THE MINIMAL ENTROPY AND MINIMAL φ-DIVERGENCE DISTANCE MARTINGALE MEASURES FOR THE TRINOMIAL SCHEME

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Abstract

In this paper we construct the martingale measures which minimize the relative entropy and reverse relative entropy with respect to a reference measure P in trinomial scheme. We also find the martingale measure which minimizes φ-divergence distance defined by convex function φ and includes relative entropy and reverse relative entropy as special cases of φ-divergence.

Key words and phrases: trinomial scheme, martingale measures, relative entropy, reverse relative entropy, φ-divergence distance.

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In this paper we investigate the problem of finding the optimal martingale measures for the trinomial scheme. In section 1 we consider the problem of finding the minimal relative entropy martingale measure. In section 2 we studied the same problem for reverse relative entropy and in section 3 we investigate the general problem of finding the optimal martingale measure in the sense of minimal φ-divergence distance including as special cases the problems which are studied in the sections 1 and 2. All results are obtained by using the optimization method under constraints based on Lagrange multipliers.

Let us consider a real valued process $S = (S_n, \mathcal{F}_n), n = 1, 2, ..., N$, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$, such that

$$S_n = S_{n-1}(1 + \rho_n), \tag{1}$$

where $S_0 > 0$ is deterministic, $(\rho_n)_{n\geq 1}$ is the sequence of independent identically distributed random variables that take only three values a, b, c

with the probabilities p, q and r respectively, $p + q + r = 1$. We assume that $a < b < c$ and $-1 < a < 0 < c$.

Such scheme is known as trinomial scheme.

Here the reference measure P is defined by p, q, and l on $\Omega = \{a, b, c\}^N$. The measure Q is a martingale measure for S if Q is equivalent to P and $S = (S_n, \mathcal{F}_n), n = 0, 1, ..., N$, is a martingale with respect to this measure. The martingale condition

$$E_Q[\Delta S_n \mid \mathcal{F}_{n-1}] = 0$$

implies that

$$a\tilde{p} + b\tilde{q} + c\tilde{r} = 0 \tag{2}$$

and the class of martingale measures $M(P)$ for S, which preserves i.i.d. of (ρ_n), $n = 1, ..., N$, is defined by $\tilde{p}, \tilde{q}, \tilde{r}$ and satisfies the equation (2), $\tilde{p} = Q(\rho_1 = a), \tilde{q} = Q(\rho_1 = b), \tilde{r} = Q(\rho_1 = c)$.

It is easy to see, that density $\frac{dQ}{dP} = Z_N(\rho_1, \rho_2, ..., \rho_N), Q \in M(P)$ has the following form ($I(\Delta)$ is the indicator of Δ):

$$\begin{gathered} Z_N = Z_N(\rho_1, \rho_2, ..., \rho_N) = \prod_{k=1}^{N} (\frac{\tilde{p}}{p} I(\rho_k = a) + \\ \frac{\tilde{q}}{q} I(\rho_k = b) + \frac{\tilde{r}}{r} I(\rho_k = c) = \prod_{k=1}^{N} \xi_k, \end{gathered} \tag{3}$$

where

$$\xi_k = \frac{\tilde{p}}{p} I(\rho_k = a) + \frac{\tilde{q}}{q} I(\rho_k = b) + \frac{\tilde{r}}{r} I(\rho_k = c). \tag{4}$$

Note, that $\xi_k, k = 1, 2, ..., N$, is the sequence of independent identically distributed random variables that take only three values $\frac{\tilde{p}}{p}, \frac{\tilde{q}}{q}, \frac{\tilde{r}}{r}$ with the probabilities p, q and r respectively and $E\xi_k = 1, EZ_N = 1$.

1. The relative entropy $I(Q, P)$ of the probability measure Q with respect to the probability measure P is defined as (see for example [1], [3])

$$I(Q, P) = \begin{cases} E_P[\frac{dQ}{dP} ln \frac{dQ}{dP}], & if Q << P \\ \\ +\infty, & otherwise. \end{cases}$$

It is well known, that $I(Q, P) = 0$ if and only if $Q = P$.

The minimal entropy martingale measure is the measure Q^*, for which

$$I(Q^*, P) = \min_{Q \in M(P)} I(Q, P).$$

The concept of relative entropy is also known as Kullback-Liebler information number.

In our case, using the independence of ξ_k and the equality $E\xi_k = 1$, we have (we use E for E_P)

$$I(Q,P) = EZ_N lnZ_N = E[\prod_{k=1}^{N} \xi_k ln \prod_{k=1}^{N} \xi_k] = E[\prod_{k=1}^{N} \xi_k (ln\xi_1 + ln\xi_2 + ... + ln\xi_N)] =$$

$$E[(\prod_{k=1}^{N} \xi_k) ln\xi_1] + E[(\prod_{k=1}^{N} \xi_k) ln\xi_2] + ... + E[(\prod_{k=1}^{N} \xi_k) ln\xi_N] = E[(\prod_{k\neq 1}^{N} \xi_k)\xi_1 ln\xi_1] +$$

$$E[(\prod_{k\neq 2}^{N} \xi_k)\xi_2 ln\xi_2] + ... + E[(\prod_{k\neq N}^{N} \xi_k)\xi_N ln\xi_N] = E[\prod_{k\neq 1}^{N} \xi_k] E[\xi_1 ln\xi_1] +$$

$$E[\prod_{k\neq 2}^{N} \xi_k] E[\xi_2 ln\xi_2] + ... + E[\prod_{k\neq N}^{N} \xi_k] E[\xi_N ln\xi_N] = [\prod_{k\neq 1}^{N} E\xi_k] E[\xi_1 ln\xi_1] +$$

$$[\prod_{k\neq 2}^{N} E\xi_k] E[\xi_2 ln\xi_2] + ... + [\prod_{k\neq N}^{N} E\xi_k] E[\xi_N ln\xi_N] = E[\xi_1 ln\xi_1] + E[\xi_2 ln\xi_2] +$$

$$E[\xi_N ln\xi_N] = NE[\xi_1 ln\xi_1].$$

Note, that

$$E[\xi_1 ln\xi_1] = p\frac{\tilde{p}}{p} ln\frac{\tilde{p}}{p} + q\frac{\tilde{q}}{q} ln\frac{\tilde{q}}{q} + r\frac{\tilde{r}}{r} ln\frac{\tilde{r}}{r} =$$

$$\tilde{p} ln\frac{\tilde{p}}{p} + \tilde{q} ln\frac{\tilde{q}}{q} + \tilde{r} \cdot ln\frac{\tilde{r}}{r}$$

and

$$EZ_N lnZ_N = N\left(\tilde{p} ln\frac{\tilde{p}}{p} + \tilde{q} ln\frac{\tilde{q}}{q} + \tilde{r} \cdot ln\frac{\tilde{r}}{r}\right). \tag{5}$$

Denote

$$f(\tilde{p},\tilde{q},\tilde{r}) = \tilde{p} ln\frac{\tilde{p}}{p} + \tilde{q} ln\frac{\tilde{q}}{q} + \tilde{r} \cdot ln\frac{\tilde{r}}{r},$$

$$\phi(\tilde{p},\tilde{q},\tilde{r}) = a\tilde{p} + b\tilde{q} + c\tilde{r},$$

and

$$\psi(\tilde{p},\tilde{q},\tilde{r}) = \tilde{p} + \tilde{q} + \tilde{r} - 1.$$

The problem of finding the minimal entropy martingale measure Q^*, $I(Q^*,P) = \min_{Q\in M(P)} I(Q,P)$ is reduced to the minimization problem of $f(\tilde{p},\tilde{q},\tilde{r})$ under conditions

$$\phi(\tilde{p},\tilde{q},\tilde{r}) = a\tilde{p} + b\tilde{q} + c\tilde{r} = 0,$$

$$\psi(\tilde{p},\tilde{q},\tilde{r}) = \tilde{p} + \tilde{q} + \tilde{r} - 1 = 0.$$

Lagrangian has the following form

$$\Phi(\tilde{p},\tilde{q},\tilde{r}) = \tilde{p} ln\frac{\tilde{p}}{p} + \tilde{q} ln\frac{\tilde{q}}{q} + \tilde{r} \cdot ln\frac{\tilde{r}}{r} +$$

$$\lambda(a\tilde{p} + b\tilde{q} + c\tilde{r}) + \mu(\tilde{p} + \tilde{q} + \tilde{r} - 1).$$

From the optimality conditions

$$\frac{\partial\Phi(\tilde{p},\tilde{q},\tilde{r})}{\partial\tilde{p}} = 0, \quad \frac{\partial\Phi(\tilde{p},\tilde{q},\tilde{r})}{\partial\tilde{q}} = 0, \quad \frac{\partial\Phi(\tilde{p},\tilde{q},\tilde{r})}{\partial\tilde{r}} = 0,$$

we obtain

$$ln\frac{\tilde{p}}{p} + 1 + \lambda a + \mu = 0, \tag{6}$$

$$ln\frac{\tilde{q}}{q} + 1 + \lambda b + \mu = 0, \tag{7}$$

$$ln\frac{\tilde{r}}{r} + 1 + \lambda c + \mu = 0, \tag{8}$$

and we also have

$$a\tilde{p} + b\tilde{q} + c\tilde{r} = 0, \tag{9}$$

$$\tilde{p} + \tilde{q} + \tilde{r} - 1 = 0. \tag{10}$$

From (6),(7), and (8) we have

$$\tilde{p} = pe^{-(\lambda a + \mu + 1)}, \tag{11}$$

$$\tilde{q} = qe^{-(\lambda b + \mu + 1)}, \tag{12}$$

$$\tilde{r} = re^{-(\lambda c + \mu + 1)}, \tag{13}$$

and if we insert these equations in (9), we get

$$ape^{-\lambda a} + bqe^{-\lambda b} + cre^{-\lambda c} = 0. \tag{14}$$

Consider the function $g(x) = ape^{-ax} + bqe^{-bx} + cre^{-cx}$. Then $g'(x) < 0$ for all x. Further, $lim_{x\to+\infty} g(x) = -\infty$ since $a < 0$, and $lim_{x\to-\infty} g(x) = +\infty$ because of $c > 0$. So, it follows, that there exists a unique λ, which satisfies (14).

From (10), using (11)-(13), we can determine μ,

$$\mu = ln(pe^{-\lambda a} + qe^{-\lambda b} + re^{-\lambda c}) - 1, \tag{15}$$

where λ is the solution of equation (14), and then we can determine $\tilde{p}, \tilde{q}$ and $\tilde{r}$ from (11)-(13).

It is easy to check that the Hessian matrix (see [5], p.148) is positively defined for our Lagrangian and therefore, $(\tilde{p}, \tilde{q}, \tilde{r})$ is a point for which it takes minimum.

Now we can find the density for minimal entropy martingale measure Q^E,

$$Z_N^* = \frac{dQ^E}{dP} = \prod_{k=1}^{N} \xi_k^*.$$

From (4) using (11)-(13) we obtain

$$\xi_k^* = e^{-(\lambda a+\mu+1)} I(\rho_k = a) + e^{-(\lambda b+\mu+1)} I(\rho_k = b) + e^{-(\lambda c+\mu+1)} I(\rho_k = c) =$$

$$e^{-(\mu+1)} [e^{-\lambda a} I(\rho_k = a) + e^{-\lambda b} I(\rho_k = b) + e^{-\lambda c} I(\rho_k = c) =$$

$$e^{-(\mu+1)} e^{-\lambda [aI(\rho_k=a)+bI(\rho_k=b)+cI(\rho_k=c)]} = ce^{-\lambda \rho_k},$$

where $c = e^{-(\mu+1)}$.

Since $\rho_k = \frac{\Delta S_k}{S_{k-1}}$, we have the following representation for ξ_k^*

$$\xi_k^* = c \cdot exp\{-\lambda \frac{\Delta S_k}{S_{k-1}}\}.$$

Density Z_N^* has the following form:

$$Z_N^* = c \cdot exp\{-\lambda \sum_{k=1}^{N} \frac{\Delta S_k}{S_{k-1}}\}.$$

We can write the minimal entropy martingale measure also in the different form.

From (6) and (7) we get

$$\lambda = \frac{ln\frac{\tilde{p}}{p} - ln\frac{\tilde{q}}{q}}{b - a}$$

and from (6) and (8) we have

$$\mu = \frac{c \cdot ln\frac{\tilde{p}}{p} - a \cdot ln\frac{\tilde{r}}{r}}{a - c} - 1.$$

After the substitution λ and μ in (6) we obtain

$$\left(\frac{\tilde{p}}{p}\right)^{b-c}\left(\frac{\tilde{q}}{q}\right)^{c-a}\left(\frac{\tilde{r}}{r}\right)^{a-b}=1.$$

From (9) and (10)

$$\tilde{q}=\frac{(a-c)\tilde{p}+c}{c-b}, \quad \tilde{r}=\frac{(b-a)\tilde{p}-b}{c-b} \tag{16}$$

and finally for determination of $\tilde{p}$ we have the following equation

$$\left(\frac{\tilde{p}}{p}\right)^{b-c}\left(\frac{(a-c)\tilde{p}+c}{q(c-b)}\right)^{c-a}\left(\frac{(b-a)\tilde{p}-b}{r(c-b)}\right)^{a-b}=1. \tag{17}$$

Denote

$$f(x)=(b-c)ln\frac{x}{p}+(c-a)ln\frac{(a-c)x+c}{q(c-b)}+(a-b)ln\frac{(b-a)x-b}{r(c-b)}.$$

On the interval $\left(\frac{b}{b-a},\frac{c}{c-b}\right)$ of determination of $f(x)$ the derivative $f'(x)<0$, since $b-a>0$ and $a-c<0$. Note, that $lim_{x\downarrow\frac{b}{b-a}}=+\infty, lim_{x\uparrow\frac{c}{c-b}}=-\infty$.

Therefore, the equation (17) for the determination of $\tilde{p}$ has a unique solution.

Thus we proved the following

Theorem 1. *The minimal entropy martingale measure Q^* for trinomial scheme (1) is unique.*

I. Q^* is determined by $\tilde{p}^*, \tilde{q}^*$ and $\tilde{r}^*$, which have the form

$$\tilde{p}^*=pe^{-(\lambda a+\mu+1)},$$

$$\tilde{q}^*=qe^{-(\lambda b+\mu+1)},$$

$$\tilde{r}^*=re^{-(\lambda c+\mu+1)},$$

where λ is a unique solution of

$$ape^{-\lambda a}+bqe^{-\lambda b}+cre^{-\lambda c}=0$$

and

$$\mu=ln[pe^{-\lambda a}+qe^{-\lambda b}+re^{-\lambda c}]-1.$$

The density $Z_N^*=\frac{dQ^*}{dP}$ has the following representation

$$Z_N^* = c \cdot exp\{-\lambda \sum_{k=1}^{N} \frac{\Delta S_k}{S_{k-1}}\},$$

where

$$c = e^{-(\mu+1)}.$$

II. Q^* is determined by $\tilde{p}^*, \tilde{q}^*$ and $\tilde{r}^*$, where $\tilde{p}^*$ is a unique solution of the equation

$$\left(\frac{\tilde{p}}{p}\right)^{b-c} \left(\frac{(a-c)\tilde{p}+c}{q(c-b)}\right)^{c-a} \left(\frac{(b-a)\tilde{p}-b}{r(c-b)}\right)^{a-b} = 1$$

and

$$\tilde{q} = \frac{(a-c)\tilde{p}+c}{c-b}, \quad \tilde{r} = \frac{(b-a)\tilde{p}-b}{c-b}.$$

Remark. The result of part I of Theorem 1 for the case $N = 1$ can be found in [1] and corresponds to general result of M.Fritelli for real adapted stochastic process X with discrete time ([1], see also [3],[4]). M.Fritelli has established, that

$$\frac{dQ^*}{dP} = c \cdot exp\{\sum_{k=1}^{N} H_k \Delta X_k\},$$

where H is a predictable process and c is normalizing constant. In our situation, as it is obvious from Theorem 1

$$c = e^{-(\mu+1)} \quad and \quad H_k = -\frac{\lambda}{S_{k-1}}.$$

Example 1. Consider the symmetrical case: $a = -\alpha, b = 0, c = \alpha$ with $\alpha > 0$. It follows from Theorem 1, that

$$\tilde{p}^s = \frac{\sqrt{pr}}{2\sqrt{pr}+q}, \quad \tilde{q}^s = \frac{q}{2\sqrt{pr}+q}, \quad \tilde{r}^s = \frac{\sqrt{pr}}{2\sqrt{pr}+q}.$$

2. The reverse relative entropy $I^r(Q, P)$ of the probability measure Q with respect to the probability measure P is defined as (see for example [2])

$$I^r(Q,P) = \begin{cases} E_P[-ln\frac{dQ}{dP}], & if \quad Q << P \\ \\ +\infty, & otherwise. \end{cases}$$

The reverse minimal entropy martingale measure is the measure Q^* for which

$$I^r(Q^*,P) = \min_{Q\in M(P)} I^r(Q,P).$$

For our trinomial scheme (1) we get

$$I^r(Q,P) = E[-lnZ_N] = E[-ln\prod_{k=1}^{N}\xi_k] = -\sum_{k=1}^{N}E[ln\xi_k] =$$
$$-NE[ln\xi_1] = -N[pln\tfrac{\tilde{p}}{p} + qln\tfrac{\tilde{q}}{q} + r\cdot ln\tfrac{\tilde{r}}{r}].$$

Consider the minimization problem of reverse relative entropy among all martingale measures $Q \in M(P)$.

Now the Lagrangian has the following form

$$\Phi(\tilde{p},\tilde{q},\tilde{r}) = -[pln\frac{\tilde{p}}{p} + qln\frac{\tilde{q}}{q} + r\cdot ln\frac{\tilde{r}}{r}]+$$
$$\lambda[a\tilde{p}+b\tilde{q}+c\tilde{r}] + \mu[\tilde{p}+\tilde{q}+\tilde{r}-1]$$

and from the optimality conditions

$$\frac{\partial\Phi(\tilde{p},\tilde{q},\tilde{r})}{\partial\tilde{p}} = 0, \quad \frac{\partial\Phi(\tilde{p},\tilde{q},\tilde{r})}{\partial\tilde{q}} = 0, \quad \frac{\partial\Phi(\tilde{p},\tilde{q},\tilde{r})}{\partial\tilde{r}} = 0,$$

we have

$$-\frac{p}{\tilde{p}} + \lambda a + \mu = 0, \tag{18}$$

$$-\frac{q}{\tilde{q}} + \lambda b + \mu = 0, \tag{19}$$

$$-\frac{r}{\tilde{r}} + \lambda c + \mu = 0. \tag{20}$$

From these three conditions and from

$$a\tilde{p}+b\tilde{q}+c\tilde{r} = 0, \tag{21}$$

$$\tilde{p}+\tilde{q}+\tilde{r}-1 = 0, \tag{22}$$

we can determine $\tilde{p}, \tilde{q}, \tilde{r}, \mu, \lambda$.

From (18)-(20) we have

$$\begin{aligned} \tilde{p} &= \frac{p}{\lambda a + \mu}, \\ \tilde{q} &= \frac{q}{\lambda b + \mu}, \\ \tilde{r} &= \frac{r}{\lambda c + \mu}. \end{aligned} \tag{23}$$

Rewrite (18)-(20) in the following form

$$\begin{aligned} \lambda a\tilde{p} + \mu\tilde{p} &= p, \\ \lambda b\tilde{q} + \mu\tilde{q} &= q, \\ \lambda c\tilde{r} + \mu\tilde{r} &= r, \end{aligned}$$

and after summation

$$\lambda(a\tilde{p} + b\tilde{q} + c\tilde{r}) + \mu(\tilde{p} + \tilde{q} + \tilde{r}) = p + q + r.$$

Using (21) and (22), we obtain

$$\mu = 1.$$

Then we substitute (23) with $\mu = 1$ in (21) and obtain the following equation for the determination of λ

$$\frac{ap}{\lambda a + 1} + \frac{bq}{\lambda b + 1} + \frac{cr}{\lambda c + 1} = 0. \tag{24}$$

We consider this equation on the interval $-\frac{1}{c} < \lambda < -\frac{1}{a}$, because in this case $\tilde{p} > 0, \tilde{q} > 0, \tilde{r} > 0$.

Let

$$G(x) = \frac{ap}{xa + 1} + \frac{bq}{xb + 1} + \frac{cr}{xc + 1}.$$

Then on the interval $(-\frac{1}{c}, -\frac{1}{a})$

$$G'(x) = -\frac{a^2p}{(xa + 1)^2} - \frac{b^2q}{(xb + 1)^2} + \frac{c^2r}{(xc + 1)^2} < 0$$

for all x. Further,

$$\lim_{x \to -\frac{1}{a}} G(x) = -\infty$$

and

$$\lim_{x \to -\frac{1}{c}} G(x) = +\infty$$

since $a < 0$ and $c > 0$.

So, it follows, that on the interval $-\frac{1}{c} < \lambda < -\frac{1}{a}$ there exists a unique λ, which satisfies (24).

The equation (24) is equivalent to the following quadratic equation

$$\lambda^2 abc + \lambda[pa(b+c) + qb(a+c) + rc(a+b)] + E\rho_1 = 0. \qquad (25)$$

It is clear, that on the interval $-\frac{1}{c} < \lambda < -\frac{1}{a}$ this quadratic equation has a unique solution.

Thus, the following theorem is valid:

Theorem 2. *The reverse minimal relative entropy martingale measure Q^* for trinomial scheme (1) is unique and is determined by $\tilde{p}^*, \tilde{q}^*$ and $\tilde{r}^*$, which have the form*

$$\tilde{p}^* = \frac{p}{\lambda a + 1},$$

$$\tilde{q}^* = \frac{q}{\lambda b + 1},$$

$$\tilde{r}^* = \frac{r}{\lambda c + 1},$$

where $\lambda \in (-\frac{1}{c}, -\frac{1}{a})$ is the unique solution of the equation (24).

Example 2. Consider the case, when $a = -\alpha, b = 0$ and $c = \alpha, (\alpha > 0)$.

From (24) we have

$$-\lambda\alpha^2(p+r) + E\rho_1 = 0$$

and since

$$E\rho_1 = \alpha(r-p),$$

the representation of λ is the following

$$\lambda = \frac{(r-p)}{\alpha(r+p)}.$$

Then from (25) we have

$$\tilde{p} = \frac{r+p}{2}, \quad \tilde{q} = q \quad \textit{and} \quad \tilde{r} = \frac{r+p}{2}.$$

3. Let $\varphi : (0, \infty) \to R$ be a convex function. Then the φ-divergence distance between Q and P is defined as

$$I_\varphi(Q,P) = \begin{cases} \int \varphi(\frac{dQ}{dP})dP, & if\ the\ integral\ exists, \\ \\ +\infty, & else, \end{cases}$$

where $\varphi(0) = \lim_{x \downarrow 0} \varphi(x)$ (see [2]).

Assume that $\varphi(x)$ satisfies the following conditions:

(A) $\lim_{x \downarrow 0} \varphi'(x) = -\infty$;

(B) $E\varphi(Z_N) = NE\varphi(Z_1)$.

Note, that the both of cases considered above ($\varphi(x) = xln(x), \varphi(x) = -ln(x)$) and also the well-known symmetric divergence distance with $\varphi(x) = (x-1)ln(x)$ satisfy the conditions (A) and (B).

The minimal φ-divergence distance martingale measure is the measure Q^*, for which

$$I_\varphi(Q^*, P) = \min_{Q \in M(P)} I(Q,P).$$

Using (3), (4) and (B) we have

$$I_\varphi(Q,P) = N\left[\varphi\left(\frac{\tilde{p}}{p}\right)p + \varphi\left(\frac{\tilde{q}}{q}\right)q + \varphi\left(\frac{\tilde{r}}{r}\right)r\right].$$

Denote

$$f(\tilde{p}, \tilde{q}, \tilde{r}) = \varphi\left(\frac{\tilde{p}}{p}\right)p + \varphi\left(\frac{\tilde{q}}{q}\right)q + \varphi\left(\frac{\tilde{r}}{r}\right)r.$$

So, the problem of finding the minimal φ-divergence martingale measure Q^* is reduced to the minimization problem of $f(\tilde{p}, \tilde{q}, \tilde{r})$ under constraints:

$$\begin{aligned} a\tilde{p} + b\tilde{q} + c\tilde{r} = 0, \\ \tilde{p} + \tilde{q} + \tilde{r} - 1 = 0. \end{aligned} \tag{26}$$

Lagrangian has the following form

$$\Phi(\tilde{p}, \tilde{q}, \tilde{r}) = \varphi\left(\frac{\tilde{p}}{p}\right)p + \varphi\left(\frac{\tilde{q}}{q}\right)q + \varphi\left(\frac{\tilde{r}}{r}\right)r + \lambda(a\tilde{p} + b\tilde{q} + c\tilde{r}) + \mu(\tilde{p} + \tilde{q} + \tilde{r} - 1).$$

From the optimality conditions

$$\frac{\partial \Phi}{\partial \tilde{p}} = 0, \quad \frac{\partial \Phi}{\partial \tilde{q}} = 0, \quad \frac{\partial \Phi}{\partial \tilde{r}} = 0$$

we have

$$\varphi'\left(\tfrac{\tilde{p}}{p}\right) + \lambda a + \mu = 0, \varphi'\left(\tfrac{\tilde{q}}{q}\right) + \lambda b + \mu = 0, \varphi'\left(\tfrac{\tilde{r}}{r}\right) + \lambda c + \mu = 0. \tag{27}$$

From (27)

$$\lambda = \frac{\varphi'\left(\frac{\tilde{p}}{p}\right) - \varphi'\left(\frac{\tilde{q}}{q}\right)}{b-a}.$$

Using (26) we have

$$\tilde{q} = \frac{(a-c)\tilde{p} + c}{c-b} \tag{28}$$

and therefore,

$$\lambda = \frac{\varphi'\left(\frac{\tilde{p}}{p}\right) - \varphi'\left(\frac{(a-c)\tilde{p}+c}{q(c-b)}\right)}{b-a}. \tag{29}$$

From (27)

$$\mu = \frac{c\varphi'\left(\frac{\tilde{p}}{p}\right) - a\varphi'\left(\frac{\tilde{r}}{r}\right)}{a-c}$$

and from (26)

$$\tilde{r} = \frac{(a-b)\tilde{p} + b}{b-c}, \tag{30}$$

so,

$$\mu = \frac{c\varphi'\left(\frac{\tilde{p}}{p}\right) - a\varphi'\left(\frac{(a-b)\tilde{p}+b}{r(b-c)}\right)}{a-c}. \tag{31}$$

After substitution λ and μ from (29) and (31) in the first equation of (27) we get the following equation for determination of $\tilde{p}$:

$$(b-c)\varphi'\left(\frac{\tilde{p}}{p}\right) - (a-c)\varphi'\left(\frac{(a-c)\tilde{p}+c}{q(c-b)}\right) - (b-a)\varphi'\left(\frac{(a-b)\tilde{p}+b}{r(b-c)}\right) = 0. \tag{32}$$

Let us show that this equation has a unique solution. Consider the function

$$g(x) = (b-c)\varphi'\left(\frac{x}{p}\right) - (a-c)\varphi'\left(\frac{(a-c)x+c}{q(c-b)}\right) - (b-a)\varphi'\left(\frac{(a-b)x+b}{r(b-c)}\right),$$
$$x \in \left(\frac{b}{b-a}, \frac{c}{c-a}\right).$$

From definition of φ and (A) we have

$$\lim_{x \downarrow b/(b-a)} g(x) = +\infty, \qquad \lim_{x \uparrow c/(c-a)} g(x) = -\infty.$$

Further,

$$g'(x) = \frac{b-c}{p}\varphi''\left(\frac{x}{p}\right) - \frac{(a-c)^2}{q(c-b)}\varphi''\left(\frac{(a-c)x+c}{q(c-b)}\right) - $$
$$-\frac{(b-a)^2}{r(c-b)}\varphi''\left(\frac{(a-b)x+b}{r(b-c)}\right)$$

and, hence, $a < b < c$ and φ is a convex function, $g'(x) < 0$. It follows that there exists a unique $\tilde{p}$, which satisfies the equation (32). So, $\tilde{q}$ and $\tilde{r}$ are also determined uniquely from (28) and (30).

It is easy to check that the Hessian matrix (see [5], p.148) is positively defined for our Lagrangian and therefore $(\tilde{p}, \tilde{q}, \tilde{r})$ is a point for which it takes a minimum.

So, we have proved the following theorem:

Theorem 3. *Let $\varphi : (0, \infty) \to R$ be a convex function and (A), (B) are fulfilled, then there exists a unique minimal φ-divergence martingale measure Q^* for trinomial scheme (1) which is determined by $\tilde{p}^*$, $\tilde{q}^*$ and $\tilde{r}^*$, where $\tilde{p}^*$ is the unique solution of the equation*

$$(b-c)\varphi'\left(\frac{\tilde{p}^*}{p}\right) - (a-c)\varphi'\left(\frac{(a-c)\tilde{p}^* + c}{q(c-b)}\right) - (b-a)\varphi'\left(\frac{(a-b)\tilde{p}^* + b}{r(b-c)}\right) = 0,$$

on the interval $(\frac{b}{b-a}, \frac{c}{c-a})$ and

$$\tilde{q}^* = \frac{(a-c)\tilde{p}^* + c}{b-a}, \quad \tilde{r}^* = 1 - \tilde{p}^* - \tilde{q}^*.$$

Remark 1. *From Theorem 3 in case when $\varphi(x) = xln(x)$ as a consequence follows the result of part II of Theorem 1. When $\varphi(x) = -ln(x)$ from Theorem 3 follows the result in the different form than the result of Theorem 2.*

Note, that the trinomial scheme (1) is popular model for description of the evolution of a stock price. The trinomial financial market in contrast to binomial is incomplete market with many martingale measures. Here we do not give the application of our results in finances. The problem of valuation of contingent claims and hedging we will consider in future.

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