COMPUTATIONAL COMPLEXITY OF THE INEQUALITY PROBLEM FOR ONE CLASS OF SEMIGROUPS

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Abstract

It is proved that the algorithmic inequality problem for the class $\mathcal{K}$ of partially ordered semigroups $\mathbf{S}$ with a finite number of defining inequalities is decidable with the space c^n, where $c = const$ for $\mathbf{S}$.

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1. *Introduction*

Let M be an alphabet. In the set W_M of all words in the alphabet M the associative operator of multiplication is introduced: if $X \in W_M$ and $Y \in W_M$, then $XY \in W_M$. Let $\{(A_i, B_i),\ i \in \mathcal{I}\}$ be a system of ordered pair of graphical distinct words from W_M. In the set W_M we make two types of elementary transforms of words: a) tautological transform; b) admissible change in a word XA_iY from W_M for any isolated word A_i (is produced by scheme $XA_iY \to XB_iY$). If there is a finite sequence of elementary transforms which transfers a word W_1 into word W_2, one writes $W_1 \geq W_2$. If $W_1 \geq W_2$ and $W_2 \geq W_1$, then one writes $W_1 = W_2$. $=$ is an equivalence relation on W_M [1]. The set of all equivalence classes $[X]$ (where $X \in W_M$) with respect to $=$ is a semigroup with respect to included operation of class product. In this semigroup induce the ordering of classes. Thus partially ordered (p.o.) semigroup $\mathbf{S}$ given in a alphabet M with defining inequalities $A_i \geq B_i$, $i \in \mathcal{I}$ is defined (see [2] too).

Defining inequality $A_{i_0} \geq B_{i_0}$ is called invertible [2] if in A the inequality $B_{i_0} \geq A_{i_0}$ holds. Then it is possible to rewrite the invertible defining inequality $A_{i_0} \geq B_{i_0}$ in the form $Ai_0 = B_{i0}$.

It is known that a representation of semigroups with defining inequalities is more general than their representation with defining equalities (in a semigroup ordering is given too) and represents a topic of much more interest.

The equality problem for a given semigroup in very general view was formulated by A. Thue [9]. First A. A. Markov [5] and E. L. Post [7] proved that there are semigroups with unsolvability of equality problem. For semigroups **S** given in a finite alphabet with a finite number of defining inequalities along with equality problem also arises the inequality problem.

The algorithmic inequality problem for p.o. semigroup **S** given in a finite alphabet M with a finite number of defining inequalities consists in finding an algorithm which decides whether $X \geq Y$ or not for an arbitrary pair of words X, Y from W_M. Obviously, if the inequality problem for p.o. semigroup **S** is decidable, then for **S** is decidable the equality problem too. Therefore the inequality problem is more general than equality problem and naturally p.o. semigroups with decidable inequality problem represents a subject of particular interest.

Torkalanov [10], basing on the papers [6,8,3], constructed a wide class of p.o. semigroups with decidable algorithmic inequality problem. This class includes the Osipova's class of semigroups with decidable equality problem, and moreover, these classes of semigroups do not coincide.

Decision theory and computational complexity theory are two very active branches of mathematical logic. In decision theory mathematicians have found a large number of decidable and undecidable problems. In the area of computational complexity the appearance of Turing machine makes it possible for mathematicians to compare different kinds of algorithms [4].

The aim of our research is to give an upper bound of the computational complexity (the storage space) [11] of the inequality problem for the class from [10]. According to our aim we rather change the decision procedure from [10].

Let **S** be p.o. semigroup in a finite alphabet $M = \{a_1, ..., a_p\}$ with a finite number of defining inequalities $A_i \geq B_i$, $i = 1, ..., q$. $\ell(X)$ denotes the length of the word X, $\doteq$ – graphic congruence, $L = \max\{\ell(A_i), \ell(B_i),\ i = 1, \ldots, q\}$.

Beginning X [end Y] of the word XY in M is called the right beginning [end] for XY, if $\ell(X) \geq 1/2\ell(XY)$ [$\ell(Y) \geq 1/2\ell(XY)$].

Definition 1. [10] *The p.o. semigroup* **S** *belongs to class* $\mathcal{K}$ *if its defining inequalities satisfy the following conditions:*

1) If a left defining word A_i and a defining word C_j (A_j or B_j) have a common part which is a right beginning [end] for some of them, then this common part is the beginning [end] for the other too;

2) If P is a right beginning of a defining word B_i and simultaneously is a right end for a defining word B_j, then $P \doteq B_i \doteq B_j$.

Definition 2. [6] *Decomposition $E = R_1R_2\cdots R_k$ of the word $E \in W_M$ is called the normal decomposition of order k, if*

a) every R_i is a part of some defining word;

b) R_1 is the right beginning and B_k is the right end of the defining word;

c) for any $i < k$ R_i is a right end of some defining word or if that is false, then R_{i+1} is a right beginning of some defining word.

If a word E has a normal decomposition of order k, then E is called the normal word of order k.

Lemma 1. *There exists an algorithm which for any word $E \in W_M$ decides whether E is a normal word or not using the space c^x, where $x = \ell(E)$, $c = const$ for* $\mathbf{S}$.

Proof. We can write out all parts of the defining words. The number of them $\leq qL^2$. For writing of these words use the space $\leq qL^3 = const$ for $\mathbf{S}$. We compose all possible products with the length x of parts above mentioned. The number of them $\leq (qL^2)^x$. By Definition 2 the given product Q of parts simply verifies whether Q is a normal word or not. It is verified whether $Q = E$ or not. If yes, then it is verified whether Q is a normal word or not. If yes, then E is the normal word. If not we come to the next product of parts and so on. If we do not meet such a word Q in this way that $Q = E$, then the word E cannot be a normal word. The writing of all products of parts of the defining words with the length x uses the space $x(qL^2)^x$, hence the space c^x, where $x = \ell(E)$, $c = const$ for $\mathbf{S}$.□

Lemma 2. *Let X be a normal word with the order k. A number of all such words Y that $X \geq Y$ in the p.o. semigroup $\mathbf{S} \in \mathcal{K}$ does not exceed $kL \cdot p^{kL}$.*

Indeed, by the induction on a length of a sequence, transferring a word X into a word Y one easily verifies that Y is the normal word of order k and hence $\ell(Y) \leq kL$. But the set of all words in the alphabet M whose length does not exceed kL, has the power p^{kL}. □

Lemma 3. *There exists an algorithm which for any normal words X_0 and Y_0 from $\mathbf{S} \in \mathcal{K}$ decides whether $X_0 \geq Y_0$ or not using the space $kL \cdot p^{kL}$, where k is the order of X_0.*

Proof. By Lemma 1 the power of the set of all such words Y that $X_0 \geq Y$, does not exceed p^{kL}. Of course, if $X_0 \geq Y_0$, then Y_0 is the member of a sequence (without repetition) of elementary transforms:

$$X_0 \to \cdots \to Y_0$$

with length $\leq p^{kL}$. It is necessary to seek the given word Y_0 among elements of the finite set above (exactly, among members of sequences of elementary transforms with the first member X_0 and the length $\leq p^{kL}$).

In the normal word X_0 we make all possible admissible changes: $X_0 \to Y_0$, $X_2 \to Y_2, \ldots, X_0 \to Y_r$ $(r \leq q)$, where $Y_1, \ldots, Y_r$ are normal words of order k and $X_0 \geq Y_i$ in $\mathbf{S}$, $i = 1, \ldots, r$. The next "step": as well as with X_0, do with words $Y_1, \ldots, Y_r$ and repeating words obtained on a previous step are canceled. The length of any member of the sequence does not exceed kL and etc. For completion of test is required less than p^{kL} "steps". The number of words that appears on last "step" is less than p^{kL}. Hence the length of the writing is less than kLp^{kL}.$\square$

Definition 3. *Words W and W' in the alphabet M of a p.o. semigroup $\mathbf{S} \in \mathcal{K}$ are expressed by concordant representation, if*

$$W = F_1P_1F_2P_2\cdots F_mP_mF_{m+1}, \quad W' = F_1P_1'F_2P_2'\cdots F_mP_m'F_{m+1}, \quad (*)$$

where $P_1 \ldots, P_m$ and $P_1', \ldots, P_m'$ are separate non-overlapping normal parts [10,8].

It is known [10] that inequality $W \geq W'$ holds iff the words W and W' have a concordant representation $(*)$ and the inequalities

$$P_i \geq P_i', \quad i = 1, \ldots, m, \quad (**)$$

are fulfilled.

Theorem. *The algorithmic inequality problem for the class $\mathcal{K}$ of p.o. semigroups* $\mathbf{S}$ *is decidable with the space c^n where n is the length of a writing of the inequality $W_1 \geq W_2$ and $c = const$ for* $\mathbf{S}$.

Proof. Let $\mathbf{S} \in \mathcal{K}$ and W_1, W_2 be the words in M. We put in order all normal parts of the word W_1 according to their beginning and, for the given beginning, according to their end in it. We represent the word W_1 consecutively with one, two, etc. separate non-overlapping normal parts. Similar action is done for the word W_2.

Let $\ell(W_1) = w_1$, $\ell(W_2) = w_2$. By Lemma 1 we decide whether a given subword E of the word W_1 is normal or not. The length of writing for a detection of all normal subwords E of the word W_1 does not exceed $w_1c^{w_1}$ and analogously for the word W_2.

Let us write out two sequences of normal subwords of the words W_1 and W_2. Then we compose representations of the words W_1 and W_2 with one, two, etc. separate non-overlapping normal parts. One easily verifies that a number of such representations does not exceed 2^{w_1} and 2^{w_2} for W_1 and W_2, respectively. Hence the length of the writing of all these representations are less than $w_12^{w_1} + w_22^{w_2}$.

From representations of the words W_1 and W_2 above we separate the pairs of concordant representations. If there are not such pairs then the words W_1 and W_2 are incomparable. For any pair of concordant representations we establish whether the inequalities $(**)$ are fulfilled or not. By Lemma 1 the length of the writing is less than $w_1Lp^{w_1L} + w_2Lp^{w_2L}$. If all the inequalities $(**)$ are fulfilled, then $W_1 \geq W_2$. Otherwise, the inequality $W_1 \geq W_2$ is not fulfilled.

Thus the inequality problem for p.o. semigroup $\mathbf{S} \in \mathcal{K}$ is decided using the space c^n.$\square$

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