{t_1}\}$, such that for $t_1 \to \infty$,

$$\begin{aligned}
&\left(\boldsymbol{\varphi}^1_{t_1}, ..., \boldsymbol{\varphi}^n_{t_1}\right) \to \left(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n\right) && \text{weakly in } \left[\mathbf{W}^{1,p}(\Omega)\right]^n, \\
&\left(\mathbf{Cof}\nabla\boldsymbol{\varphi}^1_{t_1}, ..., \mathbf{Cof}\nabla\boldsymbol{\varphi}^n_{t_1}\right) \to (\boldsymbol{G}_1, ..., \boldsymbol{G}_n) && \text{weakly in } \left[\boldsymbol{L}^q_{3\times3}(\Omega)\right]^n, \\
&\left(\det\nabla\boldsymbol{\varphi}^1_{t_1}, ..., \det\nabla\boldsymbol{\varphi}^n_{t_1}\right) \to (\delta_1, ..., \delta_n) && \text{weakly in } \left[L^r(\Omega)\right]^n.
\end{aligned} \tag{3.7}$$

Let us prove that $\left(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n\right) \in \mathcal{D}$. It suffices to show that $\det\nabla\boldsymbol{\varphi}^i \in (0, +\infty)$ almost everywhere in Ω and $\boldsymbol{\varphi}^i = \boldsymbol{id}$ almost everywhere on Γ ($i = \overline{1,n}$). The second assertion directly follows from the compactness of the trace operator $tr : W^{1,p}(\Omega) \to L^p(\Gamma)$ and, hence, we have to prove the validity of the first one.

Applying Mazur's theorem ([23]), from the third condition (3.7) we obtain, that there exists a sequence of linear combinations of ($\det\nabla\boldsymbol{\varphi}^1_{t_1}$,..., $\det\nabla\boldsymbol{\varphi}^n_{t_1}$), which strongly converges in the space $[L^r(\Omega)]^n$, i.e. there exists $j(t_1) \ge t_1$, $t_1 \le s \le j(t_1)$, $\lambda^{t_1}_s \ge 0$, $\sum\limits_{s=t_1}^{j(t_1)} \lambda^{t_1}_s = 1$,

$$\mathbf{d}^{t_1} = \sum_{s=t_1}^{j(t_1)} \lambda^{t_1}_s \left(\det\nabla\boldsymbol{\varphi}^1_s, ..., \det\nabla\boldsymbol{\varphi}^n_s\right) \to \left(\det\nabla\boldsymbol{\varphi}^1, ..., \det\nabla\boldsymbol{\varphi}^n\right)$$
$$\text{strongly in } \left[L^r(\Omega)\right]^n, \text{ as } t_1 \to \infty.$$

Consequently, there exists subsequence $\{\mathbf{d}^{t_2}\}$ of $\{\mathbf{d}^{t_1}\}$, such that

$$\sum_{s=t_2}^{j(t_2)} \lambda^{t_2}_s \det\nabla\boldsymbol{\varphi}^i_s \to \det\nabla\boldsymbol{\varphi}^i \qquad \text{a. e. in } \Omega,\ t_2 \to \infty,\ i = \overline{1,n}.$$

Since $\det\nabla\boldsymbol{\varphi}^i_s$ is almost everywhere positive, then $\det\nabla\boldsymbol{\varphi}^i \in [0, +\infty)$ almost everywhere in Ω.

Suppose, that $\det\nabla\boldsymbol{\varphi}^{i_0} = 0$ on the subset A^{i_0} of the domain Ω with positive measure meas$(A^{i_0}) > 0$, $1 \le i_0 \le n$. Then, from the third condition (3.7) and $\det\nabla\boldsymbol{\varphi}^{i_0}_{t_1} \in (0, +\infty)$ a. e. in A^{i_0}, we infer, that

$$\int\limits_{A^{i_0}} \left|\det\nabla\boldsymbol{\varphi}^{i_0}_{t_1}(x)\right| dx = \int\limits_{A^{i_0}} \det\nabla\boldsymbol{\varphi}^{i_0}_{t_1}(x) dx \to \int\limits_{A^{i_0}} \det\nabla\boldsymbol{\varphi}^{i}(x) dx = 0,$$

whence $\det\nabla\boldsymbol{\varphi}^{i_0}_{t_1} \to 0$ strongly in $L^1(A^{i_0})$, as $t_1 \to \infty$. Hence, there exists the subsequence $\{\Phi_{t_3}\}$ of $\{\Phi_{t_1}\}$ such that

$$\det\nabla\boldsymbol{\varphi}^{i_0}_{t_3} \to 0 \qquad \text{for almost all } x \in A^{i_0}, \text{ as } t_3 \to \infty.$$

As we have pointed out above $w_{t_3}(x) = \widetilde{W}\left(x, \nabla\boldsymbol{\varphi}^1_{t_3}(x), ..., \nabla\boldsymbol{\varphi}^n_{t_3}(x)\right)$ is measurable function and $w_{t_3} \geq \beta$ for all t_3, hence, according to Fatou's lemma

$$\int\limits_{A^{i_0}} \lim_{t_3\to\infty} \inf w_{t_3}(x)dx \leq \lim_{t_3\to\infty} \inf \int\limits_{A^{i_0}} w_{t_3}(x)dx.$$

Taking into account the point b) of the theorem we obtain

$$\lim_{t_3\to\infty} \inf w_{t_3}(x) = \lim_{\det \boldsymbol{F}_{i_0}\to 0^+} \widetilde{W}(x, \boldsymbol{F}_1, ..., \boldsymbol{F}_n) = +\infty$$

for almost all $x \in A^{i_0}$, and, therefore,

$$\lim_{t_3\to\infty} \int\limits_{A^{i_0}} w_{t_3}(x)dx = \lim_{t_3\to\infty} \int\limits_{A^{i_0}} \widetilde{W}\left(x, \nabla\boldsymbol{\varphi}^1_{t_3}(x), ..., \nabla\boldsymbol{\varphi}^n_{t_3}(x)\right) dx = +\infty.$$

The latter equality is in contradiction with the conditions of the theorem, since

$$\int\limits_{A^{i_0}} w_{t_3}(x)dx \leq I(\Phi_{t_3}) - \beta \text{meas}(\Omega \backslash A^{i_0}) + c_L \left\|\Phi_{t_3}\right\|_{[\mathbf{W}^{1,p}(\Omega)]^n},$$

$\lim\limits_{t_3\to\infty} I(\Phi_{t_3}) = \inf\limits_{\Psi\in\mathcal{D}} I(\Psi) < +\infty$ and weakly converging sequence $\{\Phi_{t_3}\}$ is bounded. Thus, $\text{meas}(A^{i_0}) = 0$ for all $i_0 = 1, ..., n$, and, consequently, $\Phi = \left(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n\right) \in \mathcal{D}$.

In order to prove that Φ minimizes the functional I let us show the validity of the following inequality:

$$\int\limits_{\Omega} \widetilde{W}\left(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)\right) dx \leq \lim_{t_1\to\infty} \inf \int\limits_{\Omega} \widetilde{W}(x, \nabla\boldsymbol{\varphi}^1_{t_1}(x), ..., \nabla\boldsymbol{\varphi}^n_{t_1}(x))dx.$$

Hence, we have to prove that for each subsequence $\{\Phi_{t_4}\}$ of $\{\Phi_{t_1}\}$, for which the sequence $\left\{\int\limits_{\Omega} \widetilde{W}\left(x, \nabla\boldsymbol{\varphi}^1_{t_4}(x), ..., \nabla\boldsymbol{\varphi}^n_{t_4}(x)\right) dx\right\}$ converges, the following inequality holds

$$\int\limits_{\Omega} \widetilde{W}\left(x, \nabla\boldsymbol{\varphi}^1(x), ..., \nabla\boldsymbol{\varphi}^n(x)\right) dx \leq \lim_{t_4\to\infty} \int\limits_{\Omega} \widetilde{W}\left(x, \nabla\boldsymbol{\varphi}^1_{t_4}(x), ..., \nabla\boldsymbol{\varphi}^n_{t_4}(x)\right) dx.$$

From (3.7), applying Mazur's theorem, it follows that for each t_4, there exist natural numbers $j(t_4) \geq t_4$ and real numbers $\mu^{t_4}_s$, $t_4 \leq s \leq j(t_4)$ such that $\mu_s^{t_4} \geq 0$, $\sum\limits_{s=t_4}^{j(t_4)} \mu_s^{t_4} = 1$,

$$\mathbf{D}^{t_4} = \sum_{s=t_4}^{j(t_4)} \mu_s^{t_4} \left(\nabla \Phi_s, \mathbf{Cof} \nabla \Phi_s, \mathrm{Det} \nabla \Phi_s\right) \to$$
$$\to \left(\nabla \boldsymbol{\varphi}^1, ..., \nabla \boldsymbol{\varphi}^n, \mathbf{Cof} \nabla \boldsymbol{\varphi}^1, ..., \mathbf{Cof} \nabla \boldsymbol{\varphi}^n, \det \nabla \boldsymbol{\varphi}^1, ..., \det \nabla \boldsymbol{\varphi}^n\right),$$
$$\text{strongly in } \left[\mathbf{L}^p(\Omega)\right]^n \times \left[\boldsymbol{L}^q_{3\times 3}(\Omega)\right]^n \times \left[L^r(\Omega)\right]^n, \text{ as } t_4 \to \infty,$$

where $\nabla \Phi_s = (\nabla \boldsymbol{\varphi}_s^1, ..., \nabla \boldsymbol{\varphi}_s^n)$, $\mathbf{Cof} \nabla \Phi_s = (\mathbf{Cof} \nabla \boldsymbol{\varphi}_s^1, ..., \mathbf{Cof} \nabla \boldsymbol{\varphi}_s^n)$, $\mathrm{Det} \nabla \Phi_s = (\det \nabla \boldsymbol{\varphi}_s^1, ..., \det \nabla \boldsymbol{\varphi}_s^n)$.

Therefore, there exists the subsequence $\{\mathbf{D}^{t_5}\}$ of $\{\mathbf{D}^{t_4}\}$ such that

$$\sum_{s=t_5}^{j(t_5)} \mu_s^{t_5} \left(\nabla \Phi_s(x), \mathbf{Cof} \nabla \Phi_s(x), \mathrm{Det} \nabla \Phi_s(x)\right) \to \Big(\nabla \boldsymbol{\varphi}^1(x), ..., \nabla \boldsymbol{\varphi}^n(x),$$
$$\mathbf{Cof} \nabla \boldsymbol{\varphi}^1(x), ..., \mathbf{Cof} \nabla \boldsymbol{\varphi}^n(x), \det \nabla \boldsymbol{\varphi}^1(x), ..., \det \nabla \boldsymbol{\varphi}^n(x)\Big),$$
$$\text{for almost all } x \in \Omega, \text{ as } t_5 \to \infty.$$

Since the function $W(x,.)$ is continuous on $\left[\mathbf{R}^{3\times 3}\right]^n \times \left[\mathbf{R}^{3\times 3}\right]^n \times (0,+\infty)^n$ and $\det \nabla \boldsymbol{\varphi}^i(x) \in (0,+\infty)$ for almost all $x \in \Omega$ $(i = \overline{1,n})$, we have:

$$\widetilde{W}\left(x, \nabla \Phi(x)\right) = \lim_{t_5 \to \infty} W\left(x, \sum_{s=t_5}^{j(t_5)} \mu_s^{t_5} \left(\nabla \Phi_s(x), \mathbf{Cof} \nabla \Phi_s(x), \mathrm{Det} \nabla \Phi_s(x)\right)\right)$$

for almost all $x \in \Omega$. From the latter equality, applying Fatou's lemma and taking into account convexity of $W(x,.)$, we obtain:

$$\int_\Omega \widetilde{W}\left(x, \nabla \Phi(x)\right) dx \leq \lim_{t_5 \to \infty} \inf \int_\Omega W\Bigg(x, \sum_{s=t_5}^{j(t_5)} \mu_s^{t_5} \left(\nabla \Phi_s(x), \mathbf{Cof} \nabla \Phi_s(x),\right.$$
$$\left.\mathrm{Det} \nabla \Phi_s(x)\right)\Bigg) dx \leq \lim_{t_5 \to \infty} \inf \sum_{s=t_5}^{j(t_5)} \mu_s^{t_5} \int_\Omega \widetilde{W}\left(x, \nabla \boldsymbol{\varphi}_s^1(x), ..., \nabla \boldsymbol{\varphi}_s^n(x)\right) dx =$$
$$= \lim_{t_4 \to \infty} \int_\Omega \widetilde{W}\left(x, \nabla \boldsymbol{\varphi}_{t_4}^1(x), ..., \nabla \boldsymbol{\varphi}_{t_4}^n(x)\right) dx.$$

Due to weak convergence of the sequence $\{\Phi_{t_1}\}$, we have that $L(\Phi) = \lim\limits_{t_1 \to \infty} L(\boldsymbol{\varphi}_{t_1}^1, ..., \boldsymbol{\varphi}_{t_1}^n)$ and, consequently,

$$I\left(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n\right) \leq \lim_{t_1 \to \infty} \inf I\left(\boldsymbol{\varphi}_{t_1}^1, ..., \boldsymbol{\varphi}_{t_1}^n\right) = \inf_{\Psi \in \mathcal{D}} I(\Psi).$$

Since $\Phi = \left(\varphi^1, ..., \varphi^n\right) \in \mathcal{D}$, from the latter inequality we deduce, that Φ minimizes the functional I. $\Box$

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