SOME CLASSICAL PROBLEMS OF NONLINEAR MATHEMATICAL ELASTICITY

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Abstract

In the paper two classical problems of nonlinear elasticity are considered: elastic body on a rigid support and body in an elastic hull (see [3]). The existence of solutions of stated problems is shown.

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Combining the equations of equilibrium in the reference configuration Ω, expressed in terms of the first Piola-Kirchoff stress tensor with the definition of an elastic material and assuming that the boundary condition of the place is specified on the portion $\Gamma_0 = \Gamma - \Gamma_1$ of the boundary Ω, we find that the deformation $\boldsymbol{\varphi}$ satisfies the following boundary value problem (see [2])

$$-\mathbf{div}\widehat{\mathbf{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Omega, \tag{1}$$

$$\widehat{\mathbf{T}}(x,\ \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) \cdot \mathbf{n} = \widehat{\mathbf{g}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)), \quad x \in \Gamma_1, \tag{2}$$

$$\boldsymbol{\varphi}(x) = \boldsymbol{\varphi}_0(x), \quad x \in \Gamma_0, \tag{3}$$

where $\widehat{\mathbf{T}}:\ \ \overline{\Omega} \times M^3_+ \to M^3$ is the response function for the first Piola-Kirchoff stress tensor; M_3- set of real square matrices of the third order; $M^3_+ = \{\mathbf{F} \in M^3; det\mathbf{F} > 0\}$; $\mathbf{n}$ - unit outer normal vector along $\partial\Omega$, $\widehat{\mathbf{f}}$ - density of the applied body force per unit volume in the reference configuration; $\widehat{\mathbf{g}}$ - density of the applied surface force per unit area in the reference configuration (here and below we use the same definitions and notations as in book [2]).

The first and second equations together are equivalent, at least formally, to the principle of virtual work in the reference configuration, expressed by the equations:

$$\int_{\Omega} \widehat{\mathbf{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) : \boldsymbol{\nabla}\boldsymbol{\theta}(x)dx = \int_{\Omega} \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)dx$$
$$+ \int_{\Gamma_1} \widehat{\mathbf{g}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)da, \tag{4}$$

valid for all sufficiently regular vector fields $\boldsymbol{\theta} : \ \overline{\Omega} \to R^3$, which vanish on Γ_0.

An elastic material with response function $\widehat{\mathbf{T}} : \ \overline{\Omega} \times M^3_+ \to M^3$ is hyperelastic if there exists a function

$$\widehat{W} : \overline{\Omega} \times M^3_+ \to R,$$

differentiable with respect to the variable $\mathbf{F} \in M^3_+$ for each $x \in \overline{\Omega}$, such that

$$\widehat{\mathbf{T}}(x, \mathbf{F}) = \frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) \quad for\ all \quad x \in \Omega,\ \mathbf{F} \in M^3_+,$$

i.e., componentwise

$$\widehat{T}_{ij}(x, \mathbf{F}) = \frac{\partial \widehat{W}}{\partial F_{ij}}(x, \mathbf{F}).$$

The function $\widehat{W}$ is called a stored energy function.

If we consider conservative applied body forces and conservative applied surface forces, for which the integral appearing in the right-hand side of (4) can be written as G$\hat{a}$teaux derivatives

$$\int_{\Omega} \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x))\boldsymbol{\theta}(x)dx = F^{'}(\boldsymbol{\varphi})\boldsymbol{\theta},$$

$$\int_{\Gamma_1} \widehat{\mathbf{g}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)da = G^{'}(\boldsymbol{\varphi})\boldsymbol{\theta},$$

of functionals F and G of the form

$$F(\boldsymbol{\psi}) = \int_{\Omega} \widehat{F}(x, \boldsymbol{\psi}(x))dx, \ \ G(\boldsymbol{\psi}) = \int_{\Gamma_1} \widehat{G}(x, \boldsymbol{\psi}(x)), \boldsymbol{\nabla}\boldsymbol{\psi}(x))da,$$

then the equations

$$-\mathbf{div}\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Omega,$$

$$\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\mathbf{n} = \widehat{\mathbf{g}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Gamma_1,$$

are formally equivalent to the equation

$$I^{'}(\boldsymbol{\varphi})\boldsymbol{\theta} = 0,$$

for all smooth maps $\boldsymbol{\theta}: \ \overline{\Omega} \to R^3$ that vanish on Γ_0, where the functional I is defined for sufficiently smooth mappings $\boldsymbol{\psi}: \ \overline{\Omega} \to R^3$ by

$$I(\boldsymbol{\psi}) = \int\limits_{\Omega} \widehat{W}(x, \boldsymbol{\nabla}\boldsymbol{\psi}(x))dx - \{F(\boldsymbol{\psi}) + G(\boldsymbol{\psi})\}.$$

The functional W defined for any sufficiently smooth mapping $\boldsymbol{\psi}$ by

$$W(\boldsymbol{\psi}) = \int_{\Omega} \widehat{W}(x, \boldsymbol{\nabla}\psi(x))dx$$

is called the strain energy, and the functional I is called the total energy.

Let the assumptions and notations be as above. Then any sufficiently smooth mapping $\boldsymbol{\varphi}$ that satisfies

$$\boldsymbol{\varphi} \in \boldsymbol{\Phi} : = \left\{ \boldsymbol{\psi}: \ \overline{\Omega} \to R^3, \ \boldsymbol{\psi} = \boldsymbol{\varphi}_0 \ \ on \ \ \Gamma_0 \right\}$$

and $I(\boldsymbol{\varphi}) = \underset{\boldsymbol{\psi}\in\boldsymbol{\Phi}}{inf} \ I(\boldsymbol{\psi})$,

with $I(\boldsymbol{\psi}) = \int\limits_{\Omega} \widehat{W}(x, \boldsymbol{\nabla}\psi(x))dx - \{F(\boldsymbol{\psi}) + G(\boldsymbol{\psi})\}$,

solves the following boundary value problem

$$-\mathbf{div}\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\varphi(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Omega,$$

$$\boldsymbol{\varphi}(x) = \boldsymbol{\varphi}_0(x), \quad x \in \Gamma_0,$$

$$\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\mathbf{n} = \widehat{\mathbf{g}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Gamma_1.$$

Let us now consider the following problem - an elastic body on a rigid support.

$$-\mathbf{div}\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Omega, \tag{5}$$

$$\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\mathbf{n} = \widehat{\mathbf{g}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Gamma_1. \tag{6}$$

$$\frac{\partial \widehat{W}}{\partial F_{ij}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\mathbf{n}_j = \widehat{g}_i(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)), \quad x \in \Gamma_0, \ \ i = 1, 2; \ \ j = 1, 2, 3, \tag{7}$$

$$\varphi_3(x) = \varphi_{03}(x), \quad x \in \Gamma_0, \tag{8}$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, $\widehat{\mathbf{g}} = (\widehat{\mathrm{g}}_1, \widehat{\mathrm{g}}_2, \widehat{\mathrm{g}}_3)$ (the repeated index means summation).

Problem (5)-(8) is formally equivalent to the principle of virtual work in the reference configuration

$$\int\limits_{\Omega} \frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) \cdot \boldsymbol{\nabla}\boldsymbol{\theta}(x) dx = \int\limits_{\Omega} \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x))\boldsymbol{\theta}(x) dx + \int\limits_{\Gamma_1} \widehat{\mathbf{g}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\boldsymbol{\theta}(x) da + \int\limits_{\Gamma_0} (\widehat{g}_1(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\theta_1(x) + \widehat{g}_2(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\theta_2(x)) da \tag{9}$$

valid for all sufficiently regular vector fields $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) : \overline{\Omega} \to R^3$, $\theta_3|_{\Gamma_0} = 0$.

Below we shall assume that applied body and surface forces are dead loads, i.e.,

$$\widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) = \widehat{\boldsymbol{f}}(x),$$

$$\widehat{\boldsymbol{g}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \widehat{\boldsymbol{g}}(x).$$

Problem (9) is formally equivalent to the following problem

$$I_1(\boldsymbol{\varphi}) = \underset{\boldsymbol{\psi} \in \boldsymbol{\Phi}_1}{inf}\ I_1(\boldsymbol{\psi}),$$

where

$$I_1(\boldsymbol{\psi}) = \int\limits_{\Omega} \widehat{W}(x, \boldsymbol{\nabla}\boldsymbol{\psi}(x)) dx - \{F(\boldsymbol{\psi}) + G(\boldsymbol{\psi}) + G_1(\boldsymbol{\psi})\},$$

$$G_1(\boldsymbol{\psi}) = \int\limits_{\Gamma_0} (\widehat{g}_1(x)\psi_1(x) + \widehat{g}_2(x)\psi_2(x)) da,$$

$\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$, as to set $\boldsymbol{\Phi}_1$ we will define it below.

Theorem 1. *Let Ω be a domain in R^3, and let $\widehat{W} : \ \Omega \times M^3_+ \to R$ be a stored energy function with the following properties:*

(a) Polyconvexity: For almost all $x \in \Omega$, there exists a convex function $W(x, \cdot) : M^3 \times M^3 \times]0, +\infty[\to R$ such that

$$W(x,\ \mathbf{F},\ \mathbf{Cof}\mathbf{F}, det\mathbf{F}) = \widehat{W}(x, \mathbf{F}) \quad for\ all \quad \mathbf{F} \in M^3_+,$$

the function $W(\cdot, \mathbf{F}, \mathbf{H}, \delta) : \Omega \to R$ is measurable for all $(\mathbf{F}, \mathbf{H}, \delta) \in M^3 \times M^3 \times]0, +\infty[$;

(b) Behavior as $det\mathbf{F} \to 0^+$: For almost all $x \in \Omega$

$$\underset{det\mathbf{F} \to 0^+}{\widehat{W}(x, \mathbf{F})} = +\infty;$$

(c) Coerciveness: There exist constants α, β, p, q, r *such that*

$$d > 0, \;\; p \geq 2, \;\; q \geq \frac{p}{p-1}, \;\; r > 1,$$

$$\widehat{W}(x, \mathbf{F}) \geq \alpha(\|\mathbf{F}\|^p + \|\mathbf{CofF}\|^q + (det\mathbf{F})^r) + \beta$$

for almost all $x \in \Omega$ *and for all* $\mathbf{F} \in M^3_+$.

Let $\Gamma = \Gamma_0 \cup \Gamma_1$ *be a da-measurable partition of the boundary* Γ *of* Ω *with area* $\Gamma_0 > 0$, *and let* $\varphi_0 : \;\; \Gamma_0 \to R$ *be a measurable function such that the set*

$$\mathbf{\Phi}_1 := \Big\{ \boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in W^{1,p}(\Omega); \;\; \boldsymbol{Cof}\boldsymbol{\nabla}\boldsymbol{\psi} \in L^q(\Omega), \;\; det\boldsymbol{\nabla}\boldsymbol{\psi} \in L^r(\Omega),$$

$$\psi_3 = \varphi_0 \;\; da - a.e.on \;\; \Gamma_0, det\boldsymbol{\nabla}\boldsymbol{\psi} > 0 \;\; a.e.in \;\; \Omega,$$

$$\int\limits_\Omega (\psi_1, \;\; \psi_2) dx = \mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in R^2 \Big\}$$

is nonempty. Let $f \in L^\rho(\Omega)$ *and* $g \in L^\sigma(\Gamma)$ *be such that the linear form*

$$L_1 : \;\; \boldsymbol{\psi} \in W^{1,p} \to L_1(\boldsymbol{\psi}) = F(\boldsymbol{\psi}) + G(\boldsymbol{\psi}) + G_1(\boldsymbol{\psi})$$

is continuous, let

$$I_1(\boldsymbol{\psi}) = \int\limits_\Omega \widehat{W}(x, \boldsymbol{\nabla}\boldsymbol{\psi}(x)) dx - L_1(\boldsymbol{\psi})$$

and assume that $\inf\limits_{\boldsymbol{\psi} \in \mathbf{\Phi}_1} I_1(\boldsymbol{\psi}) < +\infty$.

Then there exists at least one function $\boldsymbol{\varphi}$ *such that*

$$\boldsymbol{\varphi} \in \mathbf{\Phi}_1 \quad and \quad I_1(\boldsymbol{\varphi}) = \inf\limits_{\boldsymbol{\psi} \in \mathbf{\Phi}_1} I_1(\boldsymbol{\psi}).$$

Proof. Let us find a lower bound for $I_1(\boldsymbol{\psi})$, $\quad \boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in \mathbf{\Phi}_1$. For estimation of ψ_1 and ψ_2 we use the generalized Poincare inequality in the following form

$$\int\limits_\Omega |\; \psi_i(x) \;|^p \; dx \leq c_1 \left\{ \int\limits_\Omega |\; \mathbf{grad}\psi_i(x) \;|^p \; dx + \left| \int\limits_\Omega \psi_i(x) dx \right|^p \right\}, \quad i = 1, 2, \tag{10}$$

and for ψ_3 - Friedrichs inequality

$$\int\limits_\Omega |\; \psi_3(x) \;|^p \; dx \leq c_2 \left\{ \int\limits_\Omega |\; \mathbf{grad}\psi_3(x) \;|^p \; dx + \int\limits_{\Gamma_0} |\psi_3(x)|^p da \right\},$$

i.e.

$$\int_{\Omega} \mid \psi_3(x) \mid^p dx \leq c_2 \left\{ \int_{\Omega} \mid \mathbf{grad}\psi_3(x) \mid^p dx + \int_{\Gamma_0} \mid \varphi_0(x) \mid^p da \right\}. \tag{11}$$

By the assumed coerciveness of the function $\widehat{W}$ and by the assumed continuity of the linear form L_1,

$$\begin{aligned} &I_1(\boldsymbol{\psi}) \geq \alpha \int_{\Omega} \{\|\boldsymbol{\nabla\psi}\|^p + \|\boldsymbol{Cof\nabla\psi}\|^q + (det\boldsymbol{\nabla\psi})^r\} dx + \beta vol\Omega \\ &-(\|\mathbf{F}\| + \|\mathbf{G}\| + \|\mathbf{G_1}\|)\|\boldsymbol{\psi}\|_{1,p,\Omega}. \end{aligned} \tag{12}$$

From inequalities (11)-(13) and from the relation $\int_{\Omega}(\psi_1, \psi_2)dx = e$, as $p > 1$, we can infer that there exist c_3 and d such that

$$I(\boldsymbol{\psi}) \geq c_3 \left\{ \|\boldsymbol{\psi}\|^p_{1,p,\Omega} + |\boldsymbol{Cof\nabla\psi}|^q_{0,q,\Omega} + |det\boldsymbol{\nabla\psi}|^r_{0,r,\Omega} \right\} dx + d \tag{13}$$

for all $\boldsymbol{\psi} \in \boldsymbol{\Phi}_1$.

Let $(\boldsymbol{\varphi}^k)$ be an infimizing sequence for the functional I_1, i.e., a sequence that satisfies

$$\boldsymbol{\varphi}^k \in \boldsymbol{\Phi}_1 \quad for\ all \quad k, \quad and \quad \lim_{k\to\infty} I_1(\boldsymbol{\varphi}^k) = \inf_{\boldsymbol{\psi}\in\boldsymbol{\Phi}_1} I_1(\boldsymbol{\psi}).$$

By assumption, $\inf\limits_{\boldsymbol{\psi}\in\boldsymbol{\Phi}_1} I_1(\boldsymbol{\psi}) < +\infty$, according to (14), the sequence

$$(\boldsymbol{\varphi}^k, \ \boldsymbol{Cof\nabla\varphi}^k, \ det\boldsymbol{\nabla\varphi}^k)$$

is bounded in the reflexive Banach space $W^{1,p}(\Omega) \times L^q(\Omega) \times L^r(\Omega)$ (each number p, q, r is > 1).

And now we must check that if $\boldsymbol{\varphi}^k \rightharpoonup \boldsymbol{\varphi}$ in $W^{1,p}(\Omega)$ then

$$\int_{\Omega}(\varphi_1^k, \varphi_2^k)dx \to \int_{\Omega}(\varphi_1, \varphi_2)dx.$$

It is true because $1 \in (W^{1,p}(\Omega))^*$. The rest part of the theorem is proved analogously as in [2],[1].

Now we will consider the problem - a body in an elastic hull. For this we will introduce the following notations:

$$\widehat{T}_N := \widehat{T}_{ij}n_in_j, \quad \widehat{T}_T := \left\{\widehat{T}_{iT}\right\}, \quad \widehat{T}_{iT} := \widehat{T}_{ij}n_j - \widehat{T}_N n_i,$$

$$v_N := v_in_i; \quad v_T = v - nv_N; \quad n = \{n_i\}, \quad i = 1, 2, 3.$$

Then

$$(\widehat{T}_{ij}n_j)v_i = \widehat{T}_T v + \widehat{T}_N v_N = \widehat{T}_T v_T + \widehat{T}_N v_N.$$

The following problem

$$-\mathbf{div}\widehat{\mathbf{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Omega, \tag{14}$$

$$\widehat{\mathbf{T}}_T(x,\ \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = 0, \quad x \in \Gamma, \tag{15}$$

$$\widehat{T}_N(x,\ \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) + k u_N = 0, \quad k > 0,\ \ x \in \Gamma, \tag{16}$$

where $\mathbf{u} = \boldsymbol{\varphi} - id$ is a displacement vector, represents the above mentioned one. Problem (14)-(16) can be rewritten in the following form

$$-\mathbf{div}\widehat{\mathbf{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), \quad x \in \Omega, \tag{17}$$

$$\widehat{\mathbf{T}}_T(x,\ \nabla\varphi(x)) = 0, \quad x \in \Gamma, \tag{18}$$

$$\widehat{T}_N(x,\ \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) + k\varphi_N - k x_i n_i = 0, \quad x \in \Gamma. \tag{19}$$

Problem (17)-(19) is formally equivalent to the following principle of virtual work in the reference configuration

$$\int_\Omega \widehat{\mathbf{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) : \boldsymbol{\nabla}\boldsymbol{\theta}(x)dx = \int_\Omega \widehat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)dx + \int_\Gamma \widehat{T}_N \theta_N da,$$

or, taking into account (19),

$$\begin{aligned} \int_\Omega \widehat{\mathbf{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) : \boldsymbol{\nabla}\boldsymbol{\theta}(x)dx &= \int_\Omega \widehat{\boldsymbol{f}}(x, \boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)dx \\ -k\int_\Gamma \varphi_N \theta_N(x)da + k\int_\Gamma x_i n_i \theta_N(x)da, & \end{aligned} \tag{20}$$

valid for all sufficiently regular vector fields $\boldsymbol{\theta} : \overline{\Omega} \to R^3$.

Let as consider the functional

$$I_2(\boldsymbol{\psi}) = \int_\Omega \widehat{W}(x,\ \boldsymbol{\nabla}\boldsymbol{\psi}(x))dx + J(\boldsymbol{\psi}) - F(\boldsymbol{\psi}) - g(\boldsymbol{\psi}),$$

where

$$J(\boldsymbol{\psi}) = \frac{1}{2}k\int_\Gamma \psi_N^2(x)da,$$

$$g(\boldsymbol{\psi}) = k\int_\Gamma x_i n_i \psi_N(x)da.$$

Then, if we assume that the material is hyperelastic, problem (20) is equivalent to the equation

$$I_2'(\boldsymbol{\varphi})\boldsymbol{\theta} = 0,$$

and any sufficiently smooth mapping $\boldsymbol{\varphi}$, that satisfies

$$I_2(\boldsymbol{\varphi}) = \underset{\boldsymbol{\psi}\in\boldsymbol{\Phi}_2}{inf}\; I(\boldsymbol{\psi}),$$

solves problem (17)-(19).

At first we prove the following

Lemma. *If* $\mathbf{u} = (u_1, u_2, u_3) \in W^{1,p}(\Omega), \quad p \geq 2$, *then there exists such* $c > 0$, *that*

$$\int\limits_\Omega |\boldsymbol{\nabla u}|^p dx + \int\limits_\Gamma |\nabla \mathbf{u}_N|^p da \geq c \int\limits_\Omega |\mathbf{u}|^p dx. \tag{21}$$

Proof. If instead of function u we will consider $u|u|_{0,p,\Omega}^{-1}$, then inequality (21) is equivalent to the relation

$$|\mathbf{u}|_{0,p,\Omega} = 1, \quad \int\limits_\Omega |\boldsymbol{\nabla u}|^p dx + \int\limits_\Gamma |\boldsymbol{\nabla u}_N|^p da \geq c. \tag{22}$$

Let us assume, that relation (22) is not valid. Then there exists such a sequence $(\boldsymbol{u}_\alpha)$, that

$$|\boldsymbol{u}_\alpha|_{0,p,\Omega} = 1, \quad \int\limits_\Omega |\boldsymbol{\nabla u}_\alpha|^p dx + \int\limits_\Gamma |\boldsymbol{\nabla u}_{\alpha N}|^p da \to 0. \tag{23}$$

From (23) it follows, that the sequence (u_α) is bounded in $W^{1,p}(\Omega)$. Therefore, we can say that

$$\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u} \quad in \quad W^{1,p}(\Omega).$$

As $W^{1,p}(\Omega)$ is compactly embedding in $L^p(\Omega)$, hence it follows, that

$$\boldsymbol{u}_\alpha \to \boldsymbol{u} \quad in \quad L^p(\Omega).$$

Therefore $|\boldsymbol{u}|_{0,p,\Omega} = 1$.

Since $|x|^p, \quad p \geq 2$, is a convex function, therefore the continuous functional

$$\int\limits_\Omega |\boldsymbol{\nabla u}|^p dx + \int\limits_\Gamma |\boldsymbol{u}_N|^p da$$

in $W^{1,p}(\Omega)$ is convex. Hence follows, that this functional is weakly continuous and

$$\int\limits_\Omega |\boldsymbol{\nabla u}_\alpha|^p dx + \int\limits_\Gamma |\boldsymbol{u}_{\alpha N}|^p da \to \int\limits_\Omega |\boldsymbol{\nabla u}|^p dx + \int\limits_\Gamma |\boldsymbol{u}_N|^p da = 0.$$

From this relation we obtain that $\boldsymbol{\nabla u} = 0$, i.e. $u = const$. From the condition $u_N|_\Gamma = 0$, and as Γ cannot be a plane, it follows that $\boldsymbol{u} \equiv 0$. So, we have obtained a contradiction. Thus, the Lemma is proved (for $p = 2$ this Lemma is proved in [3].)

Using this Lemma we will prove the following

Theorem 2. *Let Ω be a domain in R^3, and let $\widehat{W}: \quad \Omega \times M_+^3 \to R$ be a stored energy that satisfies assumption (a),(b),(c) of Theorem 1 (polyconvexity, behavior as $det\boldsymbol{F} \to 0^+$, coerciveness). Let*

$$\boldsymbol{\Phi}_2 := \Big\{\boldsymbol{\psi} = \{\psi_1, \psi_2, \psi_3\} \in W^{1,p}(\Omega), \quad \boldsymbol{Cof\nabla\psi} \in L^q(\Omega),$$

$$det\boldsymbol{\nabla\psi} \in L^r(\Omega), \quad det\boldsymbol{\nabla\psi} > 0 \quad a.e.in \quad \Omega\}$$

then $p = 2$,

$$\boldsymbol{\Phi}_2 := \Big\{\boldsymbol{\psi} = \{\psi_1, \psi_2, \psi_3\} \in W^{1,p}(\Omega), \quad \boldsymbol{Cof\nabla\psi} \in L^q(\Omega),$$

$$det\boldsymbol{\nabla\psi} \in L^r(\Omega), \quad det\boldsymbol{\nabla\psi} > 0 a.e.in\Omega, \quad \|\boldsymbol{\psi}\|_{L^\infty} \le M, \quad M = const > 0\}$$

then $p > 2$.

Let $\inf\limits_{\boldsymbol{\psi}\in\boldsymbol{\Phi}_2} I_2(\psi) < +\infty$. Then there exists at least one function $\boldsymbol{\varphi} \in \boldsymbol{\Phi}_2$ such that

$$\boldsymbol{\varphi} \in \boldsymbol{\Phi}_2 \quad and \quad I_2(\boldsymbol{\varphi}) = inf I_2(\boldsymbol{\psi}).$$

Proof. First we will consider the case $p = 2$. From the condition of coerciveness we obtain

$$I_2(\boldsymbol{\psi}) \ge \alpha \int\limits_\Omega \Big\{\|\boldsymbol{\nabla\psi}\|^2 + \|\boldsymbol{Cof\nabla\psi}\|^q + (det\boldsymbol{\nabla\psi})^r\Big\} dx + \beta vol\Omega$$

$$+\frac{1}{2}k \int\limits_\Gamma \psi_n^2 da - \|\boldsymbol{F}\|\|\boldsymbol{\psi}\|_{1,2,\Omega} - \|\boldsymbol{g}\|\|\boldsymbol{\psi}\|_{1,2,\Omega}.$$

According to inequality (21)

$$I_2(\boldsymbol{\psi}) \ge \alpha_1 \Big\{\|\boldsymbol{\psi}\|^2_{1,2,\Omega} + |\boldsymbol{Cof\nabla\psi}|^q_{0,r,\Omega} + |det\boldsymbol{\nabla\psi}|^r_{0,r,\Omega}\Big\} + d,$$

where $\alpha_1 > 0$. Hence we can state, that if $(\boldsymbol{\varphi}^k)$ is an infimizing sequence for the functional I_2, then the sequence $(\boldsymbol{\varphi}^k, \ \boldsymbol{Cof\nabla\varphi}^k, \ det\boldsymbol{\nabla\varphi}^k)$ is bounded in the reflexive Banach space $W^{1,p}(\Omega) \times L^q(\Omega) \times L^r(\Omega)$. It remains to show, that if $\boldsymbol{\varphi}^k \rightharpoonup \boldsymbol{\varphi} \quad in \quad W^{1,p}(\Omega)$, then

$$\int\limits_\Gamma (\varphi_N^k)^2 da \to \int\limits_\Gamma \varphi_N^2 da. \tag{24}$$

This follows from the compactness of the trace operator $tr \in L(W^{1,2}(\Omega),\ L^2(\Gamma))$.

Now we will consider the case $p > 2$. Let $\boldsymbol{u} \in \boldsymbol{\Phi}_2$ and $\boldsymbol{v} = \boldsymbol{u}/M$, then $\|\boldsymbol{v}\|_{L^\infty} \leq 1$. From inequality (21) and relation $|v_N| = |\boldsymbol{v}| \cdot |\boldsymbol{n}||cos\alpha| \leq |\boldsymbol{v}|$, i.e., $\|\boldsymbol{v_N}\|_{L^\infty(\Gamma)} \leq 1$, we obtain that

$$c\int\limits_\Omega |\boldsymbol{v}|^p dx \leq \int\limits_\Omega |\boldsymbol{\nabla v}|^p dx + \int\limits_\Gamma |\boldsymbol{v_N}|^p da \geq \int\limits_\Omega |\boldsymbol{\nabla v}|^p dx + \int\limits_\Gamma |\boldsymbol{v_N}|^2 da.$$

Hence

$$c\int\limits_\Omega \left|\frac{\boldsymbol{u}}{M}\right|^p dx \leq \int\limits_\Omega \left|\nabla \frac{\boldsymbol{u}}{M}\right|^p dx + \int\limits_\Gamma \left|\left(\frac{\boldsymbol{u}}{M}\right)_N\right|^2 da,$$

or

$$\frac{c}{M^p}\int\limits_\Omega |\boldsymbol{u}|^p\, dx \leq \frac{1}{M^p}\int\limits_\Omega |\boldsymbol{\nabla u}|^p\, dx + \frac{1}{M^2}\int\limits_\Gamma |\boldsymbol{u_N}|^2\, da.$$

Thus,

$$\int\limits_\Omega |\boldsymbol{\nabla u}|^p\, dx + \int\limits_\Gamma |\boldsymbol{u_N}|^2\, da \geq c_1 \int\limits_\Omega |\boldsymbol{u}|^p\, dx.$$

Hence, in an analogous manner as above, we can state that if (φ^k) is an infimizing sequence for the functional I_2, then the sequence $(\boldsymbol{\varphi}^k,\ \boldsymbol{Cof\nabla\varphi}^k,\ det\boldsymbol{\nabla\varphi}^k)$ is bounded in the reflexive Banach space $W^{1,p}(\Omega) \times L^q(\Omega) \times L^(\Omega)$.

Let $\boldsymbol{\varphi}^l \rightharpoonup \boldsymbol{\varphi}$ in $W^{1,p}(\Omega)$, then $\boldsymbol{\varphi}^l \to \boldsymbol{\varphi}$ in $L^2(\Omega)$. Therefore, there exists such subsequence $(\boldsymbol{\varphi}^{l_k})$ that converges almost everywhere to $\boldsymbol{\varphi}$. As $\|\boldsymbol{\varphi}^{l_k}\|_{L^\infty(\Gamma)} \leq M$, so $\|\boldsymbol{\varphi}\|_{L^\infty(\Gamma)} \leq M$.

As to relation (24) it is proved in the same manner.

References

1. Ball J. M. *Convexity Conditions and Existence Theorems in Nonlinear Elasticity.* Arch. Rational Mech. Anal., v. 63, pp. 337-403.

2. Ciarlet P. G. *Mathematical Elasticity, Volume 1: Three Dimensional Elasticity.* North-Holland, Amsterdam, 1988.

3. Duvaut G., Lions J.-L. *Les Inequations en Mecanique et en Physique.* Dunod, Paris, 1972.