NON-CLASSICAL ASYMPTOTIC EXPANSIONS FOR EIGENVALUE PROBLEMS WITH A PARAMETER IN THE BOUNDARY CONDITION

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Abstract

The aim of this work is to obtain exponentially convergent asymptotic expansions for eigenvalues and eigenfunctions of an eigenvalue problem containing the eigenvalue parameter in the boundary condition. Such eigenvalue problems are used when solving sloshing problems by analytical-numerical methods.

Key words and phrases: asymptotic expansion, boundary eigenvalue parameter, sloshing analysis.

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1. *Introduction*

The following eigenvalue problem

$$\begin{aligned} \Delta u(x,y) - q(x,y)u(x,y) &= 0, \quad x \in \Omega \\ \frac{\partial u(x,y)}{\partial n} &= 0, \quad (x,y) \in \Gamma_1, \\ \frac{\partial u(x,y)}{\partial n} &= \lambda u(x,y), \quad (x,y) \in \Sigma_0 \end{aligned} \tag{1.1}$$

is of the great importance for the theory and applications. Here Ω is a bounded domain with the boundary $\Gamma = \Gamma_1 \cup \Sigma_0, : \Gamma_1 \cap \Sigma_0 \neq \emptyset, : q(x,y)$ is

a given function, λ is the eigenvalue parameter. Such eigenvalue problems arise when modeling oscillations of fluids in tanks ([2],[5],[7]). The eigenvalue asymptotics for the case $q(x, y) = 0$ has been studied in [6].
The aim of this paper is to investigate convergence conditions of non-classical asymptotic expansions for the eigenvalues of the problem (1.1) in the case

$$\begin{aligned} \Omega(x, y) &= \{(x, y) : \ 0 < x, y < 1\}, \\ \Sigma_0 &= \{(x, 1) : 0 < x < 1\} \end{aligned} \tag{1.2}$$

and to generalize these results to a common case. Let us define the non-classical asymptotic expansions.

Definition 1. *We say that the series*

$$\lambda_n \sim \sum_{j=0}^{\infty} \lambda_n^{(j)} \tag{1.3}$$

is a non-classical asymptotic expansion for the eigenvalues of the problem (1.1) if

$$\lambda_n^{(j)} = \frac{b_n^{(j)}}{n^j}, \ |b_n^{(j)}| \le c \cdot b^j$$

with some constants c, b independent of n, j.

Note, that $b_n^{(j)}$ in the classical asymptotic expansions does not depend on n. The second part of the paper is devoted to construction of non-classical asymptotic expansions by the FD-method (in the case $q(x, y) = q_1(x) + q_2(y)$, see [5-8]) and to the investigation of the asymptotic convergence conditions. These conditions for $\overline{q}(x, y) = 0$ are

$$\beta_n^{(0)} = \frac{4(3 + \sqrt{8})}{\pi n} max\{\|q_2\|_\infty, \frac{1}{4}\|q_1\|_\infty\} < 1, \ n = 1, 2, ..., \tag{1.4}$$

where $\lambda_n^{(j)}$ corresponding to the FD-method are estimated by

$$|\lambda_n^{(j)}| \le \frac{3 + \sqrt{5}}{2\sqrt{5}} max\{\|q_2\|_\infty, \frac{1}{4}\|q_1\|_\infty\}[\beta_n^{(0)}]^j = c(b/n)^j$$

with

$$c = \frac{3 + \sqrt{5}}{2\sqrt{5}} max\{\|q_2\|_\infty, \frac{1}{4}\|q_1\|_\infty\}, b = \frac{4(3 + \sqrt{8})}{\pi} max\{\|q_2\|_\infty, \frac{1}{4}\|q_1\|_\infty\}.$$

If the condition (1.4) is not fulfilled for a fixed n we use another version of the FD-method ($\overline{q}(x,y) \not\equiv 0$)) and give the following definition.

Definition 2. *The series (1.3) is called a generalized non-classical expansion for the eigenvalues of the problem (1.1) with $q(x,y) = q_1(x) + q_2(y)$, if*

$$\begin{aligned} \lambda_n^{(j)} &= \frac{b_n^{(j)}}{n^j}, |b_n^{(j)} \le c_1(b_1\|q - \vec{q}\|_\infty)^j, \\ \|q - \vec{q}\|_\infty &= max\{\|q_2 - \overline{q}_2\|_\infty, \frac{1}{4}\|q_1 - \overline{q}_1\|_\infty\}, \end{aligned}$$

where $\overline{q}_1(x), \overline{q}_2(x)$ are piece-wise constant approximations for $q_1(x), q_2(x)$, respectively, c_1, b_1 are constants independent of n, j.

In the third part of the paper we will find sufficient conditions providing exponential convergence of generalized non-classical expansions for the eigenvalues of the problem(1.1),(1.2) for $q(x,y) = q_1(x) + q_2(y)$. These conditions are

$$\delta_n(\vec{q}(\cdot)) = \frac{\nu_n}{n} < 1 \tag{1.5}$$

with a bounded ν_n such that

$$\lim_{n\to\infty} \nu_n = \frac{1}{\pi}(2+\sqrt{2})(12+9\sqrt{2})\|q - \vec{q}\|_\infty.$$

Note, that one can fulfill the condition (1.4) beginning with some sufficient large n whereas the condition (1.5) can be fulfilled for any n by an appropriate approximation of $q(x,y)$ by piece-wise constant functions $\overline{q}(x,y)$.

The usual technique of FD-method for the classical eigenvalue problems is not appropriate for the problem (1). One can see it on the example of the following problem. Let $q(x,y) = q_1(x) + q_2(y)$, then by separation of variables in (1.1)-(1.2) we get the following two eigenvalue problems connecting through the parameter μ_n:

$$\begin{aligned} &X_n''(x) + [\mu_n - q_1(x)]X_n(x) = 0, \ 0 < x < 1, \\ &X_n'(0) = X_n'(1) = 0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} &Y_n''(y) + [\mu_n + q_2(y)]Y_n(y) = 0, \ 0 < y < 1, \\ &Y_n'(0) = 0, \quad Y_n'(1) = \lambda_n Y_n(1). \end{aligned} \tag{1.7}$$

First one has to solve the problem (1.6) (the classical one) and then after setting μ_n into (1.7) the problem (1.7) (non-classical one) must be solved.

Thus, the objectives of this paper is to use the FD-method in order to get non-classical asymptotic expansions for the problem (1.1) which represent a basis for effective numerical algorithms.
The correspondent problem to (6), (7) abstract problem is

$$(A+B)u+\lambda u=0,$$

where A, B are some self-adjoint closed densely defined operators in Hilbert space H. Suppose that there exists an self-adjoint closed densely defined operator B_0 such that the eigenvalue problem for $A+B_0$ is "simpler" than that for $A+B$ and $\|B-B_0\|$ is small. We consider the problem

$$(A+B_0+t(B-B_0))u(t)+\lambda(t)u(t)=0, \tag{1.8}$$

and seek the solution in the form of the Taylor series

$$\begin{aligned} u &= u(t)|_{t=1}=\sum_{j=0}^{\infty}u^{(j)}t^j|_{t=1}=\sum_{j=0}^{\infty}u^{(j)}, \\ \lambda &= \lambda(t)|_{t=1}=\sum_{j=0}^{\infty}\lambda^{(j)}, \end{aligned} \tag{1.9}$$

where

$$u^{(j)}=\frac{1}{j!}\frac{d^j u(t)}{dt^j}|_{t=0}, \lambda^{(j)}=\frac{1}{j!}\frac{d^j\lambda(t)}{dt^j}|_{t=0}.$$

Inserting (1.9) into (1.8) we get the following sequence of equations

$$\begin{aligned} &(A+B_0)u^{(j+1)}+\lambda^{(0)}u^{(j+1)} \\ = &-(B-B_0)u^{(j)}-\sum_{p=0}^{j}\lambda^{(j+1-p)}u^{(p)}, j=0,1,... \end{aligned} \tag{1.10}$$

$$(A+B_0)u^{(0)}+\lambda^{(0)}u^{(0)}=0. \tag{1.11}$$

We call (1.11) the basic problem and suppose that it is explicitly solvable. For convenience the eigenfunctions can be normalized by

$$\|u^{(0)}\|=1.$$

Suppose that $\lambda^{(0)}$ is a simple eigenvalue of the operator $A+B_0$, then the solution of (1.10) exists under the condition that the right-hand side is orthogonal to $u^{(0)}$ what leads to

$$\lambda^{(j+1)}=-((B-B_0)u^{(j)},u^{(0)})-\sum_{p=1}^{j}\lambda^{(j+1-p)}(u^{(p)},u^{(0)}).$$

In this case $u^{(j+1)}$ is not unique ($u^{(j+1)} + cu^{(0)}$ obviously also is a solution). For convenience we choose that $u^{(j+1)}$ for which

$$(u^{(j+1)}, u^{(0)}) = 0$$

and we have

$$\lambda^{(j+1)} = -((B - B_0)u^{(j)}, u^{(0)}), j = 0, 1, ...$$

The smallness of $\|B - B_0\|$ will play an important role in our convergence analysis.
This scheme of the FD-method is not appropriate for the problem (1.1). One can see that it as an example of the following abstract problem (1.6),(1.7)

$$\begin{aligned} (A + B)u + \mu u &= 0, \\ (C + D)v - \mu v &= 0, \qquad (1.12) \\ Fv &= \lambda P v. \end{aligned}$$

We are looking for μ, λ for which (**??**) have non-trivial solutions. The problem (**??**) is imbedded into the following parametric problem

$$\begin{aligned} (A + B_0 + t(B - B_0))u(t) + \mu(t)u(t) &= 0, \\ (C + D_0 + t(D - D_0))v(t) - \mu(t)v(t) &= 0, \qquad (1.13) \\ Fv(t) &= \lambda(t)Pv(t), \qquad (1.14) \end{aligned}$$

so that

$$u = u(1), v = v(1), \mu = \mu(1), \lambda = \lambda(1).$$

We represent

$$\begin{aligned} u &= u(t)|_{t=1} = \sum_{j=0}^{\infty} u^{(j)} t^j|_{t=1} = \sum_{j=0}^{\infty} u^{(j)}, \\ v &= v(t)|_{t=1} = \sum_{j=0}^{\infty} v^{(j)} t^j|_{t=1} = \sum_{j=0}^{\infty} v^{(j)}, \qquad (1.15) \\ \lambda &= \lambda(t)|_{t=1} = \sum_{j=0}^{\infty} \lambda^{(j)}, \\ \mu &= \mu(t)|_{t=1} = \sum_{j=0}^{\infty} \mu^{(j)}, \end{aligned}$$

and setting it into (1.13) we get

$$(A + B_0)u^{(j+1)} + \mu^{(0)} u^{(j+1)} = -(B - B_0)u^{(j)} - \sum_{p=0}^{j} \mu^{(j+1-p)} u^{(p)},$$

$$
\begin{aligned}
(C + D_0)v^{(j+1)} - \mu^{(0)}v^{(j+1)} &= -(B - B_0)u^{(j)} \\
&- \sum_{p=0}^{j} \mu^{(j+1-p)}v^{(p)}, \qquad (1.16)
\end{aligned}
$$

$$
Fv^{(j+1)} - \lambda^{(0)}Pv^{(j+1)} = \sum_{p=0}^{j} \lambda^{(j+1-p)}Pv^{(p)}, j = 0, 1, ...
$$

with the basic problem

$$
\begin{aligned}
(A + B_0)u^{(0)} + \mu^{(0)}u^{(0)} &= 0, \\
(C + D_0)v^{(0)} - \mu^{(0)}v^{(0)} &= 0, \qquad (1.17) \\
Fv^{(0)} &= \lambda^{(0)}Pv^{(0)}.
\end{aligned}
$$

And the operators B_0, D_0 must approximate the operators B, D and therefore ought to be quite simple so as the problem (23) would be more simple than the initial problems (19), (20). We shall see in the following analysis that the smallness of $\|B - B_0\|, \|D - D_0\|$ provide the exponential convergence of the numerical method for eigenvalues with "small" numbers and these norms for such eigenvalues represent the only parameter which influences the accuracy. For eigenvalues with "great" numbers the second influence parameter which accelerates the convergence and improves the accuracy is the eigenvalue number. Thus, our approach has common features with traditional discretization methods (B_0, D_0 can be interpreted e.g. as a finite-difference or FEM approximation) on the one hand and with the asymptotics methods on the other hand which work better for eigenvalues with great numbers. The generalization of this results see in the last section.

2. *Non-classical asymptotic series for the case* $q(x, y) = q_1(x) + q_2(y)$

We apply the FD-method ([8],[9],[1]) to the Sturm-Liouville problem (1.6) and to the problem (1.7) with the eigenvalue parameter in the boundary condition, i.e. we are looking for the solutions of (1.6), (1.7) as the series

$$
\mu_n = \sum_{j=0}^{\infty} \mu_n^{(j)}, X_n(x) = \sum_{j=0}^{\infty} X_n^{(j)}(x), \qquad (2.1)
$$

$$
\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, Y_n(y) = \sum_{j=0}^{\infty} Y_n^{(j)}(y), \qquad (2.2)
$$

with terms satisfying the following recurrence sequence

$$\frac{d^2}{dx^2}\left[X_n^{(j+1)}(x)\right] + (n\pi)^2 X_n^{(j+1)}(x) = q_1(x) X_n^{(j)}(x)$$

$$-\sum_{p=0}^{j} \mu_n^{(j+1-p)} X_n^{(p)}(x) \equiv -F_n^{(j+1)}(x), x \in (0,1), \tag{2.3}$$

$$\frac{d}{dx} X_n^{(j+1)}(0) = \frac{d}{dx} X_n^{(j+1)}(1) = 0,$$

$$\frac{d^2}{dx^2}\left[Y_n^{(j+1)}(y)\right] - (n\pi)^2 Y_n^{(j+1)}(y) = q_2(y) Y_n^{(j)}(y)$$

$$+\sum_{p=0}^{j} \mu_n^{(j+1-p)} Y_n^{(p)}(y) \equiv -G_n^{(j+1)}(y), y \in (0,1), \tag{2.4}$$

$$\frac{d}{dy} Y_n^{(j+1)}(0) = 0,$$

$$\frac{d}{dy} Y_n^{(j+1)}(1) - \lambda_n^{(0)} Y_n^{(j+1)}(1) = \sum_{p=0}^{j} \lambda_n^{(j+1-p)} Y_n^{(p)}(1).$$

The base problems is

$$\begin{aligned} \frac{d^2}{dx^2}\left[X_n^{(0)}(x)\right] + \mu_n^{(0)} X_n^{(0)}(x) &= 0, x \in (0,1), \\ \frac{d}{dx} X_n^{(0)}(0) &= \frac{d}{dx} X_n^{(0)}(1) = 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{d^2}{dx^2}\left[Y_n^{(0)}(y)\right] - \mu_n^{(0)} Y_n^{(0)}(y) &= 0, y \in (0,1), \\ \frac{d}{dy} Y_n^{(0)}(0) = 0, \frac{d}{dy} Y_n^{(0)}(1) &= \lambda_n^{(0)} Y_n^{(0)}(1) \end{aligned} \tag{2.6}$$

with the solution

$$\mu_n^{(0)} = (n\pi)^2,\ X_n^{(0)}(x) = \sqrt{2}\cos n\pi x,$$

$$\lambda_n^{(0)} = n\pi \tanh n\pi,\ Y_n^{(0)}(y)$$

$$= \sqrt{\frac{2}{1+\sinh(2n\pi)/(2n\pi)}} \cosh n\pi y, n = 0, 1, ... \tag{2.7}$$

The following statement holds for the problem (1.4) (see [13]).

Theorem 1. *The FD-method converges exponentially as a geometric progression with a denominator β_n and estimates*

$$|\mu_n - \overset{m}{\mu}_n| \leq \frac{\|q_1\|_\infty \beta_n^m}{1-\beta_n} \frac{(2m-1)!!}{2(2m+2)!!} \tag{2.8}$$

hold provided that the condition

$$\beta_n = \frac{4}{\pi^2(2n-1)}\|q_1\|_\infty < 1 \tag{2.9}$$

and the normalization condition

$$\int_0^1 X_n^{(p)}(x)X_n^{(0)}(x)dx = \delta_{p,0},\ p = 0, 1, ...$$

are fulfilled.

Now we are in position to investigate the FD-method applied to the problem (1.7). First of all we seek the solvability conditions for the problem (2.4). For this aim we multiply both sides of (2.4) by $Y_n^{(0)}(y)$ (it is the solution of the homogeneous problem (2.6)) and integrate over the interval $(0, 1)$:

$$\frac{dY_n^{(j+1)}(1)}{dy}Y_n^{(0)}(1) - Y_n^{(j+1)}(1)\frac{dY_n^{(0)}(1)}{dy} = \int_0^1 q_2(y)Y_n^{(j)}(y)Y_n^{(0)}(y)dy$$

$$+\sum_{p=0}^{j} \mu_n^{(j+1-p)} \int_0^1 Y_n^{(p)}(y)Y_n^{(0)}(y)dy. \tag{2.10}$$

Taking into account the boundary condition at the point $x = 1$ we get

$$Y_n^{(0)}(1)\sum_{p=0}^{j} \lambda_n^{(j+1-p)}Y_n^{(p)}(1) = \int_0^1 q_2(y)Y_n^{(j)}(y)Y_n^{(0)}(y)dy$$

$$+\sum_{p=0}^{j} \mu_n^{(j+1-p)} \int_0^1 Y_n^{(p)}(y)Y_n^{(0)}(y)dy. \tag{2.11}$$

Denoting

$$Y_n^{(p)}(y)/Y_n^{(0)}(y) = \hat{Y}_n^{(p)}(y)$$

we can write the solvability condition (2.9) as follows

$$\lambda_n^{(j+1)} = -\sum_{p=1}^{j} \lambda_n^{(j+1-p)}\hat{Y}_n^{(p)}(1)$$

$$+\int_0^1 q_2(y)\hat{Y}_n^{(j)}(y)\hat{Y}_n^{(0)}(y)\left[\frac{\cosh n\pi y}{\cosh n\pi}\right]^2 dy \tag{2.12}$$
$$+\sum_{p=0}^{j}\mu_n^{(j+1-p)}\int_0^1 \hat{Y}_n^{(p)}(y)\hat{Y}_n^{(0)}(y)\left[\frac{\cosh n\pi y}{\cosh n\pi}\right]^2 dy.$$

Taking into account the solvability condition (2.9) one can see that the solution of the problem (2.4) can be represented by

$$\begin{aligned} Y_n^{(j+1)}(y) &= \int_0^y \frac{\sinh n\pi(y-\eta)}{n\pi}\Big[q_2(y)Y_n^{(j)}(\eta) \\ &+ \sum_{p=0}^{j}\mu_n^{(j+1-p)}Y_n^{(p)}(\eta)Y_n^{(p)}(\eta)\Big]\, d\eta. \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} \hat{Y}_n^{(j+1)}(y) &= \frac{1}{n\pi}\int_0^y K_n(y,\eta)\Big[q_2(\eta)\hat{Y}_n^{(j)}(\eta) \\ &+ \sum_{p=0}^{j}\mu_n^{(j+1-p)}\hat{Y}_n^{(p)}(\eta)\Big]\, d\eta, \end{aligned} \tag{2.14}$$

where

$$K_n(y,\eta) = \frac{1}{2}[1-e^{-2n\pi(y-\eta)}]\frac{1+e^{-2n\pi\eta}}{1+e^{-2n\pi y}}.$$

In order to estimate $\lambda_n^{(j)}, \hat{Y}_n^{(j)}(y)$ we use (2.9),(2.11),(2.14). Then we get

$$\begin{aligned} |\lambda_n^{(j+1)}| &\le \sum_{p=1}^{j}|\lambda_n^{(j+1-p)}||\hat{Y}_n^{(p)}(1)| + \|q_2\|_\infty\|\hat{Y}_n^{(j)}\| \\ &+ \|q_1\|_\infty\sum_{p=0}^{j}\beta_n^{j-p}\frac{(2j-2p-1)!!}{2(2j-2p+2)!!}\|\hat{Y}_n^{(p)}\|, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \|\hat{Y}_n^{(j+1)}\| &\le \frac{1}{n\pi}\Big[\|q_2\|_\infty\|\hat{Y}_n^{(j)}\| \\ &+ \|q_1\|_\infty\sum_{p=0}^{j}\beta_n^{j-p}\frac{(2j-2p-1)!!}{2(2j-2p+2)!!}\|\hat{Y}_n^{(p)}\|\Big], \end{aligned} \tag{2.16}$$

$$\begin{aligned} |\hat{Y}_n^{(j+1)}| &\le \frac{1}{n\pi}\Big[\|q_2\|_\infty\|\hat{Y}_n^{(j)}\| \\ &+ \|q_1\|_\infty\sum_{p=0}^{j}\beta_n^{j-p}\frac{(2j-2p-1)!!}{2(2j-2p+2)!!}\|\hat{Y}_n^{(p)}\|\Big]. \end{aligned} \tag{2.17}$$

Now we must solve the nonlinear system of inequalities (2.15)-(2.17). First of all we will solve the inequality (2.16) which will yield the estimate for $|\hat{Y}_n^{(j)}|$ and then we will solve (2.15) with respect to $|\lambda_n^{(j)}|$. Let us denote

$$\begin{aligned} q &= max\{\|q_2\|_\infty, \frac{1}{4}\|q_1\|_\infty\}, a = \{[3+\sqrt{8}]\|q_1\|_\infty/(2q)\}^{-1}, \\ v_p &= \beta_n^{-p}\|\hat{Y}_n^{(p)}\|, p > 0, v_0 = 1. \end{aligned}$$

Then the inequality (2.16) takes the form

$$v_{j+1} \le a(v_j + \sum_{p=0}^{j} v_p), j = 0, 1, ... \tag{2.18}$$

The majorant equation for the inequality (2.18) is

$$V_{j+1} = a(V_j + \sum_{p=0}^{j} V_p), j = 0, 1, ..., \; V_0 = 1. \tag{2.19}$$

We solve this equation by the generating functions method. Let

$$f(z) = \sum_{j=0}^{\infty} z^j V_j,$$

then it follows from (**??**):

$$\begin{aligned} f(z) &= \frac{1-z}{1-(2a+1)z+az^2} = (1-z)\sum_{j=0}^{\infty} z^j \left(\frac{2a}{2a+1-\sqrt{4a^2+1}}\right)^j \\ &\times \left[1-\left(\frac{2a+1-\sqrt{4a^2+1}}{2a+1+\sqrt{4a^2+1}}\right)^{j+1}\right]\left[1-\frac{2a+1-\sqrt{4a^2+1}}{2a+1+\sqrt{4a^2+1}}\right]^{-1} \end{aligned}$$

what yields

$$\begin{aligned} V_j &= \frac{2a+1+\sqrt{4a^2+1}}{2\sqrt{4a^2+1}}\left(\frac{2a}{2a+1-\sqrt{4a^2+1}}\right)^{j-1} \\ &\times \left[\frac{\sqrt{4a^2+1}-1}{2a+1-\sqrt{4a^2+1}} + \left(\frac{2a+1-\sqrt{4a^2+1}}{2a+1+\sqrt{4a^2+1}}\right)^j \frac{\sqrt{4a^2+1}+1}{2a+1+\sqrt{4a^2+1}}\right] \\ &\le (2a+1)^j. \end{aligned}$$

Thus, we get

$$\|\hat{Y}_n^{(p)}\| \le [\beta_n(2a+1)]^p, p = 0, 1, ..., \tag{2.20}$$

$$|\hat{Y}_n^{(p)}(1)| \le [\beta_n(2a+1)]^p, p = 0, 1, ... \tag{2.21}$$

These inequalities together with (2.15)- (2.17) imply

$$|\lambda_n^{(j+1)}| \le \sum_{p=1}^{j} |\lambda_n^{(j+1-p)}|[\beta_n(2a+1)]^p + \pi n[\beta_n(2a+1)]^{j+1}$$

$$= \sum_{p=1}^{j} |\lambda_n^{(j+1-p)}|[\beta_n(2a+1)]^p$$

$$+(3+\sqrt{8})\frac{n(2a+1)}{2n-1}\|q_1\|_\infty[\beta_n(2a+1)]^j. \tag{2.22}$$

We solve this inequality by the generating functions method denoting

$$\begin{aligned} |\lambda_n^{(j+1)}|[\beta_n(2a+1)]^{-j} &\le w_{j+1}, \\ b_n &= (3+\sqrt{8})\frac{n}{2n-1}(2a+1)\|q_1\|_\infty, \\ w_{j+1} &= \sum_{p=1}^{j} w_{j+1-p} + b_n,\ j=0,1,... \end{aligned}$$

Let

$$g(z) = \sum_{j=0}^{\infty} z^j w_{j+1},$$

then we have

$$g(z) = \frac{b_n}{1-2z} = b_n \sum_{j=0}^{\infty} z^j 2^j,$$

which implies

$$|\lambda_n^{(j+1)}| \le b_n[2\beta_n(2a+1)]^j,\ j=0,1,... \tag{2.23}$$

Now we are in position to prove the following convergence result.

Theorem 2. *The FD-method for the problem (1.1)- (1.2) converges exponentially with the explicit estimate*

$$|\lambda_n - \overset{m}{\lambda}_n| = |\sum_{j=m+1}^{\infty} \lambda_n^{(j)}| \le b_n \frac{\gamma_n^m}{1-\gamma_n}, \tag{2.24}$$

provided that

$$\gamma_n = 2(2a+1)\beta_n < 1. \tag{2.25}$$

Proof. The condition (2.22) yields (2.8) which provides the convergence of the FD-method for the problem (1.6) (see [13]). In order to prove the convergence of the FD-method for problem (1.7) we note that (2.20),(2.21),(2.23),(2.22) imply the convergence of the series (2.2) as geometrical progressions with the denominators γ_n and $(2a+1)\beta_n$.We must show that these series converge to the solution of the problem (1.7).
Let us consider the series

$$Y_n(y,t) \equiv \sum_{j=0}^{\infty} t^j Y_n^{(j)}(y). \tag{2.26}$$

According to (2.4),(2.11),(2.20) every term in (2.26) is twice differentiable with respect to t for any fixed $t \in [0,1]$ and the series

$$\sum_{j=0}^{\infty} t^j \frac{d^2 Y_n^{(j)}(y)}{dy^2}$$

is uniformly convergent for all $t \in [0,1]$. Thus, we can multiply both sides of (2.4) by t^{j+1}, sum up over j from 0 to ∞ and taking into account (2.4), we get:

$$\frac{\partial^2}{\partial y^2} Y_n(y,t) - \left[\sum_{j=0}^{\infty} t^j \mu_n^{(j)} + t q_2(y) \right] Y_n(y,t) = 0.$$

Due to Theorem 1 the value

$$\mu_n(t) = \sum_{j=0}^{\infty} t^j \mu_n^{(j)}$$

is the eigenvalue of the Sturm-Liouville problem

$$\begin{aligned} \frac{\partial^2}{\partial y^2} X_n(x,t) \quad &+ \quad [\mu_n(t) - t q_1(x)]\, X_n(x,t) = 0, \; 0 < x < 1, \\ \frac{\partial X_n(0,t)}{\partial x} \quad &= \quad \frac{\partial X_n(1,t)}{\partial x} = 0 \end{aligned} \tag{2.27}$$

which contains the problem (1.6) in the sense that

$$\mu_n(1) = \mu_n, \; X_n(x,1) = X_n(x).$$

Thus, the function (2.26) satisfies the equation

$$\frac{\partial^2}{\partial y^2} Y_n(y,t) - \left[\mu_n^(t) + t q_2(y) \right] Y_n(y,t) = 0, \; \forall y \in (0,1) \tag{2.28}$$

and besides the boundary condition

$$\frac{\partial Y_n(0,t)}{\partial y} = 0. \tag{2.29}$$

On the right end of the interval we have

$$\begin{aligned}\frac{\partial Y_n(1,t)}{\partial y} &= \sum_{j=0}^{\infty} t^j \frac{\partial Y_n^{(j)}(1)}{\partial y} = \sum_{j=0}^{\infty} t^j \sum_{p=0}^{j} \lambda_n^{(j-p)} Y_n^{(p)}(1) \\ &= \sum_{j=0}^{\infty} t^j \lambda_n^{(j)} \sum_{j=0}^{\infty} t^j Y_n^{(j)}(1) = \lambda_n(t) Y_n(1,t),\end{aligned} \tag{2.30}$$

i.e. the function (2.26) is the solution of the problem (2.28)-(2.30) $\forall t \in [0,1]$ and, in particular, for $t = 1$. The problem (2.28)-(2.30) for $t = 1$ takes the look

$$\begin{aligned}\frac{\partial^2}{\partial y^2} Y_n(y,1) &- [\mu_n + q_2(y)]\, Y_n(y,1) = 0,\ 0 < y < 1, \\ \frac{\partial Y_n(0,1)}{\partial y} &= 0,\ \frac{\partial Y_n(1,1)}{\partial y} = \lambda_n(1) Y_n(1,1).\end{aligned} \tag{2.31}$$

Comparing (2.31) and (1.7) we conclude that

$$Y_n(y,1) = Y_n(y),\ \lambda_n(1) = \lambda_n,$$

and it completes the proof.

The series for λ_n are non-classical asymptotic series with $c = (3 + \sqrt{8})(2a+1)\|q_1\|_\infty,\ b = \frac{2}{\pi}c.$

Example 1. Let $q_1 = x,\ q_2 = y$. Then we have

$$\mu_n^{(1)} = \frac{1}{2}, \mu_n^{(2)} = \frac{n^3\pi^3 + 21n\pi}{48(n\pi)^5}, X_n^{(1)}(x) = \frac{\sqrt{2}}{2n\pi}\left[\frac{x^2 - x}{2}\sin n\pi x\right.$$

$$\left. + \frac{x}{2n\pi}\cos n\pi x - \frac{2}{(2n\pi)^2}\sin n\pi x\right] - \frac{\sqrt{2}}{2(2n\pi)^2}\cos n\pi x$$

$$\lambda_n^{(1)} = \frac{1}{(2n\pi)^2\cosh^2 n\pi}\left[\frac{3}{2}n\pi\sinh 2n\pi + 2(n\pi)^2 - \cosh^2 n\pi + 1\right],$$

$$Y_n^{(1)}(y) = \sqrt{\frac{2}{1 + (2\pi n)^{-1}\sinh 2\pi n}}(2\pi n)^{-3}\left[-2yn\pi\cosh n\pi y + 2\sinh n\pi y\right.$$

$$\left. + 2y^2(n\pi)^2\sinh n\pi y + 2y(n\pi)^2\sinh n\pi y\right],$$

$$\int_0^1 y \frac{Y_n^{(1)}(y) Y_n^{(0)}(y)}{[Y_n^{(0)}(1)]^2} dy =$$

$$= -\frac{1}{6(2n\pi)^5 \cosh^2 n\pi}[-24(n\pi)^3 \cosh 2n\pi + 42(n\pi)^2 \sinh 2n\pi$$

$$-48n\pi \cosh 2n\pi + 21 \sinh 2n\pi + 8(n\pi)^3 + 6n\pi],$$

$$\lambda_n^{(2)} = \frac{-2\lambda_n^{(1)}}{(2n\pi)^3}[-n\pi \cosh n\pi$$

$$+ \sinh n\pi + 2(n\pi)^2 \sinh n\pi]/\cosh n\pi$$

$$-\frac{1}{6(2n\pi)^5 \cosh n\pi}[-24(n\pi)^3 \cosh 2n\pi$$

$$+42(n\pi)^2 \sinh 2n\pi - 48n\pi \cos 2n\pi$$

$$+21 \sinh 2n\pi + 8(n\pi)^3 + 6n\pi] + \mu_n^{(2)} \frac{1 + \frac{\sinh 2n\pi}{2n\pi}}{2\cosh^2 n\pi}$$

$$-\frac{1}{4(2n\pi)^4 \cosh^2 n\pi}[4n^2\pi^2$$

$$+10 \sinh 2n\pi - 8 \cosh 2n\pi - 8(n\pi)^2 \cosh 2n\pi + 8].$$

In this case we have

$$\|q_1\|_\infty = \|q_2\|_\infty = 1, q = 1, a = \frac{2}{3+\sqrt{8}},$$

$$b_n = (3+\sqrt{8})(\frac{4}{3+\sqrt{8}} + 1)n(2n-1), \gamma_n = 2(\frac{4}{3+\sqrt{8}} + 1)\beta_n = \frac{2(7+\sqrt{8})}{\pi(2n-1)},$$

$$|\lambda_n - \hat{\lambda_n}| \le (7+\sqrt{8}) \frac{n}{2n-1} \frac{2(7+\sqrt{8})}{\pi(2n-1)}[1 - \frac{2(7+\sqrt{8})}{\pi(2n-1)}]^{-1}$$

$$= \frac{2(7+\sqrt{8})^2 n}{\pi(2n-1)^2}[1 - \frac{2(7+\sqrt{8})}{\pi(2n-1)}]^{-1}.$$

3. *Generalized non-classical asymptotic series in the case* $q(x,y) = q_1(x) + q_2(y)$,

If the assumptions (2.4),(2.22) for a given n are not fulfilled then we apply the FD-method with $\bar{q}_1(x) \neq 0, \bar{q}_2(y) \neq 0$ for the problems (1.6),(1.7) (see

e.g. [8], [13]).
We consider two grids

$$\begin{aligned} \omega_1 &= \{0 = x_0 < x_1 < ... < x_{N_1} < 1\}, \\ \omega_2 &= \{0 = y_0 < y_1 < ... < y_{N_2} < 1\} \end{aligned}$$

and two step-functions $\bar{q}_1(x), \bar{q}_2(y)$ which approximate the functions $q_1(x), q_2(y)$ and are defined by

$$\begin{aligned} \bar{q}_1(x) &= q_1(x_{i-1/2}), \quad x \in (x_{i-1}, x_i], \\ i &= 1, 2, ..., N_1, \quad x_{i-1/2} = \frac{1}{2}(x_{i-1} + x_i), \\ \bar{q}_2(y) &= q_2(y_{j-1/2}), \quad y \in (y_{j-1}, y_j], \\ j &= 1, 2, ..., N_2, \quad y_{j-1/2} = \frac{1}{2}(y_{j-1} + y_i). \end{aligned}$$

If the functions $q_1(x), q_2(y)$ are piece-wise smooth with finite number of discontinuity points then we include these points into the grids ω_1, ω_2.
We are looking for the solutions of (1.6),(1.7) as the expansions

$$\mu_n = \sum_{j=0}^{\infty} \mu_n^{(j)}\left(\bar{q}_1(\cdot)\right), \quad X_n(x) = \sum_{j=0}^{\infty} X_n^{(j)}(x, \bar{q}_1(\cdot)), \tag{3.1}$$

$$\begin{aligned} \lambda_n &= \sum_{j=0}^{\infty} \lambda_n^{(j)}(\vec{q}(\cdot)), \quad Y_n(y) = \sum_{j=0}^{\infty} Y_n^{(j)}(y, \vec{q}(\cdot)), \\ \vec{q}(\cdot) &= \{\bar{q}_1(\cdot), \bar{q}_2(\cdot)\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \mu_n^{(j)}(0) &= \mu_n^{(j)}, \quad X_n^{(j)}(x, 0) = X_n^{(j)}(x) \\ \lambda_n^{(j)}(\vec{0}) &= \lambda_n^{(j)}, \quad Y_n^{(j)}(y, 0) = Y_n^{(j)}(y). \end{aligned}$$

The series terms of (3.1),(3.2) satisfy the following recurrent sequence of differential equations

$$\begin{aligned} \frac{d^2}{dx^2}\left[X_n^{(j+1)}(x, \bar{q}_1(\cdot))\right] &+ \left[\mu_n^{(0)}(q_1(\cdot)) - \bar{q}_1(x)\right] X_n^{(j+1)}(x, \bar{q}_1(\cdot)) \\ &= \left[q_1(x) - \bar{q}_1(x)\right] X_n^{(j)}(x, \bar{q}_1(\cdot)) \\ -\sum_{p=0}^{j} \mu_n^{(j+1-p)}(\bar{q}_1(\cdot)) X_n^{(p)}(x, \bar{q}_1(\cdot)) &\equiv -F_n^{(j+1)}(x) \end{aligned}$$

$$\frac{dX_n^{(j+1)}}{dx}(0,\bar{q}_1(\cdot))=0, \quad \frac{dX_n^{(j+1)}(1,\bar{q}_1(\cdot))}{dx}=0, \tag{3.3}$$

$$\frac{d^2}{dy^2}\left[Y_n^{(j+1)}(y,\vec{q}_1(\cdot))\right]-\left[\mu_n^{(0)}(\vec{q}_1(\cdot))+\bar{q}_2(y)\right]Y_n^{(j+1)}(y,\vec{q}(\cdot))$$
$$=\left[q_2(y)-\bar{q}_2(y)\right]Y_n^{(j)}(y,\vec{q}(\cdot))$$
$$+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))Y_n^{(p)}(y,\vec{q}(\cdot))$$
$$\frac{dY_n^{(j+1)}(0,\vec{q}(\cdot))}{dy}=0, \tag{3.4}$$
$$\frac{dY_n^{(j+1)}(1,\vec{q}(\cdot))}{dy}-\lambda_n^{(0)}(\vec{q}(\cdot))Y_n^{(j+1)}(1,\vec{q}(\cdot))=$$
$$\sum_{p=0}^{j}\lambda_n^{(j+1-p)}(\vec{q}(\cdot))Y_n^{(p)}(1,\vec{q}(\cdot)), j=0,1,...$$

for which the basic system is

$$\frac{d^2}{dx^2}\left[X_n^{(0)}(x,\bar{q}_1(\cdot))\right] + \left[\mu_n^{(0)}(\bar{q}_1(\cdot))-\bar{q}_1(x)\right]X_n^{(0)}(x,\bar{q}_1(\cdot))=0,$$
$$\frac{dX_n^{(0)}(0,\bar{q}_1(\cdot))}{dx} = \frac{dX_n^{(0)}(1,\bar{q}_1(\cdot))}{dx}=0, \tag{3.5}$$

$$\frac{d^2}{dy^2}\left[Y_n^{(0)}(y,\vec{q}(\cdot))\right]-\left[\mu_n^{(0)}(\bar{q}_1(\cdot))+\bar{q}_2(y)\right]Y_n^{(0)}(y,\vec{q}(\cdot))=0,$$
$$\frac{dY_n^{(0)}(0,\vec{q}(\cdot))}{dy}=0,$$
$$\frac{dY_n^{(0)}(q,\vec{q}(\cdot))}{dy}=\lambda_n^{(0)}(\vec{q}(\cdot))Y_n^{(0)}(1,\vec{q}(\cdot)). \tag{3.6}$$

In order to begin the recurrence process (3.3),(3.4) first of all we have to solve the basic system (3.5),(3.6) of differential equations with piece-wise constant coefficients. Contrary to the basic system (2.5),(2.6) we can not write down its solution explicitly since we can not find $\mu_n^{(0)}(\bar{q}_1(\cdot))$ explicitly. But one can find the eigenvalues $\mu_n^{(0)}(\bar{q}_1(\cdot))$ using the following algorithm. First, we can write down the general solution of the equation (3.5) with constant coefficients on every interval $[x_{i-1},x_i]$. This solution contains $2N_1$ arbitrary parameters. Supposing the continuity of this solution and its derivative (flux) at the grid nodes $x_i,\ i=1,2,...,N_1-1$ we get $2N_1-2$ linear algebraic equations plus two equations due to the boundary conditions. Denoting by $\Delta_1\left(\mu_n^{(0)}(\bar{q}_1(\cdot))\right)$ the determinant of this system we get the equation

$$\Delta_1\left(\mu_n^{(0)}(\bar{q}_1(\cdot))\right)=0, \tag{3.7}$$

for determining of all $\mu_n^{(0)}(\bar{q}_1(\cdot))$. Substituting these $\mu_n^{(0)}(\bar{q}_1(\cdot))$ in (3.6) we can analogously get a system of linear algebraic equations with $2N_2$ arbitrary parameters defining $Y_n^{(0)}(y,\bar{q}(\cdot))$. Let $\Delta_2\left(\lambda_n^{(0)}(\vec{q}(\cdot))\right)$ be its determinant, then the equation

$$\Delta_2\left(\lambda_n^{(0)}(\vec{q}(\cdot))\right)=0, \tag{3.8}$$

defines all eigenvalues $\lambda_n^{(0)}(\vec{q}(\cdot))$. The following convergence theorem for the case $\bar{q}_1(x)\not\equiv 0$ was proved in [13].

Theorem 3. *Under the condition*

$$\beta_n(\bar{q}_1(\cdot))=4\|q_1-\bar{q}_1\|_\infty$$
$$\times\max\left\{\left[\mu_{n+1}^{(0)}(\bar{q}_1(\cdot))-\mu_n^{(0)}(\bar{q}_1(\cdot))\right]^{-1},\right.$$
$$\left.\left[\mu_n^{(0)}(\bar{q}_1,(\cdot))-\mu_{n-1}^{(0)}(\bar{q}_1(\cdot))\right]^{-1}\right\}<1, \tag{3.9}$$

and the normalizing condition

$$\int_0^1 X_n^{(p)}(x,\bar{q}_1(\cdot))X_n^{(0)}(x,\bar{q}_1(\cdot))=\delta_{p,0},\quad p=0,1,...$$

the FD-method for the problem (1.6) converges as a geometric progression with the denominator $\beta_n(\bar{q}_1(\cdot))$ and the following estimates hold

$$\left|\mu_n\left(q_1(\cdot)\right)-\overset{m}{\mu}_n\left(\bar{q}_1(\cdot)\right)\right|\le\frac{\|q_1-\bar{q}_1\|_\infty[\beta_n\left(\bar{q}_1(\cdot)\right)]^m}{1-\beta_n\left(\bar{q_1(\cdot)}\right)}\frac{(2m-1)!!}{2(2m+2)!!} \tag{3.10}$$

Remark 1. *In order to be able to use the estimate (3.10) one has first to solve the basic problem and find $\mu_n^{(0)}(\bar{q}_1(\cdot))$. In order to get an explicit a priori estimate we use a method according to [1].*
Let us rewrite the problem (3.3) as

$$X_n^{(j+1)}(x,\bar{q}_1(\cdot))=X_n^{(j+1)}(0,\bar{q}_1(\cdot))\cos n\pi x$$
$$-\int_0^x\frac{\sin n\pi(x-\xi)}{n\pi}\left[\mu_n^{(0)}(\bar{q}_1(\cdot))-(\pi n)^2-\bar{q}_1(\xi)\right]X_n^{(j+1)}(\xi,\bar{q}_1(\cdot))d\xi$$

$$+\int_0^x \frac{\sin n\pi(x-\xi)}{n\pi}\Big\{[q_1(\xi)-\bar q_1(\xi)]X_n^{(j)}(\xi,\bar q_1(\cdot))-$$

$$\sum_{p=0}^{j}\mu_n^{(j+1-p)}\times(\bar q_1(\cdot))X_n^{(p)}(\xi,\bar q_1(\cdot))\Big\}\,d\xi$$

and choose the normalizing condition

$$X_n^{(j+1)}(0,\bar q_1(\cdot))=0.$$

Then we get

$$\left|X_n^{(j+1)}(x,\bar q_1(\cdot))\right|\le \frac{d_n}{n\pi}\sqrt{\frac{1}{2}\left(x+\frac{\sin 2\pi nx}{2\pi n}\right)}\left\{\int_0^x\left[X_n^{(j+1)}(\xi,\bar q(\cdot))\right]^2 d\xi\right\}^{1/2}$$

$$+\sqrt{\frac{1}{2}\left(x+\frac{\sin 2\pi nx}{2\pi n}\right)}\left\{\int_0^x\left[F_n^{(j+1)}(\xi)\right]^2 d\xi\right\}^{1/2},$$

$$d_n=|\mu_n^{(0)}(\bar q_1(\cdot))-(\pi n)^2-\bar q_1(\xi)\|_\infty,$$

from where

$$\left[X_n^{(j+1)}(x,\bar q_1(\cdot))\right]^2\le\frac{1}{(\pi n)^2}\left(x+\frac{\sin 2\pi nx}{2\pi n}\right)$$

$$\times\left\{d_n^2\int_0^x\left[X_n^{(j+1)}(\xi,\bar q_1(\cdot))\right]^2 d\xi+\|F_n^{(j+?)}\|^2\right\}.$$

Using the Gronwall lemma we arrive at the following estimate

$$\left[X_n^{(j+1)}(x,\bar q_1(\cdot))\right]^2\le\frac{1}{(\pi n)^2}\left(x+\frac{\sin 2\pi nx}{2\pi n}\right)$$

$$\times\exp\left[\frac{d_n^2}{(\pi n)^2}\left(\frac{1}{2}+\frac{2}{(2\pi n)^2}\right)\right]\|F_n^{(j+1)}\|^2$$

which yields

$$\left\|X_n^{(j+1)}\right\|\le\frac{1}{\sqrt{2}\pi n}\exp\left[\frac{d_n^2}{2(\pi n)^2}\left(\frac{1}{2}+\frac{2}{(2\pi n)^2}\right)\right]\|F_n^{(j+1)}\|$$

$$\le M_n\left\{\|q_1-\bar q_1\|_\infty\|X_n^{(j)}\|+\sum_{p=0}^{j}\left|\mu_n^{(j+1-p)}\right|\left\|X_n^{(p)}\right\|\right\},$$

$$M_n=\frac{1}{\sqrt{2}\pi n}\exp\left[\frac{d_n^2}{2(\pi n)^2}\left(\frac{1}{2}+\frac{2}{(2\pi n)^2}\right)\right\}.$$

The constant d_n contains the unknown $\mu_n^{(0)}(\bar{q}_1(\cdot))$. To get rid of it we increase the constant M_n. It follows from (3.22),(3.25) that

$$0 \le \mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 \le \|\bar{q}_1\|_\infty,$$

thus,

$$d_n \le 2\|\bar{q}_1\|_\infty$$

and

$$M_n \le \frac{1}{\sqrt{2}\pi n} \exp\left\{ \frac{(\|q_1\|_\infty)^2}{(\pi n)^2} \left[2 + \frac{1}{(\pi n)^2}\right]\right\} = M_n'.$$

As the result we arrive at the following system of inequalities

$$\begin{aligned} \|X_n^{(j+1)}\| &\le M_n' \left\{ \|q_1 - \bar{q}_1\|_\infty \|X_n^{(j)}\| + \sum_{p=0}^{j} \left|\mu_n^{(j+1-p)}\right| \|X_n^{(p)}\| \right\}, \\ \left|\mu_n^{(j+1)}\right| &\le \sum_{p=1}^{j} \left|\mu_n^{(j+1-p)}\right| \left\|X_n^{(p)}\right| + \|q_1 - \bar{q}_1\|_\infty \|X_n^{(j)}\|, \end{aligned}$$

which can be solved analogously to [1]. As a consequence we get the following statement.

Theorem 3*. *Under the conditions*

$$\beta_n^*(\bar{q}_1(\cdot)) = 2(3+\sqrt{8})M_n'\|q_1 - \bar{q}_1\|_\infty < 1 \tag{3.11}$$

and

$$X_n^{(j+1)}(0, \bar{q}_1(\cdot)) = 0,$$

the FD-method for the problem (1.6) converges as a geometric progression with the denominator $\beta_n^*(\bar{q}_1(\cdot))$ and the following estimates hold

$$\left|\mu_n\left(q_1(\cdot)\right) - \overset{m}{\mu}_n\left(\bar{q}_1(\cdot)\right)\right| \le \frac{\|q_1 - \bar{q}_1\|_\infty [\beta_n^*\left(\bar{q}_1(\cdot)\right)]^m}{1 - \beta_n^*\left(\bar{q}_1(\cdot)\right)} \frac{(2m-1)!!(3+\sqrt{8})}{2(2m+2)!!} \tag{3.12}$$

Now we turn to the analysis of the FD-method applied to problem (1.7). In order to get the solvability conditions of (3.4) we multiply this equation by $Y_n^{(0)}(y, \vec{q}(\cdot))$ and integrate over (0,1):

$$Y_n^{(0)}(1, \vec{q}(\cdot)) = \sum_{p=0}^{j} \lambda_n^{(j+1-p)}(\vec{q}(\cdot)) Y_n^{(p)}(1, \vec{q}(\cdot)) = \int_0^1 [q_2(y) - \bar{q}_2(y)] Y_n^{(j)}(y, \vec{q}(\cdot))$$

$$\times Y_n^{(0)}(y,\vec{q}(\cdot))dy+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))\int_0^1 Y_n^{(p)}(y,\vec{q}(\cdot))Y_n^{(0)}(y,\vec{q}(\cdot))dy. \quad (3.13)$$

Denoting

$$\hat{Y}_n^{(p)}(y,\vec{q}(\cdot)) = Y_n^{(p)}(y,\vec{q}(\cdot))/Y_n^{(0)}(y,\vec{q}(\cdot))$$

the equality (3.13) takes the form

$$\lambda_n^{(j+1)}(\vec{q}(\cdot)) = -\sum_{p=1}^{j}\lambda_n^{(j+1-p)}(\vec{q}(\cdot))\hat{Y}_n^{(p)}(1,\vec{q}(\cdot))$$

$$+\int_0^1[q_2(y)-\bar{q}_2(y)]\frac{Y_n^{(0)}(y,\vec{q}(\cdot))}{Y_n^{(0)}(1,\vec{q}(\cdot))}\hat{Y}_n^{(j)}(y,\vec{q}(\cdot))\hat{Y}_n^{(0)}(y,\vec{q}(\cdot))dy$$

$$+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))\int_0^1 \hat{Y}_n^{(p)}(y,\vec{q}(\cdot))\hat{Y}_n^{(0)}(y,\vec{q}(\cdot))\frac{Y_n^{(0)}(y,\vec{q}(\cdot))}{Y_n^{(0)}(1,\vec{q}(\cdot))}dy \qquad (3.14)$$

On analyzing the problem (3.6) it is easy to note that the function $\frac{d}{dy}\left[Y_n^{(0)}(y,\vec{q}(\cdot))\right] \neq 0$ is of constant signs on the interval [0,1]. Actually, assuming the opposite there exists $\eta \in (0,1]$ for which

$$\frac{dY_n^{(0)}(\eta,\vec{q}(\cdot))}{dy} = 0.$$

But in this case (3.6) yields

$$\frac{d^2}{dy^2}\left[Y_n^{(0)}(y,\vec{q}(\cdot))\right] - \left[\mu_n^{(0)}(\bar{q}_1(\cdot))+\bar{q}_2(y)\right]Y_L^{(0)}(y,\vec{q}(\cdot)) = 0, \quad y \in (0,\eta),$$

$$\frac{dY_n^{(0)}(y,\vec{q}(\cdot))}{dy}\Big|_{y=0,\eta} = 0,$$

with the null-solution on $[0,\eta]$, in particular $Y_n^{(0)}(\eta,\vec{q}(\cdot)) = \frac{dY_n^{(0)}(\eta,\vec{q}(\cdot))}{dy} = 0$. This implies the following Cauchy problem

$$\frac{d^2}{dy^2}\left[Y_n^{(0)}(y,\vec{q}(\cdot))\right] - \left[\mu_n^{(0)}(\bar{q}_1(\cdot))+\bar{q}_2(y)\right]Y_n^{(0)}(y,\vec{q}(\cdot)) = 0, \quad \eta < y \leq 1$$

$$Y_n^{(0)}(\eta,\bar{q}(\cdot)) = \frac{dY_n^{(0)}(\eta,\vec{q}(\cdot))}{dy} = 0,$$

with the trivial solution which contradicts the fact that

$Y_n^{(0)}(y,\vec{q}(\cdot))$ is not identically equal to 0 for $y \in [0,1]$. The property we have proved implies

$$0 \le \frac{Y_n^{(0)}(\eta,\vec{q}(\cdot))}{Y_n^{(0)}(y,\vec{q}(\cdot))} \le 1, \quad 0 \le \eta \le y \le 1. \tag{3.15}$$

We can write the problem (3.4) in the following equivalent form

$$Y_n^{(j+1)}(y,\vec{q}(\cdot)) = \int_0^y \frac{shn\pi(y-\eta)}{n\pi}\left[\mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 + \bar{q}_2(\eta)\right]$$

$$\times Y_n^{(j+1)}(\eta,\vec{q}(\cdot))dy + \int_0^y \frac{shn\pi(y-\eta)}{n\pi}\Big\{[q_2(\eta)-\bar{q}_2(\eta)]Y_n^{(j)}(\eta,\vec{q}(\cdot))$$

$$+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))Y_n^{(p)}(\eta,\vec{q}(\cdot))\Big\}dy$$

or taking into account notations we have introduced

$$\hat{Y}_n^{(j+1)}(y,\vec{q}(\cdot)) = \frac{1}{n\pi}\int_0^y K_n(y,\eta,\vec{q}(\cdot))\left[\mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 + \bar{q}_2(\eta)\right]$$

$$\times \hat{Y}_n^{(j+1)}(\eta,\vec{q}(\cdot))dy + \frac{1}{n\pi}\int_0^y K_n(y,\eta,\vec{q}(\cdot))\left\{[q_2(\eta)-\bar{q}_2(\eta)]\hat{Y}_n^{(j)}(\eta,\vec{q}(\cdot))\right.$$

$$\left.+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))\hat{Y}_n^{(p)}(\eta,\vec{q}(\cdot))\right\}dy, \tag{3.16}$$

$$K_n(y,\eta,\vec{q}(\cdot)) = \sinh n\pi(y-\eta)\frac{Y_n^{(0)}(\eta)}{Y_n^{(0)}(y)}. \tag{3.17}$$

From (3.14),(3.15),(3.16) we get

$$\begin{aligned}\left|\lambda_n^{(j+1)}(\vec{q}(\cdot))\right| \quad &\le \quad \sum_{p=1}^{j}\left|\lambda_n^{(j+1-p)}(\vec{q}(\cdot))\right|\left|\hat{Y}_n^{(p)}(1,\vec{q}(\cdot))\right| + \|q_2-\bar{q}_2\|_\infty\|\hat{Y}_n^{(j)}\| \\ &+\sum_{p=0}^{j}\left|\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))\right|\left\|\hat{Y}_n^{(p)}\right\|. \end{aligned}\tag{3.18}$$

In order to estimate $\hat{Y}_n^{(j+1)}(y,\vec{q}(\cdot))$ we need the following auxiliary statement.

Lemma 1. *It holds*

$$0 \le \frac{Y_n^{(0)}(\eta,\vec{q}(\cdot))}{Y_n^{(0)}(y,\vec{q}(\cdot))} \le \frac{\cosh n\pi\eta}{\cosh n\pi y}, \quad 0 \le \eta \le y. \tag{3.19}$$

Proof. It is easy to see that the function

$$v(\eta) = \frac{Y_n^{(0)}(\eta, \vec{q}(\cdot))}{Y_n^{(0)}(y, \vec{q}(\cdot))} - \frac{\cosh n\pi\eta}{\cosh n\pi y}$$

satisfies the boundary value problem

$$\begin{aligned} v''(\eta) - (\pi n)^2 v(\eta) &= \left[\mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 + \bar{q}_2(\eta)\right] \frac{Y_n^{(0)}(\eta, \vec{q}(\cdot))}{Y_n^{(0)}(y, \vec{q}(\cdot))}, \\ 0 &< \eta < y, \quad v'(0) = 0, \quad v(y) = 0. \end{aligned} \tag{3.20}$$

Using the Green function

$$G(\eta, \mu) = \frac{1}{\pi n \cosh \pi n y} \begin{cases} \cosh \pi n \eta \cdot \sinh \pi n (y - \mu), & \eta \le \mu \\ \cosh \pi n \mu \cdot \sinh \pi n (y - \eta), & \mu \le \eta \end{cases}$$

we can write down

$$v(\eta) = -\int_0^y G(\eta, \mu) \left[\mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 + \bar{q}_2(\mu)\right] \frac{Y_n^{(0)}(\mu, \vec{q}(\cdot))}{Y_n^{(0)}(y, \vec{q}(\cdot))} d\mu. \tag{3.21}$$

One can see that the right inequality in (3.19) holds true provided that the expression in the square brackets is not negative. The left inequality in (3.19) was proved earlier (see (3.15)).
The FD-method implies (see [8])

$$\mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 = \int_0^1 \frac{d\mu_n(t, \bar{q}_1(\cdot))}{dt} dt, \tag{3.22}$$

where $\mu_n(t, \bar{q}_1(\cdot))$ is an eigenvalue of the following parametrized Sturm-Liouville problem

$$\begin{aligned} \frac{\partial^2}{\partial x^2} X_n(x, t; \bar{q}_1(\cdot)) &+ \{\mu_n(t, \bar{q}_1(\cdot)) - \bar{q}_1(x) t\} X_n(x, t; \bar{q}_1(\cdot)) = 0 \\ \frac{\partial X_n(0, t; \bar{q}_1(\cdot))}{\partial x} &= \frac{\partial X_n(1, t; \bar{q}_1(\cdot))}{\partial x} = 0. \end{aligned} \tag{3.23}$$

It follows from (3.23)

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial t} X_n(x, t; \bar{q}_1(\cdot))\right] + \{\mu_n(t, \bar{q}_1(\cdot)) - t\bar{q}_1(x)\}\left[\frac{\partial}{\partial t} X_n(x, t; \bar{q}_1(\cdot))\right]$$

$$= \left\{\frac{d\mu_n(t, \bar{q}_1(\cdot))}{dt} - \bar{q}_1(x)\right\} X_n(x, t; \bar{q}_1(\cdot)),$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}X_n(x,t;\bar{q}_1(\cdot))\right]_{x=0,1} = 0. \tag{3.24}$$

The necessary and sufficient solvability condition for this problem is

$$\frac{d\mu_n(t,\bar{q}_1(\cdot))}{dt} = \int_0^1 \bar{q}_1(x)\left[X_n(x,t;\bar{q}_1(\cdot))\right]^2 dx$$

$$\times\left\{\int_0^1 \left[X_n(x,t;\bar{q}_1(\cdot))\right]^2 dz\right\}^{-1} > 0\ , \tag{3.25}$$

Together with the inequality $\bar{q}_2(y) \geq 0$ and (3.22) it proves the nonnegativity of the expression in the square brackets in (3.21). The lemma is proved.

Now we are in the position to estimate $\hat{Y}_n^{(j+1)}(y,\vec{q}(\cdot))$. Using Lemma 1 we get

$$0 \leq K_n(y,\eta,\vec{q}(\cdot)) \leq \sinh n\pi(y-\eta)\frac{\cosh n\pi\eta}{\cosh n\pi y} \leq \frac{1-e^{-2n\pi(y-\eta)}}{1+e^{-2n\pi y}}. \tag{3.26}$$

Taking into account (3.25) we get from (3.16)

$$\left[\hat{Y}_n^{(j+1)}(y;\vec{q}(\cdot))\right]^2 \leq \frac{2}{(n\pi)^2}\int_0^y\left[\frac{1-e^{2n\pi(y-\eta)}}{1+e^{-2n\pi y}}\right]^2 dy$$

$$\left\{d_n^2\int_0^y\left[\hat{Y}_n^{(j+1)}(\eta,\vec{q}(\cdot))\right]^2 d\eta + \int_0^y\left[(q_2(\eta)-\bar{q}_2(\eta))\hat{Y}_n^{(j)}(\eta,\vec{q}(\cdot))\right.\right.$$

$$\left.\left.+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))\hat{Y}_n^{(p)}(\eta,\vec{q}(\cdot))\right]^2 d\eta\right\}. \tag{3.27}$$

Using the estimate

$$\int_0^y\left[\frac{1-e^{2n\pi(y-\eta)}}{1+e^{-2n\pi y}}\right]^2 d\eta \leq y + \frac{5}{4n\pi}$$

and the Gronwall lemma (see [3]) we get from (3.26)

$$\left[\hat{Y}_n^{(j+1)}(y,\vec{q}(\cdot))\right]^2 \leq \frac{2}{(n\pi)^2}\left(y+\frac{5}{4n\pi}\right)\int_0^1\left\{\left[q_2(\eta)-\bar{q}_2(\eta)\right]\hat{Y}_n^{(j)}(\eta,\vec{q}(\cdot))\right.$$

$$\left.+\sum_{p=0}^{j}\mu_n^{(j+1-p)}(\bar{q}_1(\cdot))\hat{Y}_n^{(p)}(\eta,\vec{q}(\cdot))\right\}^2 d\eta$$

$$\times\exp\left\{\frac{2d_n^2}{(n\pi)^2}\left(\frac{1}{2}+\frac{5}{4n\pi}\right)\right\}, \tag{3.28}$$

where

$$d_n = \|\mu_n^{(0)}(\bar{q}_1(\cdot)) - (\pi n)^2 + \bar{q}_2(y)\|_\infty.$$

As consequences of (3.28) we have the following two inequalities

$$\|\hat{Y}_n^{(j+1)}\| \le \frac{1}{n\pi}\sqrt{1+\frac{5}{2n\pi}}\exp\left\{\frac{d_n^2}{(n\pi)^2}\left(\frac{1}{2}+\frac{5}{4n\pi}\right)\right\}$$
$$\times\left\{\|q_2-\bar{q}_2\|_\infty\|\hat{Y}_n^{(j)}\| + \sum_{p=0}^{j}|\mu_n^{(j+1-p)}|\|\hat{Y}_n^{(p)}\|\right\}, \qquad (3.29)$$

and

$$|\hat{Y}_n^{(j+1)}(1,\vec{q}(\cdot))| \le \frac{\sqrt{2}}{n\pi}\sqrt{1+\frac{5}{4n\pi}}\exp\left\{\frac{d_n^2}{(n\pi)^2}\left(\frac{1}{2}+\frac{5}{4n\pi}\right)\right\}$$
$$\times\left\{\|q_2-\bar{q}_2\|_\infty\|\hat{Y}_n^{(j)}\| + \sum_{p=0}^{j}|\mu_n^{(j+1-p)}|\|\hat{Y}_n^{(p)}\|\right\}. \qquad (3.30)$$

Now we have to solve the inequalities (3.18),(3.29),(3.30). First we will solve (3.29) making use of Theorem 3. It will make possible to solve (3.30) and then, finally, we will solve (3.18).
Denoting

$$M_n'' = \frac{1}{n\pi}\sqrt{1+\frac{5}{2n\pi}}\exp\left\{\frac{d_n^2}{(n\pi)^2}\left(\frac{1}{2}+\frac{5}{4n\pi}\right)\right\},$$

and using (3.10) we get from (3.29)

$$\|\hat{Y}_n^{(j+1)}\| \le M_n''\left\{\|q_2-\bar{q}_2\|_\infty\|\hat{Y}_n^{(j)}\| + \|q_1-\bar{q}_1\|_\infty\frac{1}{4}\sum_{p=0}^{j}[\beta_n(\bar{q}_1(\cdot))]^{j-p}\|\hat{Y}_n^{(p)}\|\right\}.$$

Proceeding as done for solving of the inequality (2.16) we arrive at the equations

$$\upsilon_{j+1} = a_1\left\{\upsilon_j + \sum_{p=0}^{j}\upsilon_p\right\}, \quad j = 0,1,..., \qquad (3.31)$$

where

$$\begin{aligned} a_1 &= \frac{M_n''}{\beta_n^*(\bar{q}_1(\cdot))}\max\left\{\|q_2-\bar{q}_2\|_\infty, \frac{1}{4}\|q_1-\bar{q}_1\|_\infty\right\}, \\ (\beta_n^*)^{-j}(\bar{q}_1(\cdot))\|\hat{Y}_n^{(j)}\| &\le \upsilon_j, \quad \upsilon_0 = 1. \end{aligned} \qquad (3.32)$$

The solution of the equation (3.31) can be estimated by

$$v_j \le (2a_1+1)^j \tag{3.33}$$

Thus, returning to (3.32) we have

$$\|\hat{Y}_n^{(j)}\| \le [(2a_1+1)\beta_n^*(\bar{q}_1(\cdot))]^j \,. \tag{3.34}$$

Now it follows from (3.30) that

$$|\hat{Y}_n^{(j+1)}(1,\vec{q}(\cdot))| \le \sqrt{2}\,[(2a_1+1)\beta_n^*(\bar{q}_1(\cdot))]^j \,. \tag{3.35}$$

Finally it remains to estimate $\lambda_n(\vec{q}(\cdot))$. The inequalities (3.10),(3.18),(3.34),(3.35) imply

$$\begin{aligned} |\lambda_n^{(j+1)}| &\le \sqrt{2}\sum_{p=1}^{j}|\lambda_n^{(j+1-p)}|(\gamma_n)^p + \|q_2-\bar{q}_2\|_\infty(\gamma_n)^j \\ &+ \frac{1}{4}\|q_1-\bar{q}_1\|_\infty\sum_{p=0}^{j}(\beta_n^*)^{j-p}(\gamma_n)^p \,, \end{aligned} \tag{3.36}$$

where

$$\gamma_n = \gamma_n(\vec{q}(\cdot)) = (2a_1+1)\beta_n^*(\bar{q}_1(\cdot)).$$

Solving this inequality by the method of generating functions analogously as done above we get

$$\left|\lambda_n^{(j+1)}(\vec{q}(\cdot))\right| \le \left(1+\frac{1}{2a_1}\right)\left[(1+\sqrt{2})\gamma_n(\vec{q}(\cdot))\right]^j \,. \tag{3.37}$$

Thus, we arrive at the following statement.

Theorem 4. *Let the condition*

$$\delta_n(\vec{q}(\cdot)) = (1+\sqrt{2})\gamma_n(\vec{q}(\cdot)) = (1+\sqrt{2})(2a_1+1)\beta_n^*(\bar{q}_1(\cdot)) < 1 \tag{3.38}$$

holds and the approximations $\bar{q}_1(x),\bar{q}_2(x)$ of the functions $q_1(x),q_2(x)$ are chosen so that

$$\frac{\|q_2-\bar{q}_2\|_\infty}{\|q_1-\bar{q}_1\|_\infty} \le C, \quad \forall N_1,N_2. \tag{3.39}$$

Then the FD-method for the problem (1.1),(1.2) converges as a geometric progression with the denominator $\delta_n(\vec{q}(\cdot))$ and the following estimate holds

$$\begin{aligned} |\lambda_n - \lambda_n^m(\vec{q}(\cdot))| &= \left|\lambda_n - \sum_{j=0}^{m}\lambda_n^{(j)}(\vec{q}(\cdot))\right| \\ &\le \left(1+\frac{1}{2a_1}\right)[\delta_n(\vec{q}(\cdot))]^m[1-\delta_n(\vec{q}(\cdot))]^{-1} \end{aligned} \tag{3.40}$$

The proof can be performed analogously to that of Theorem 2

The series

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}(\vec{q}(\cdot))$$

in this case is a generalized asymptotic expansion for eigenvalues of the problem (1.1),(1.2) in the sense of Definition 2 with

$$\begin{aligned} c_1 &= 1 + \frac{1}{2a_1^-}, \\ b_1 &= (1+\sqrt{2})(2a_1^+ + 1)2(3+\sqrt{8})\frac{1}{\sqrt{2\pi}} \exp\left\{\frac{\|q_1\|_\infty^2}{\pi^2}\left(1+\frac{1}{\pi^2}\right)\right\}, \end{aligned}$$

where

$$\begin{aligned} a_1^- &= \max\left\{\|q_2 - \bar{q}_2\|_\infty, \frac{1}{4}\|q_1 - \bar{q}_1\|_\infty\right\} \min_n \frac{M_n''}{\beta_n^*(\bar{q}_1(\cdot))} \\ &\geq [4\sqrt{2}(3+\sqrt{8}]^{-1} \\ a_2^+ &= \max\left\{\|q_2 - \bar{q}_2\|_\infty, \frac{1}{4}\|q_1 - \bar{q}_1\|_\infty\right\} \max_n \frac{M_n''}{\beta_n^*(\bar{q}_1(\cdot))} \\ &\leq \frac{\max(C, 1/4)}{\sqrt{2}(3+\sqrt{8})}\sqrt{1+\frac{5}{2\pi}} \exp\left\{\frac{\|q_1\|_\infty^2}{\pi^2}\left(1+\frac{5}{\pi}\right)\right\}. \end{aligned}$$

4. *A case of an arbitrary function $q(x, y)$*

In this part we consider the following 3D-problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) - q(x,y)u(x,y,z) = 0, \quad (x,y,z) \in Q$$

$$\frac{\partial u(x,y,z)}{\partial n} = 0, \quad (x,y,z) \in \Gamma_Q \backslash \Sigma_Q,$$

$$\frac{\partial u(x,y,z)}{\partial y} - \lambda u(x,y,z) = 0, \quad (x,y,z) \in \Sigma_Q, \tag{4.1}$$

where $Q = (0,1) \times (0,1) \times (0,1)$ is the unit cube with the boundary Γ_Q, $\Sigma_Q = \{(x,1,z) : 0 < x, z < 1\}$, n is the outer normal to Γ_Q. We look for the solution of (4.1) in the form

$$u(x,y,z) = u_{n,m}(x,y) \cos m\pi z. \tag{4.2}$$

The function $u_{n,m}(x,y)$ is the solution of the problem

$$\Delta u_{n,m}(x,y) - \Big[(m\pi)^2 + q(x,y)\Big]\, u_{n,m}(x,y) = 0, \quad (x,y) \in \Omega,$$

$$\frac{\partial u_{n,m}(x,y)}{\partial n_1} = 0, \quad (x,y) \in \Gamma_\Omega \backslash \Sigma_\Omega,$$

$$\frac{\partial u_{n,m}(x,y)}{\partial n_1} - \lambda_{n,m} u_{n,m}(x,y) = 0, \quad (x,y) \in \Sigma_\Omega, \tag{4.3}$$

where $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \Omega = (0,1) \times (0,1)$ is the unit square with the boundary Γ_Ω, $\Sigma_\Omega = \{(x,1) : 0 < x < 1\}$, n_1 is the outer normal to Γ_Ω. Contrary to (1.2) the function $q(x,y)$ can be arbitrary but the term $(m\pi)^2$ allows to apply the exponentially convergent FD-method which will be described below.
In accordance with the FD-method we represent (non-classical expansions)

$$u_{n,m}(x,y) = \sum_{j=0}^{\infty} u_{n,m}^{(j)}(x,y), \quad \lambda_{n,m} = \sum_{j=0}^{\infty} \lambda_{n,m}^{(j)}, \tag{4.4}$$

where the terms of the series satisfy the recurrent system

$$\Delta u_{n,m}^{(j+1)}(x,y) - (m\pi)^2 u_{n,m}^{(j+1)}(x,y) = q(x,y) u_{n,m}(j)(x,y), \quad (x,y) \in \Omega,$$

$$\frac{\partial u_{n,m}^{(j+1)}(x,y)}{\partial n_1} = 0, \quad (x,y) \in \Gamma_\Omega \backslash \Sigma_\Omega,$$

$$\frac{\partial u_{n,m}^{(j+1)}(x,y)}{\partial n_1} - \lambda_{n,m}^{(0)} u_{n,m}^{(j+1)}(x,y) = \sum_{p=0}^{j} \lambda_{n,m}^{(j+1-p)} u_{n,m}^{(p)}(x,y), \tag{4.5}$$

$$(x,y) \in \Sigma_\Omega, \quad j = 0, 1, ...$$

The initial values for this recurrence procedure are the solutions of the problems

$$\Delta u_{n,m}^{(0)}(x,y) - (m\pi)^2 u_{n,m}^{(0)}(x,y) = 0, \quad (x,y) \in \Omega,$$

$$\frac{\partial u_{n,m}^{(0)}(x,y)}{\partial n_1} = 0 \quad (x,y) \in \Gamma_\Omega \backslash \Sigma_\Omega,$$

$$\frac{\partial u_{n,m}^{(0)}(x,y)}{\partial n_1} - \lambda_{n,m}^{(0)} u_{n,m}^{(0)}(x,y) = 0, \quad (x,y) \in \Sigma_\Omega, \tag{4.6}$$

which one can write down explicitly as

$$u_{n,m}^{(0)}(x,y) = A_{n,m} \cos n\pi x \cosh \sqrt{(n\pi)^2 + (m\pi)^2} y,$$

$$\lambda_{n,m}^{(0)} = \sqrt{(n\pi)^2 + (m\pi)^2 ?} \tanh \sqrt{(n\pi)^2 + (m\pi)^2}. \tag{4.7}$$

Here $A_{m,n}$ is the normalizing constant

$$A_{n,m}^2 = 4 \left[1 + \frac{\sinh 2\sqrt{(n\pi)^2 + (m\pi)^2}}{2\sqrt{(n\pi)^2 + (m\pi)^2}} \right]^{-1}, \tag{4.8}$$

such that

$$\|u_{n,m}^{(0)}\| = 1.$$

In order to find the solvability condition for (4.5) let us multiply (4.5) by $u_{n,m^{(0)}}$ and integrate over Ω. Then we get

$$\int_0^1 \left[\frac{\partial u_{n,m}^{(j+1)}(x,1)}{\partial y} u_{n,m}^{(0)}(x,1) - u_{n,m}^{(j+1)}(x,1) \frac{\partial u_{n,m}^{(0)}(x,1)}{\partial y} \right] dx$$
$$= \int \int_\Omega q(x,y) u_{n,m}^{(j)}(x,y) u_{n,m}^{(0)}(x,y) dxdy$$

and further taking into account (4.5),(4.6)

$$\sum_{p=o}^{j} \lambda_{n,m}^{(j+1-p)} \int_0^1 u_{n,m}^{(p)}(x,1) u_{n,m}^{(0)}(x,1) dx$$

$$= \int \int_\Omega q(x,y) u_{n,m}^{(j)}(x,y) u_{n,m}^{(0)}(x,y) dxdy.$$

This implies

$$\lambda_{n,m}^{(j+1)} = \left\{ \int_0^1 \left[u_{n,m}^{(0)}(x,1) \right]^2 dx \right\}^{-1}$$

$$\times \Big\{ - \sum_{p=1}^{j} \lambda_{n,m}^{(j+1-p)} \int_0^1 u_{n,m}^{(p)}(x,1) u_{n,m}^{(0)}(x,1) dx$$

$$+ \int \int_\Omega q(x,y) u_{n,m}^{(j)}(x,y) u_{n,m}^{(0)}(x,y) dxdy \Big\}. \tag{4.9}$$

The condition (4.9) provides the solvability of (4.5) within a term

$$B_{n,m}^{(j+1)} u_{n,m}^{(0)}(x,y).$$

In order to determine it we demand that

$$\int_0^1 u_{n,m}^{(j+1)}(x,1) u_{n,m}^{(0)}(x,1) dx = 0, \quad j = 0, 1, ..., \tag{4.10}$$

from where $B_{n,m}^{(j+1)} = 0$.
We look for the solution of (4.5) in the form of the series

$$u_{n,m}^{(j+1)}(x,y) = \sum_{p=0,p\neq n}^{\infty} u_{n,m,(p)}^{(j+1)}(y)\cos p\pi x \tag{4.11}$$

The terms of this series satisfy the following equations

$$\frac{d^2}{dy^2}u_{n,m,(p)}^{(j+1)}(y) - \left[m\pi)^2 + (p\pi)^2\right] u_{n,m,(p)}^{(j+1)}(y)$$

$$= \int_0^1 q(x,y)u_{n,m}^{(j)}(x,y)\cos p\pi x dx$$

$$\frac{du_{n,m,(p)}^{(j+1)}(0)}{dy} = 0, \quad \frac{du_{n,m,(p)}^{(j+1)}(1)}{dy} - \lambda_{n,m}^{(j+1)}u_{n,m,(p)}^{(j+1)}(1) = 0, \tag{4.12}$$

$$p = 0, 1, ..., \quad p \neq n.$$

We represent the solution of each problem (4.12) as

$$u_{n,m,(p)}^{(j+1)}(y) = -\int_0^1 G_{n,m,(p)}(y,\eta)\int_0^1 q(x,\eta)u_{n,m}^{(j)}(x,\eta)\cos p\pi x dx d\eta, \tag{4.13}$$

where the Green function $G_{n,m,(p)}(y,\eta)$ possesses the following explicit representation

$$G_{n,m,(p)}(y,\eta) = \begin{cases} A(\eta)\cosh\ \sqrt{(m\pi)^2+(p\pi)^2}y, & y \leq \eta \\ A(y)\cosh\ \sqrt{(m\pi)^2+(p\pi)^2}\eta, & \eta \leq y, \end{cases} \tag{4.14}$$

where

$$\begin{aligned} A(y) &= \frac{\cosh\kappa_{m,p}(1-y) - \frac{\kappa_{m,n}}{\kappa_{m,p}}\tanh\kappa_{m,n}\sinh\kappa_{mp}(1-y)}{\kappa_{m,p}\sinh\kappa_{m,p} - \kappa_{m,n}\tanh\kappa_{m,n}\sinh\kappa_{m,p}}, \\ \kappa_{m,p} &= \pi\sqrt{m^2+n^2}. \end{aligned}$$

Due to

$$\left\|u_{n,m}^{(j+1)}\right\| = \left\{\sum_{p=0,p\neq n}\frac{1+\delta_{0,p}}{2}\left\|u_{n,m,(p)}^{(j+1)}\right\|^2\right\}^{1/2},$$

we get from (4.13)

$$\left\|u_{n,m}^{(j+1)}\right\|^2 \leq \max_p \int_0^1\int_0^1 G_{n,m,(p)}^2(y,\eta)d\eta dy\left\|qu_{n,m}^{(j)}\right\|^2$$

$$\leq \|q\|_\infty^2 \max_p \int_0^1\int_0^1 G_{n,m,(p)}^2(y,\eta)d\eta dy\left\|u_{n,m}^{(j)}\right\|^2. \tag{4.15}$$

Further we need an estimate for the Green function (4.14). Let $\eta \le y$, then

$$|G_{n,m,(p)}(y,\eta)| = \frac{e^{-\kappa_{m,p}|y-\eta|}}{2\kappa_{m,p}}$$

$$\times \left| \frac{1-\frac{\kappa_{m,n}}{\kappa_{m,p}}\tanh\kappa_{m,n} + e^{-2\kappa_{m,p}(1-y)}\left[1+\frac{\kappa_{m,n}}{\kappa_{m,p}}\tanh\ \kappa_{m,n}\right]}{1-\frac{\kappa_{m,n}}{\kappa_{m,p}}\tanh\ \kappa_{m,n} - e^{-2\kappa_{m,p}}\left[1+\frac{\kappa_{m,n}}{\kappa_{m,p}}\tanh\ \kappa_{m,n}\right]} \right|$$

$$\times \left[1+e^{-2\kappa_{m,p}\eta}\right] \le \frac{e^{-\kappa_{m,p}|y-\eta|}}{\kappa_{m,p}}, \quad p \le n-1. \tag{4.16}$$

Here we have used the estimates

$$1-\frac{\kappa_{m,n}}{\kappa_{m,p}}th\kappa_{m,n} \le 1-\frac{\kappa_{m,n}}{\kappa_{m,n-1}}th\kappa_{m,n}$$

$$= \frac{e^{\kappa_{m,n}}\{\kappa_{m,n-1}-\kappa_{m,n}+e^{-2\kappa_{m,n}}(\kappa_{m,n-1}+\kappa_{m,n})\}}{\kappa_{m,n-1}(e^{\kappa_{m,n}}+e^{-\kappa_{m,n}})}$$

$$= \frac{e^{\kappa_{m,n}}\{-(2n-1)+e^{-2\kappa_{m,n}}(\kappa_{m,n-1}+\kappa_{m,n})^2\}}{\kappa_{m,n-1}(e^{\kappa_{m,n}}+e^{-\kappa_{m,n}})(\kappa_{m,n-1}+\kappa_{m,n})}$$

$$\le \frac{e^{\kappa_{m,n}}\{-(2n-1)+4\pi^2 e^{-2\pi}\}}{\kappa_{m,n-1}(e^{\kappa_{m,n}}+e^{-\kappa_{m,n}})(\kappa_{m,n-1}+\kappa_{m,n})} < 0.$$

Further we have

$$\left|G_{n,m,(p)}(y,\eta)\right| \le \frac{e^{-\kappa_{m,p}|y-\eta|}}{\kappa_{m,p}}$$

$$\times \left\{1-\frac{\kappa_{m,n}}{\kappa_{m,p}}\tanh\kappa_{m,n} - e^{-2\kappa_{m,p}}\left[1+\frac{\kappa_{m,n}}{\kappa_{m,p}}th\kappa_{m,n}\right]\right\}^{-1}$$

$$\le e^{-\kappa_{m,p}|y-\eta|}\frac{1+e^{-2\kappa_{m,n}}}{1-e^{-2(\kappa_{m,p}+\kappa_{m,n})}}\ \frac{p}{\kappa_{m,p}-\kappa_{m,n}}$$

$$\le \frac{e^{-\kappa_{m,p}|y-\eta|}}{1-e^{-2\kappa_{m,n}}}\ \frac{1}{\kappa_{m,p}-\kappa_{m,n}}, \quad p \ge n+1 \tag{4.17}$$

In the case $y \le \eta$ we get the same estimates (4.16),(4.17). These estimates imply for $m, n \ge 1$

$$\left|G_{n,m,(p)}(y,\eta)\right| \le \frac{e^{-\kappa_m|y-\eta|}}{\kappa_m}, \quad p \le n-1, \tag{4.18}$$

$$\left|G_{n,m,(p)}(y,\eta)\right| \le \frac{e^{-\kappa_{m,n+1}|y-\eta|}}{1-e^{-2\kappa_{m,n}}}\ \frac{1}{\kappa_{m,n+1}-\kappa_{m,n}}, \quad p \ge n+1. \tag{4.19}$$

The following obvious estimate

$$\begin{aligned}\int_0^1\int_0^1 e^{-a|y-\eta|}d\eta dy &= \int_0^1\left\{\int_0^y e^{-(y-\eta)a}dy+\int_y^1 e^{-a(\eta-y)}d\eta\right\}dy\\ &= \frac{1}{a}\int_0^1\left\{1-e^{-ay}+1-e^{-a(1-y)}\right\}dy\\ &\le \frac{2}{a}(1-e^{-a/2})\end{aligned}$$

together with (4.18),(4.19) yields

$$\int_0^1\int_0^1 G^2_{n,m,(p)}(y,\eta)d\eta dy \le \begin{cases}\frac{1}{\kappa_m^3}, & p\le n-1\\ \frac{1}{\kappa_{m,n+1}[\kappa_{m,n+1}-\kappa_{m,n}]^2} & p\ge n-1.\end{cases}$$

Assuming further that $m\le n$ we have

$$\begin{aligned}\kappa_{m,n+1}[\kappa_{m,n+1}-\kappa_{m,n}]^2 &= \pi^3\left(\frac{2n+1}{\kappa_{m,n+1}/\pi+\kappa_{m,n}/\pi}\right)^2\frac{\kappa_{m,n+1}}{\pi}\\ &\ge \frac{\pi^3}{2}\sqrt{m^2+(n+1)^2}\end{aligned}$$

and, thus, for $m\le n$

$$\begin{aligned}\int_0^1\int_0^1 G^2_{n,m,(p)}(y,\eta)d\eta dy &\le \frac{1}{\pi^3}\max\left\{\frac{1}{m^3},\frac{2}{\sqrt{m^2+(n+1)^2}}\right\}\\ &\equiv M^2_{n,m}. \qquad (4.20)\end{aligned}$$

The estimate (4.15) takes now the form

$$\left\|u^{(j+1)}_{n,m}\right\|\le\|q\|_\infty M_{n,m}\left\|u^{(j)}_{n,m}\right\|. \qquad (4.21)$$

This together with (4.9),(4.10) implies

$$\left\|u^{(j+1)}_{n,m}\right\|\le\{\|q\|_\infty M_{n,m}\}^{j+1},$$

$$\left|\lambda^{(j+1)}_{n,m}\right|\le\left[1+\frac{\sinh\ 2\sqrt{(n\pi)^2+(m\pi)^2}}{2\sqrt{(n\pi)^2+(m\pi)^2}}\right]$$

$$\times\frac{\|q\|_\infty}{\cosh^2\ \sqrt{(n\pi)^2+(m\pi)^2}}\{\|q\|_\infty M_{n,m}\}^j,\quad j=0,1,... \qquad (4.22)$$

Now we are at the point to formulate the following statement.

Theorem 5. *If*

$$\|q\|_\infty M_{n,m}<1 \qquad (4.23)$$

then the FD method for the problem (4.1) is exponentially convergent and the following estimates hold

$$\|u_{n,m} - u_{n,m}^N\| \equiv \|u_{n,m} - \sum_{j=0}^{N} u_{n,m}^{(j)}\| \le \frac{\{\|q\|_\infty M_{n,m}\}^{N+1}}{1-\|q\|_\infty M_{n,m}}, \tag{4.24}$$

$$|\lambda_{n,m} - \lambda_{n,m}^N| \equiv |\lambda_{n,m} - \sum_{j=0}^{N} \lambda_{n,m}^{(j)}|$$

$$\le \left[1 + \frac{sh 2\pi\sqrt{n^2+m^2}}{2\pi\sqrt{n^2+m^2}}\right] \frac{\|q\|_\infty}{ch^2\pi\sqrt{n^2+m^2}} \frac{\{\|q\|_\infty M_{n,m}\}^{N}}{1-\|q\|_\infty M_{n,m}}. \tag{4.25}$$

Proof. *Let us consider the series*

$$\tilde{u}_{n,m}(x,y,t) = \sum_{j=0}^{\infty} t^j u_{n,m}^{(j)}(x,y). \tag{4.26}$$

In accordance with (4.5) each term of the last series is twice differentiable with respect to x and y and for any fixed $t \in [0,1]$ the series

$$\Delta\tilde{u}_{n,m}(x,q,t) = \sum_{j=0}^{\infty} t^j \Delta u_{n,m}^{(j)}(x,y)$$

$$= \sum_{j=0}^{\infty} t^j \left[(m\pi)^2 u_{n,m}^{(j)}(x,y) + q(x,y) u_{n,m}^{(j-1)}(x,y)\right]$$

$$u_{n,m}^{(-1)}(x,y) = 0$$

is uniformly convergent. Thus, we can multiply both sides of the equation and boundary conditions from (4.5) by t^{j+1} and sum up over j from 0 to ∞:

$$\Delta\tilde{u}_{n,m}(x,y,t) - \left[(m\pi)^2 + tq(x,y)\right]\tilde{u}_{n,m}(x,y,t) = 0,$$

$$\frac{\partial\tilde{u}_{n,m}(x,y,t)}{\partial n} = 0, \quad (x,y) \in \Gamma_\Omega \backslash \Sigma_\Omega,$$

$$\frac{\partial\tilde{u}_{n,m}(x,y,t)}{\partial n} - \tilde{\lambda}_{n,m}\tilde{u}_{n,m}(x,y,t) = 0, \quad (x,y) \in \Sigma_\Omega, \tag{4.27}$$

where

$$\tilde{\lambda}_{n,m}(t) = \sum_{j=0}^{\infty} t^j \lambda_{n,m}^{(j)}.$$

It means that the pair $\tilde{\lambda}_{n,m}(t), \tilde{u}_{n,m}(x,y,t)$ solves the eigenvalue problem (4.27) for any $t \in [0,1]$ and , in particularly, for $t = 1$. On the other hand the problem (4.27) for $t = 1$ coincides with the problem (4.3) which yields

$$\tilde{\lambda}_{n,m}(1) = \lambda_{n,m}, \quad \tilde{u}_{n,m}(x,y,1) = u_{n,m}(x,y).$$

The proof is complete.

*If the condition (4.23) of Theorem 1 for concrete m,n is not fulfilled then the FD-method with $\bar{q}(x,y) \neq 0$ for the problem (**??**) can be applied. To this end we use a domain decomposition so that*

$$\Omega = \bigcup_{i=1}^{p} \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad i \neq j.$$

Let us denote by $\partial\Omega_i$ the boundary of Ω_i and choose the function $\bar{q}(x,y)$ such that

$$\bar{q}(x,y) = q(\xi_i, \eta_i), \quad \forall (x,y) \in \Omega_i,\ i = \overline{i,p},$$

where the point $(\xi_i, \eta_i) \in \Omega_i$ satisfy the condition

$$\delta_i = \min_{(\xi,\eta)\in\Omega_i} \max_{(x,y)\in\Omega_i} |q(x,y) - q(\xi,\eta)| = \max_{(x,y)\in\Omega_i} |q(x,y) - q(\xi_i,\eta_i)|.$$

Remark 2. *Let Π_p be the set of all possible decompositions of the domain Ω in p subdomains with Lipschitz boundary and $\pi_{p,\alpha}$ be a fixed decomposition, i.e.*

$$\Pi_p = \{\pi_{p,\alpha} : \alpha \in I\}$$

where I is the index set. One can consider the problem of optimal domain decomposition in the following sense: given a fixed domain Ω with a boundary Γ, a continuous function $q(x,y) \in C(\bar{\Omega})$ and a natural number p find the decomposition $\pi_{p,\alpha}$ such that

$$\inf_{\pi_{p,\alpha}\in\Pi_p} \left\|q - P_{\pi_{p,\alpha}} q\right\|_\infty = \left\|q - P_{\pi_{p,\beta}} q\right\|_\infty,$$

where $\left(P_{\pi_{p,\beta}} q\right)(x,y)$ is a piece-wise constant function on the decomposition $\pi_{p,\beta}$. We do not consider this problem here and refer only to [12]-[10].

For a given domain decomposition and $\bar{q}(x,y) \not\equiv 0$ we look for the solution of (4.3) in the form (the generalized non-classical expansions)

$$\begin{aligned} u_{n,m}(x,y) &= \sum_{j=0}^{\infty} u_{n,m}^{(j)}(x,y;\bar{q}(\cdot)), \\ \lambda_{n,m} &= \sum_{j=0}^{\infty} \lambda_{n,m}^{(j)}(\bar{q}(\cdot)), \\ (u_{n,m}^{(j)}(x,y,0) &\equiv u_{n,m}^{(j)}(x,y), \lambda_{n,m}^{(j)}(0) = \lambda_{n,m}^{(j)}), \end{aligned} \tag{4.28}$$

where the series terms are the solutions of

$$\begin{gathered} \Delta u_{n,m}^{(j+1)}(x,y;\bar{q}(\cdot)) - \left[(m\pi)^2 + \bar{q}(x,y)\right] u_{n,m}^{(j+1)}(x,y;\bar{q}(\cdot)) \\ = [q(x,y) - \bar{q}(x,y)] u_{n,m}^{(j)}(x,y;\bar{q}(\cdot)), \quad (x,y) \in \Omega, \\ \frac{\partial u_{n,m}^{(j+1)}(x,y;\bar{q}(\cdot))}{\partial \vec{n}_1} = 0, \quad (x,y) \in \Gamma_\Omega \backslash \Sigma_\Omega, \\ \frac{\partial u_{n,m}^{(j+1)}(x,y;\bar{q}(\cdot))}{\partial \vec{n}_1} - \lambda_{n,m}^{(0)}(\bar{q}(\cdot)) u_{n,m}^{(j+1)}(x,y;\bar{q}(\cdot)) \\ = \sum_{j=0}^{j} \lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)) u_{n,m}^{(j)}(x,y;\bar{q}(\cdot)) \\ (x,y) \in \Sigma_\Omega, j = 0,1,... \end{gathered} \tag{4.29}$$

The basic problem for (4.29) is

$$\begin{gathered} \Delta u_{n,m}^{(0)}(x,y;\bar{q}(\cdot)) - \left[(m\pi)^2 + \bar{q}(x,y)\right] u_{n,m}^{(0)}(x,y;\bar{q}(\cdot)) = 0, \quad (x,y) \in \Omega, \\ \frac{\partial u_{n,m}^{(0)}(x,y;\bar{q}(\cdot))}{\partial \vec{n}_1} = 0, \quad (x,y) \in \Gamma_\Omega \backslash \Sigma_\Omega, \\ \frac{\partial u_{n,m}^{(0)}(x,y;\bar{q}(\cdot))}{\partial \vec{n}_1} - \lambda_{n,m}^{(0)}(\bar{q}(\cdot)) u_{n,m}^{(0)}(x,y;\bar{q}(\cdot)) = 0, \quad (x,y) \in \Sigma_\Omega. \end{gathered} \tag{4.30}$$

The solution of the last problem is normalized by

$$\left\| u_{n,m}^{(0)} \right\| = 1. \tag{4.31}$$

In order to get the solvability condition for the problem (4.29) we multiply the both sides of (4.29) by $u_{n,m}^{(0)}$ and integrate over Ω. Then we get

$$\int_0^1 u_{n,m}^{(0)}(x,1;\bar{q}(\cdot)) \sum_{s=0}^{j} \lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)) u_{n,m}^{(s)}(x,1;\bar{q}(\cdot)) dx$$

$$= \int_0^1 \int_0^1 [q(x,y) - \bar{q}(x,y)] u_{n,m}^{(j)}(x,y;\bar{q}(\cdot)) u_{n,m}^{(0)}(x,y;\bar{q}(\cdot)) dxdy$$

or

$$\lambda_{n,m}^{(j+1)}(\bar{q}(\cdot)) = \left\{ \int_0^1 \left[u_{n,m}^{(0)}(x,1;\bar{q}(\cdot)) \right]^2 dx \right\}^{-1}$$

$$\times \left\{ - \sum_{s=1}^{j} \lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)) \int_0^1 u_{n,m}^{(0)}(x,1;\bar{q}(\cdot)) \right.$$

$$\times\, u_{n,m}^{(s)}(x,1;\bar{q}(\cdot))dx + \int_0^1 \int_0^1 [q(x,y) - \bar{q}(x,y)]\, u_{n,m}^{(j)}(x,y;\bar{q}(\cdot))$$

$$\left. \times\, u_{n,m}^{(0)}(x,y;\bar{q}(\cdot)) dxdy \right\}. \tag{4.32}$$

Let us represent the solution of the problem (4.29) as

$$u_{n,m}^{(j+1)}(x,y;\bar{q}(\cdot)) = w(x,y) + v(x,y) \tag{4.33}$$

where the functions $w(x,y), v(x,y)$ are solutions of the problems

$$\Delta w(x,y) - \left[(m\pi)^2 + \bar{q}(x,y) \right] w(x,y) = - \left[\bar{q}(x,y) - q(x,y) \right] u_{n,m}^{(j)}(x,y;\bar{q}(\cdot)),$$

$$(x,y) \in \Omega$$

$$\frac{\partial w(x,y)}{\partial \vec{n}} = 0, \quad (x,y) \in \Gamma \tag{4.34}$$

and

$$\Delta v(x,y) - \left[(m\pi)^2 + \bar{q}(x,y) \right] v(x,y) = 0, \quad (x,y) \in \Omega,$$

$$\frac{\partial v(x,y)}{\partial \vec{n}} = 0, \quad (x,y) \in \Gamma \backslash \Sigma_\Omega,$$

$$\frac{\partial v(x,y)}{\partial \vec{n}} - \lambda_{n,m}^{(0)}(\bar{q}(\cdot)) v(x,y) = \lambda_{n,m}^{(0)}(\bar{q}) w(x,y) \tag{4.35}$$

$$+ \sum_{s=0}^{j} \lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)) u_{n,m}^{(s)}(x,y;\bar{q}(\cdot)), (x,y) \in \Sigma_\Omega.$$

Further we need an estimate for $w(x,y)$. We multiply the equation (4.34) by $w(x,y)$ and after integration over Ω taking into account the Neumann boundary conditions we get

$$\|w\|_{1,2,\Omega} \le \frac{1}{(m\pi)^2} \|(q-\bar{q}) u_{n,m}^{(j)}\|_{0,2,\Omega}, \quad (\bar{q}(x,y) \ge 0) \tag{4.36}$$

where

$$(\|w\|_{1,2,\Omega})^2 = \int\int_\Omega \left\{ [w(x,y)]^2 + \left[\frac{\partial w(x,y)}{\partial x} \right]^2 + \left[\frac{\partial w(x,y)}{\partial y} \right]^2 \right\} dxdy.$$

Using the inequality

$$(a+b)^2 \le (1+\varepsilon)a^2 + \left(1+\frac{1}{\varepsilon}\right)b^2, \quad \forall \varepsilon > 0$$

with $\varepsilon = 2$ *one can get the estimate*

$$\int_0^1 [w(x,1)]^2 dx \le \frac{3}{2}\left(\|w\|_{1,2,\Omega}\right)^2 \le \frac{3}{2} \quad \left(\|(q-\bar{q})u_{n,m}^j\|_{0,2,\Omega}\right)^2. \qquad (4.37)$$

Since the system of functions

$$\left\{u_{p,m}^{(0)}(x,1)\right\}_{p=\overline{1,\infty}}$$

is complete and orthonormal on the interval $[0,1]$, *one easily get*

$$v(x,1) = \sum_{p=1,p\ne n}^{\infty} u_{p,m}^{(0)}(x,1;\bar{q}(\cdot)) \left(\frac{\lambda_{n,m}^{(0)}(\bar{q}(\cdot)) \int_0^1 w(\xi,1) u_{p,m}^{(0)}(\xi,1,\bar{q})d\xi}{\lambda_{p,m}^{(0)}(\bar{q}(\cdot)) - \lambda_{n,m}^{(0)}(\bar{q}(\cdot))} \right.$$
$$\left. + \frac{\sum_{s=0}^{j} \lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)) \int_0^1 u_{n,m}^{(s)}(\xi,1;\bar{q}(\cdot)) u_{p,m}^{(0)}(\xi,1,\bar{q})d\xi}{\lambda_{p,m}^{(0)}(\bar{q}(\cdot)) - \lambda_{n,m}^{(0)}(\bar{q}(\cdot))} \right) \qquad (4.38)$$

from where

$$\left[\int_0^1 v^2(\kappa,1)dx\right]^{1/2}$$
$$\le \sqrt{2}\max\left\{ \left[\lambda_{n,m}^{(0)}(\bar{q}(\cdot)) - \lambda_{n-1,m}^{(0)}(\bar{q}(\cdot))\right]^{-1}, \left[\lambda_{n+1,m}^{(0)}(\bar{q}(\cdot)) - \lambda_{n,m}^{(0)}(\bar{q}(\cdot))\right]^{-1} \right\}$$
$$\times \left\{ \lambda_{n,m}^{(0)}(\bar{q}(\cdot))^2 \|w(\cdot,1)\|_{0,2,(0,1))}^2 + \left(\|g(\cdot))\|_{0,2,(0,1)}\right)^2 \right\}^{1/2}. \qquad (4.39)$$

On the other hand, multiplying (4.35) by $v(x,y)$ *and integrating over* Ω *we get*

$$\int\int_\Omega \left\{ \left[\frac{\partial v(x,y)}{\partial x}\right]^2 + \left[\frac{\partial v(x,y)}{\partial y}\right]^2 + \left[(m\pi)^2 + \bar{q}(x,y)][v(x,y)\right]^2 \right\} dxdy$$
$$= \lambda_{n,m}^{(0)}(\bar{q}(\cdot)) \int_0^1 v^2(x,1)dx + \lambda_{n,m}^{(0)}(\bar{q}(\cdot)) \int_0^1 w(x,1)v(x,1)dx$$
$$+ \sum_{s=0}^{j} \lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)) \int_0^1 u_{n,m}^{(s)}(x,1,\bar{q}(\cdot))v(x,1)dx, \qquad (4.40)$$

which leads to

$$\int_\Omega \int [v(x,y)]^2 dxdy$$

$$\le \frac{1}{(m\pi)^2}\left\{\lambda^{(0)}_{n,m}(\bar q(\cdot))\left[\int_0^1 v^2(x,1)dx + \|v(\cdot,1)\|\|w(\cdot,1)\|\right]\right.$$

$$\left.+\sum_{s=0}^{j}\left|\lambda^{(j+1-s)}_{n,m}\right|\left\|u^{(s)}_{n,m}(\cdot,1;\bar q(\cdot))\right\|\|v(\cdot,1)\|\right\}$$

$$=\frac{1}{(m\pi)^2}M_{n,m}\left\{\left(\lambda^{(0)}_{n,m}(\bar q(\cdot))\|w(\cdot,1)\|\right)^2\right.$$

$$\left.+\left(\sum_{s=0}^{j}|\lambda^{(j+1-s)}_{n,m}(\bar q)|\|u^{(s)}_{n,m}(\cdot,1,\bar q)\|\right)^2\right\}^{1/2}$$

$$\times\left\{\left|\lambda^{(0)}_{n,m}(\bar q(\cdot))\right|\left[M_{n,m}\left((\lambda^{(0)}_{n,m}(\bar q)\|w(\cdot,1)\|\right)^2\right.\right.$$

$$\left.+\left(\sum_{s=0}^{j}\left|\lambda^{(j+1-s)}_{n,m}(\bar q)\right|\left\|u^{(s)}_{n,m}(\cdot,1,\bar q)\right\|\right)^2\right)^{1/2}$$

$$\left.+\ \|w(\cdot,1)\|+\sum_{s=0}^{j}\left|\lambda^{(j+1-s)}_{n,m}(\bar q)\right|\left\|u^{(s)}_{n,m}(\cdot,1,\bar q)\right\|\right\}$$

$$\le\frac{M_{n,m}}{(m\pi)^2}\left[\lambda^{(0)}_{n,m}(\bar q)\|w(\cdot,1)\|+\sum_{s=0}^{j}\left|\lambda^{(j+1-s)}_{n,m}(\bar q)\right|\left\|u^{(s)}_{n,m}(\cdot,1,\bar q)\right\|\right]$$

$$\times\left\{\left(\left[\lambda^{(0)}_{n,m}(\bar q(\cdot))\right]^2 M_{n,m}+\lambda^{(0)}_{n,m}(\bar q(\cdot))\right)\|w(\cdot,1)\|\right.$$

$$\left.+\left(\lambda^{(0)}_{n,m}(\bar q)M_{n,m}+1\right)\sum_{s=0}^{j}\left|\lambda^{(j+1-s)}_{n,m}(\bar q)\right|\|u^{(s)}_{n,m}(\cdot,1,\bar q)\|\right\}$$

$$=\frac{M_{n,m}}{(m\pi)^2}\left[\lambda^{(0)}_{n,m}(\bar q)M_{n,m}+1\right]$$

$$\times\left[\lambda^{(0)}_{n,m}(\bar q)\|w(\cdot,1)\|+\sum_{s=0}^{j}\left|\lambda^{(j+1-s)}_{n,m}(\bar q)\right|\|u^{(s)}_{n,m}(\cdot,1,\bar q)\|\right]^2$$

or

$$\left\{\int_\Omega\int[v(x,y)]^2dxdy\right\}^{1/2}$$

$$\leq \frac{\sqrt{M_{n,m}}}{m\pi}\left[\lambda_{n,m}^{(0)}(\bar{q})M_{n,m}+1\right]^{1/2}\left[\lambda_{n,m}^{(0)}(\bar{q})\|w(\cdot,1)\|\right.$$

$$\left.+\sum_{s=0}^{j}\left|\lambda_{n,m}^{(j+1-s)}(\bar{q})\right|\left\|u_{n,m}^{(s)}(\cdot,1;\bar{q})\right\|^{2}\right] \tag{4.41}$$

where

$$M_{n,m}=\sqrt{2}\max\left\{\left[\lambda_{n,m}^{(0)}(\bar{q}(\cdot))-\lambda_{n-1,m}^{(0)}(\bar{q}))\right]^{-1},\right.$$

$$\left.\left[\lambda_{n+1,m}^{(0)}(\bar{q}(\cdot))-\lambda_{n,m}^{(0)}[\bar{q}(\cdot))\right]^{-1}\right\}.$$

Now, it follows from (4.36),(4.37),(4.39),(4.40) that

$$\left\|u_{n,m}^{(j+1)}\right\|_{0,2,\Omega}\leq\|w\|_{0,2,\Omega}+\|v\|_{0,2,\Omega}\leq$$

$$\frac{\|q-\bar{q}\|_{\infty}}{(m\pi)^{2}}\|u_{n,m}^{(j)}\|_{0,2,\Omega}+\frac{\sqrt{M_{n,m}}}{m\pi}\left[\lambda_{n,m}^{(0)}(\bar{q})M_{n,m}+1\right]^{1/2}\times$$

$$\left\{\lambda_{n,m}^{(0)}(\bar{q})\sqrt{\frac{3}{2}}\|q-\bar{q}\|_{\infty}\|u_{n,m}^{(j)}\|_{0,2,\Omega}+\sum_{s=0}^{j}\left|\lambda_{n,m}^{(j+1-s)}(\bar{q})\right|\left\|u_{n,m}^{(s)}(\cdot,1;\bar{q})\right\|\right\}$$

or

$$\left\|u_{n,m}^{(j+1)}\right\|_{0,2,\Omega}\leq$$

$$\|q-\bar{q}\|_{\infty}P_{n,m}^{(1)}\|u_{n,m}^{(j)}\|_{0,2,\Omega}+P_{n,m}^{(2)}\sum_{s=0}^{j}\left|\lambda_{n,m}^{(j+1-s)}(\bar{q})\right|\left\|u_{n,m}^{(s)}(\cdot,1;\bar{q})\right\| \tag{4.42}$$

where

$$P_{n,m}^{(1)}=\frac{1}{(m\pi)^{2}}+\frac{\sqrt{M_{n,m}}}{m\Pi}\left[\lambda_{n,m}^{(0)}(\bar{q})M_{n,m}+1\right]^{1/2}\lambda_{n,m}^{(0)}(\bar{q})\sqrt{\frac{3}{2}}$$

$$P_{n,m}^{(2)}=\frac{\sqrt{M_{n,m}}}{m\pi}\left[\lambda_{n,m}^{(0)}(\bar{q})M_{n,m}+1\right]^{1/2}.$$

Analogously one gets

$$\left\|u_{n,m}^{(j+1)}(\cdot,1,\bar{q}(\cdot))\right\|_{0,2,(0,1)}\leq\|w(\cdot,1)\|+\|v(\cdot,1)\|\leq$$

$$\frac{\sqrt{3}}{2}\|q-\bar{q}\|_{\infty}\|u_{n,m}^{(j)}\|_{0,2,\Omega}+M_{n,m}\times$$

$$\left\{\lambda_{n,m}^{(0)}(\bar{q})\sqrt{\frac{3}{2}}\|q-\bar{q}\|_{\infty}\|u_{n,m}^{(j)}\|_{0,2,\Omega}\right.$$

$$+\sum_{s=0}^{j}\left|\lambda_{n,m}^{(j+1-s)}(\bar{q})\right|\left\|u_{n,m}^{(s)}(\cdot,1,\bar{q})\right\|_{0,2,(0,1)}\Bigg\}$$

or

$$\left\|u_{n,m}^{(j+1)}(\cdot,1,\bar{q}(\cdot))\right\|_{0,2,(0,1)} \leq \sqrt{\frac{3}{2}}\|q-\bar{q}\|_{\infty}\left[1+M_{n,m}\lambda_{n,m}^{(0)}(\bar{q})\right]\|u_{n,m}^{(j)}\|_{0,2,\Omega}+$$

$$M_{n,m}\sum_{s=0}^{j}\left|\lambda_{n,m}^{(j+1-s)}(\bar{q})\right|\left\|u_{n,m}^{(s)}(\cdot,1,\bar{q})\right\|_{0,2,(0,1)} \tag{4.43}$$

The inequalities (4.42),(4.43) should be considered together with

$$\left|\lambda_{n,m}^{(j+1)}(\bar{q}(\cdot)\right| \leq \sum_{s=1}^{j}\left|\lambda_{n,m}^{(j+1-s)}(\bar{q}(\cdot)\right|\frac{\|u_{n,m}^{(s)}(\cdot,1,\bar{q})\|_{0,2,(0,1)}}{\|u_{n,m}^{(0)}(\cdot,1,\bar{q})\|_{0,2,(0,1)}}$$

$$+\|q-\bar{q}\|_{\infty}\|u_{n,m}^{(j)}\|_{0,2,\Omega}\frac{\|u_{n,m}^{(0)}\|_{0,2,\Omega}}{\|u_{n,m}^{(0)}(\cdot,1,\bar{q})\|_{0,2,(0,1)}}. \tag{4.44}$$

The nonlinear system of inequalities (4.42)-(4.44) is majorized by the following system of linear equations

$$\begin{aligned}
U_{j+1} &= a\left[\|q-\bar{q}\|_{\infty}U_j+\sum_{s=0}^{j}\Lambda_{j+1-s}U_s^{\#}\right],\\
U_{j+1}^{\#} &= b\left[\|q-\bar{q}\|_{\infty}U_j+\sum_{s=0}^{j}\Lambda_{j+1-s}U_s^{\#}\right],\\
\Lambda_{j+1} &= c\left[\|q-\bar{q}\|_{\infty}U_j+\sum_{s=1}^{j}\Lambda_{j+1-s}U_s^{\#}\right],\\
j &= 0,1,...,
\end{aligned} \tag{4.45}$$

where

$$\begin{aligned}
a &= \max\left\{P_{n,m}^{(1)},P_{n,m}^{(2)}\right\}, \quad b=\max\left\{\sqrt{\frac{3}{2}}[1+M_{n,m}\lambda_{n,m}^{(0)}(\bar{q})],M_{n,m}\right\},\\
c &= \max\left\{1,\|u_{n,m}^{(0)}\|_{0,2,\Omega}\right\}\\
U_0^{\#} &= 1, \quad U_0=\|u_{n,m}^{(0)}\|_{0,2,\Omega}.
\end{aligned}$$

Since (4.45) yields

$$U_j^{\#}=\frac{b}{a}U_j, \tag{4.46}$$

it is sufficient to consider the following system instead of (4.45)

$$\begin{aligned} U_{j+1} &= a\left[\|q-\bar{q}\|_\infty U_j + \frac{b}{a}\sum_{s=0}^{j}\Lambda_{j+1-s}U_s\right], \\ \Lambda_{j+1} &= c\left[\|q-\bar{q}\|_\infty U_j + \frac{b}{a}\sum_{s=1}^{j}\Lambda_{j+1-s}U_s\right] \end{aligned}$$

or

$$\begin{aligned} U_{j+1} &= b\left[\frac{a}{b}\|q-\bar{q}\|_\infty U_j + \sum_{s=0}^{j}\Lambda_{j+1-s}U_s\right], \\ \Lambda_{j+1} &= \frac{cb}{a}\left[\frac{a}{b}\|q-\bar{q}\|_\infty U_j + \sum_{s=1}^{j}\Lambda_{j+1-s}U_s\right]. \end{aligned} \tag{4.47}$$

Changing variables in (4.47) by

$$U_j = [a\|q-\bar{q}\|_\infty]^j\,\frac{a}{cb}U_j^*, \quad \Lambda_j = \frac{a}{b}\|q-\bar{q}\|_\infty\,[a\|q-\bar{q}\|_\infty]^{j-1}\,\Lambda_j^*, \tag{4.48}$$

we get

$$\begin{aligned} U_{j+1}^* &= U_j^* + \sum_{p=0}^{j}\Lambda_{j+1-p}^*U_p^*, \\ \Lambda_{j+1}^* &= U_j^* + \sum_{p=1}^{j}\Lambda_{j+1-p}^*U_p^*, \\ j=0,1,\ldots \qquad & U_0^* = \frac{cb}{a}\|u_{n,m}^{(0)}\|_{0,2,\Omega}. \end{aligned} \tag{4.49}$$

We use the method of generating functions in order to solve the nonlinear recurrent system of equations (4.49). Let

$$f(z) = \sum_{j=0}^{\infty} z^j U_j^*, \quad g(z) = \sum_{j=0}^{\infty} z^j \Lambda_{j+1}^*,$$

then we get from (4.49)

$$\begin{aligned} f(z) - U_0^* &= zf(z)[1+g(z)], \\ g(z) &= f(z) + [f(z)-U_0^*]g(z) \end{aligned}$$

which yields

$$f(z) = \frac{1}{2}\{1 + 2U_0^* - z(1+U_0^*)$$

$$-\sqrt{[1 + 2U_0^* - z(1+U_0^*)]^2 - 4U_0^*(1+U_0^*)}\Big\}$$

$$= \frac{1}{2}\{1 + 2U_0^* - z(1+U_0^*)$$

$$-\left[1 - \frac{(1+U_0^*)z}{\beta}\right]^{1/2}[1-(1+U_0^*)z\beta]^{1/2}\Bigg\},$$

$$g(z) = \frac{f(z)-U_0^*}{z(1+U_0^*)}, \quad \beta = \frac{1}{1+2U_0^*+2\sqrt{U_0^*(1+U_0^*)}}.$$

Using the Laurent expansion for $\sqrt{1-x}$ *we can write down the last two equations in the form*

$$\begin{aligned} f(z) &= U_0^* + U_0^*(1+U_0^*)z - \frac{1}{2}\sum_{j=2}^{\infty}\left[\frac{1+U_0^*}{\beta}z\right]^j \sum_{p=0}^{j}\alpha_p\alpha_{j-p}\beta^{2p}, \\ g(z) &= U_0^* - \frac{1}{2\beta}\sum_{j=2}^{\infty}\left[\frac{1+U_0^*}{\beta}z\right]^{j-1}\sum_{p=0}^{j}\alpha_p\alpha_{j-p}\beta^{2p}, \end{aligned} \tag{4.50}$$

where

$$\alpha_p = -\frac{(2p-3)!!}{(2p)!!}, \quad p \geq 2, \quad \alpha_0 = 1, \quad \alpha_1 = -\frac{1}{2}.$$

It follows from (4.50) that

$$U_j^* = -\frac{1}{2}\left[\frac{1+U_0^*}{\beta}\right]^j \sum_{p=0}^{j}\alpha_p\alpha_{j-p}\beta^{2p}$$

$$= \frac{1}{2}\left[\frac{1+U_0^*}{\beta}\right]^j \frac{(2j-3)!!}{(2j)!!}$$

$$\times\left\{1 + \beta^{2j} - \frac{(2j)!!}{(2j-3)!!}\sum_{p=1}^{j-1}\frac{(2p-3)!!}{(2p)!!}\frac{(2j-2p-3)!!}{(2j-2p)!!}\beta^{2p}\right\}$$

$$\leq \frac{1}{2}\left(\frac{1+U_0^*}{\beta}\right)^j \frac{(2j-3)!!}{(2j)!!}, \tag{4.51}$$

$$\Lambda_j^* = \frac{1}{2\beta}\left(\frac{1+U_0^*}{\beta}\right)^{j-1}\frac{(2j-3)!!}{(2j)!!}$$

$$\times \left\{ 1 + \beta^{2j} - \frac{(2j)!!}{(2j-3)!!} \sum_{p=1}^{j-1} \frac{(2p-3)!!}{(2p)!!} \frac{(2j-2p-3)!!}{(2j-2p)!!} \beta^{2p} \right\}$$

$$\leq \frac{1}{2\beta} \left(\frac{1+U_0^*}{\beta} \right)^{j-1} \frac{(2j-3)!!}{(2j)!!}. \quad (4.52)$$

Having in mind the relations (4.48) we get

$$\|u_{n,m}^{(j)}\|_{0,2,\Omega} \leq U_j =$$

$$[a\|q-\bar{q}\|_\infty]^j \frac{a}{cb} U_j^* \leq \frac{a}{2cb} \frac{(2j-3)!!}{(2j)!!} \left[a\|q-\bar{q}\|_\infty \frac{1+U_0^*}{\beta} \right]^j, \quad (4.53)$$

and

$$|\lambda_{n,m}^{(j+1)}| \leq \Lambda_{j+1} = \frac{1}{b}[a\|q-\bar{q}\|_\infty]^{j+1} \Lambda_{j+1}^*$$

$$\leq \frac{a\|q-\bar{q}\|_\infty}{2b\beta} \frac{(2j-1)!!}{(2j+2)!!} \left[a\|q-\bar{q}\|_\infty \frac{1+U_0^*}{\beta} \right]^j. \quad (4.54)$$

The estimates (4.53),(4.54) allow us to get the following statement.

Theorem 6. *If*

$$\nu_{n,m} = a\|q-\bar{q}\|_\infty \frac{1+U_0^*}{\beta} < 1, \quad (4.55)$$

then the FD-method for the problem (4.1) converges exponentially and the following error estimate holds true

$$\left| \lambda_{n,m} - \lambda_{n,m}^N(\bar{q}(\cdot)) \right| = \left| \lambda_{n,m} - \sum_{j=0}^{N} \lambda_{n,m}^{(j)}(\bar{q}(\cdot)) \right|$$

$$\leq \frac{1}{2} \left[\sqrt{U_0^*} + \sqrt{1+U_0^*} \right]^2 \frac{(2N-1)!!}{(2N+2)!!} \frac{(\nu_{n,m})^N}{1-\nu_{n,m}} \quad (4.56)$$

The proof is completely analogous to that of Theorem 1.

The implementation of the FD-method for the basic problem (4.30) can be performed by the formulation of (4.30) as an interface problem analogously to [4].

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