CONTROL OF GAS FLOW IN THE MAIN PIPE-LINE BY BOUNDARY CONDITIONS

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(Received: 03.02.02; revised: 15.07.02)

Abstract

The mathematical model of filtration including low-order derivatives towards spatial coordinates is considered. A problem of gas flow control by means of gas flux at the beginning of the pipe-line is studied.

Relying on the method of variational inequalities, the existence of the solution and convergence of approximate solutions are established.

Key words and phrases: Gas filtration, optimal control, variational inequalities.

AMS subject classification: 49J20, 76N25.

1. *Introduction*

In consolidation theory of the medium comprising fluid and gas main objects of research are: hydrotechnical construction, optimal projecting of pipe-lines, their exploitation and so on.

We consider the problem of gas flow control in the main pipe-line (cylindrical area). In particular, to determine the value of flux at the beginning of the pipe-line that ensures the desired, named beforehand, value of gas pressure at the end of the pipe-line during a certain period of time. It is assumed that the value of the flux at the end of the main pipe-line are known values. Problems of similar type and methods of their numerical solution are studied in many papers [(4), (5), (7), (8)].

Generally, filtration processes are described by means of nonlinear partial differential equations [(1), (6)]. The study of real processes frequently requires solving two-phase filtration problems. In corresponding mathematical models, by retaining high-order precision members, a parabolic equation is received that also includes lower-order derivatives. In this work a linear mathematical model including such equation is considered.

Let us introduce the following notations:

$$\left.\begin{array}{l} \Omega \ is\ a\ cylinder\ in\ R^3,\ the\ axis\ of\ which\ coincides\ with \\ OX_1\ coordinate\ line,\ left\ and\ right\ surfaces\ are\ \sigma_1 \\ and\ \sigma_2\ correspondingly\ and\ lateral\ surface\ is\ \sigma_3; \\ \Gamma = \sigma_1 \cup \sigma_2 \cup \sigma_3; \\ Q_T = \Omega \times (O,T)\ is\ open\ cylinder; \\ \Sigma_T = \Gamma \times (O,T)\ is\ lateral\ surface\ of\ cylinder\ Q_T. \end{array}\right\} \quad (1.1)$$

Consider the following parabolic equation:

$$\frac{\partial y}{\partial t} + A\left(x, t, \frac{\partial}{\partial x}\right) y = f(x,t), \quad (1.2)$$

where $x = (x_1, x_2, x_3)$,

$$A\left(x, t, \frac{\partial}{\partial x}\right) = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{3} a_i(x,t)\frac{\partial}{\partial x_i} + a_0(x,t)I, \quad (1.3)$$

$a_{ij},\ \ a_i,\ \ a_0 \in L^\infty(\Omega_T)$ and

$$\left\{\begin{array}{l} \sup\limits_{i}(\operatorname*{supess}\limits_{(x,t)\in\Omega_T} \mid a_i(x,t) \mid) \le c_1, \\ \operatorname*{supess}\limits_{(x,t)\in\Omega_T} \mid a_i(x,t) \mid \le c_2; \end{array}\right. \quad (1.4)$$

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_i \ge \alpha \sum_{i=1}^{3} \mid \xi_i \mid^2, \quad (1.5)$$

$\alpha > 0,\ \ \xi_i \in R$, almost everywhere on Ω_T and α does not depend on x and t.

For equation (2) consider the following Cauchy-Neyman problem:

$$y(x,0) = y_0, \quad (1.6)$$

$$\frac{\partial y}{\partial \nu_A}|_{\Sigma_1} = v,\ \ \frac{\partial y}{\partial \nu_A}|_{\Sigma_2} = w,\ \ \frac{\partial y}{\partial \nu_A}|_{\Sigma_3} = 0, \quad (1.7)$$

where

$$\Sigma_i = \sigma_i \times (O,T),\ \ i = 1,2,3,$$

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{3} a_{ij}(x,t) cos(n, x_i) \frac{\partial y}{\partial x_j}$$

($cos(n, x_i)$ is the cosine of direction i of exterior normal $\mathbf{n}$).

Assume that $\exists v_1(x,t)$ and $v_2(x,t)$ such that

$$v_1(x,t) \leq v(x,t) \leq v_2(x,t).$$

Below we consider the case when y_0 and w are fixed functions, and v is variable. Therefore we will denote the solution of the problem (2), (6), (7) in this way: $y(x,t) = y(x,t;v)$.

Suppose $p(x_2, x_3, t)$ is a given function on $\sum_2$. Introduce the cost function

$$J(v) = \int\limits_{\Sigma_2} (y(v)\mid_{\Sigma_2} - p)^2 d\Sigma.$$

Our objective is: to find function u such that

$$J(u) = \underset{v \in U_\partial}{inf}\ J(v),$$

where v is called control, U_∂ is the set of possible control, $y(v)$ is state function of the system.

Let us formulate the set problem in variational terms and establish under what conditions the solution exists.

Consider the following quadratic form:

$$\begin{aligned} a(t,u,v) = \sum_{i,j=1}^{3} \int\limits_{\Omega} a_{ij}(x,t) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx \\ + \sum_{i=1}^{3} \int\limits_{\Omega} a_i(x,t) \frac{\partial u}{\partial x_i}(x) v(x) dx + \int\limits_{\Omega} a_0(x,t) u(x) v(x) dx. \end{aligned} \tag{1.8}$$

Let us show that the form given by equality (8) is coercive on $H^1(\Omega)$. According to conditions (4) and (5):

$$\sum_{i,j=1}^{3} \int\limits_{\Omega} a_{ij}(x,t) \frac{\partial v}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx \geq \alpha \sum_{i=1}^{3} \int\limits_{\Omega} \left(\frac{\partial v}{\partial x_i}\right)^2 dx, \tag{1.9}$$

$$\begin{aligned} |\sum_{i=1}^{3} \int\limits_{\Omega} a_i(x,t) \frac{\partial v}{\partial x_i}(x) v(x) dx| \leq c_1 \left(\int\limits_{\Omega} \sum_{i=1}^{3} |\frac{\partial v}{\partial x_i}|^2 dx \right)^{\frac{1}{2}} \\ \times \left(\int\limits_{\Omega} |v(x)|^2 dx \right)^{\frac{1}{2}} \leq \frac{\alpha}{2} \|gradv\|^2_{L^2(\Omega)} + \frac{c_1^2}{2\alpha} \|v\|^2_{L^2(\Omega)}, \end{aligned} \tag{1.10}$$

$$|\int_\Omega a_0(x,t)|v(x)|^2dx| \leq c_2\|v\|^2_{L^2(\Omega)}. \tag{1.11}$$

If we select λ so that $\lambda - c_2 - \frac{c_1^2}{2\alpha} > 0$, from inequalities (9), (10) and (11) we receive:

$$a(t,v,v) + \lambda\|v\|^2 \geq \frac{\alpha}{2}\|gradv\|^2_{L^2(\Omega)}$$

$$+(\lambda - c_2 - \frac{c_1^2}{2\alpha})\|v\|^2_{L^2(\Omega)} \geq \alpha_1\|v\|^2_{H^1(\Omega)}. \tag{1.12}$$

Problem (2), (6), (7) can be written in the following vector form:

$$\frac{dy(v)}{dt} + A(t)y(v) = F(t) + Bv, \tag{1.13}$$

$$y(v)|_{t=0} = y_0, \tag{1.14}$$

where operator $A(t)$ is determined by form (8),

$$(F(t),\varphi) = \int_\Omega f(t)\varphi dx + \int_{\sigma_2} w\varphi d\sigma,$$

$$(Bv,\varphi) = \int_{\sigma_1} v\varphi d\sigma.$$

Find the solution of problem (3), (4) in the following vector space

$$y(v) \in L^2(0,T;H^1(\Omega)) \ \left(y, \ \frac{\partial y}{\partial x_i} \in L^2(Q_T)\right).$$

If we demand that $f \in L^2(Q_T), \ v \in L^2(\Sigma_1), \ \omega \in L^2(\Sigma_2)$, then

$$F(t) \in L^2(0,T;(H^1(\Omega))^{'}), \ B \in \mathcal{L}\left(L^2(\Sigma_1), \ (H^1(\Omega))^{'}\right).$$

The following theorem is widely known (see [3],[7]):

Theorem 1. Let conditions (4) and (5) be fulfilled and let $f \in L^2(Q_T), \ v \in L^2(\Sigma_1), \ \omega \in L^2(\Sigma_2)$, then for arbitrary initial value

$$y(0) = y_0 \in L^2(\Omega)$$

problem (13), (14) has the unique solution $y(v) \in L^2(0,T;H^1(\Omega))$ and this solution is continuously dependent on the initial values. Mapping $f,v,\omega,y_0 \to y: \ L^2(0,T;H^1(\Omega)^{'}) \times L^2(\Sigma_1) \times L^2(\Sigma_2) \to \ L^2(0,T;(H^1(\Omega))$ is continuous.

Let us move to the solution of the set problem. Suppose $v_1, v_2 \in L^2(\Sigma_1)$ and $U_\partial = \{v : \ v \in L^2(\Sigma_1), \ v_1 \le v \le v_2\}, \quad p \in L^2(\Sigma_2)$. Consider cost function

$$J(v) = \int\limits_{\Sigma_2} (y(v)|_{\Sigma_2} - p)^2 \, d\Sigma$$

on $U_\partial \subset L^2(\Sigma_1)$ convex set.

Let us show that the cost function is a convex functional. Indeed ,

$$J(\alpha v_1 + (1-\alpha)v_2) = \int\limits_{\Sigma_2} (y(\alpha v_1 + (1-\alpha)v_2) - p)^2 \, d\Sigma =$$

$$\int\limits_{\Sigma_2} (\alpha y(v_1) + (1-\alpha)y(v_2) - p)^2 d\Sigma = \int\limits_{\Sigma_2} (\alpha^2 y^2(v_1) + (1-\alpha)^2 y^2(v_2) +$$

$$p^2 + 2\alpha(\alpha-1)y(v_1)y(v_2) - 2\alpha y(v_1)p - 2(1-\alpha)y(v_2)p)d\Sigma =$$

$$\int\limits_{\Sigma_2} (\alpha^2 y^2(v_1) + \alpha y^2(v_1) - \alpha y^2(v_1) + (1-\alpha)^2 y^2(v_2) +$$

$$(1-\alpha)y^2(v_2) - (1-\alpha)y^2(v_2) + p^2 +$$

$$2\alpha(\alpha-1)y(v_1)y(v_2) - 2\alpha y(v_1)p - 2(1-\alpha)y(v_2)p)d\Sigma =$$

$$\int\limits_{\Sigma_2} (-\alpha(1-\alpha)y^2(v_1) - \alpha(1-\alpha)y^2(v_2) + 2\alpha(1-\alpha)y(v_1)y(v_2) +$$

$$\alpha(y^2(v_1) - 2py(v_1) + p^2) + (1-\alpha)(y^2(v_2) - 2py(v_2) + p^2))d\Sigma =$$

$$\int\limits_{\Sigma_2} (-\alpha(1-\alpha)(y^2(v_1) - 2y(v_1)y(v_2) + y^2(v_2)) +$$

$$(1-\alpha)(y^2(v_2) - 2py(v_2) + p^2) + \alpha(y^2(v_1) - 2py(v_1) + p^2))d\Sigma =$$

$$\int\limits_{\Sigma_2} (-\alpha(1-\alpha)(y(v_1) - y(v_2))^2 + \alpha(y(v_1) - p)^2 + (1-\alpha)(y(v_2) - p)^2)d\Sigma \le$$

(because $0 \le \alpha \le 1$)

$$\le \int\limits_{\Sigma_2} (\alpha(y(v_1) - p)^2 + (1-\alpha)(y(v_2) - p)^2)d\Sigma = \alpha J(v_1) + (1-\alpha)J(v_2).$$

Now let us show that the cost function is differentiable in space $L^2(\Sigma_1)$

$$J(v+h) - J(v) = \int\limits_{\Sigma_2} (y(v+h) - p)^2 d\Sigma - \int\limits_{\Sigma_2} (y(v) - p)^2 d\Sigma =$$

$$\int_{\Sigma_2}(y(v+h)-y(v)+y(v)-p)^2 d\Sigma - \int_{\Sigma_2}(y(v)-p)^2 d\Sigma =$$

$$\int_{\Sigma_2}(y(v+h)-y(v))^2 d\Sigma + 2\int_{\Sigma_2}(y(v+h)-y(v))(y(v)-p)d\Sigma +$$

$$\int_{\Sigma_2}(y(v)-p)^2 d\Sigma - \int_{\Sigma_2}(y(v)-p)^2 d\Sigma =$$

$$2\int_{\Sigma_2}(y(v+h)-y(v))(y(v)-p)d\Sigma + \int_{\Sigma_2}(y(v+h)-y(v))^2 d\Sigma =$$

$$2\int_{\Sigma_2}(y(v+h)-y(v))(y(v)-p)d\Sigma + O(\|h\|^2).$$

Therefore

$$J^{'}(v)h = 2\int_{\Sigma_2}(y(v+h)-y(v))(y(v)-p)d\Sigma.$$

So, we have shown that the cost function is convex and differentiable. It follows that it is weakly semi-continuous in $L^2(\Sigma_1)$ (see [2]). That is, if $u_k \rightharpoonup u$ in $L^2(\Sigma_1)$ then

$$J(u) \le \lim_{k\to\infty} \inf_k J(u_k).$$

So, relying on the results of paper [7] it can be concluded that the following theorem holds true:

Theorem 2. Let conditions of theorem 1 be fulfilled, then set X of functions $u \in U_\partial$ such that

$$J(u) = \inf_{v\in U_\partial} J(v)$$

is a nonempty convex set. Besides, for $u \in X$ it is necessary and enough that

$$\int_{\Sigma_2}(y(v)-y(u))(y(u)-p)d\Sigma \ge 0 \ \ \forall v \in U_\partial. \tag{1.15}$$

Theorem 2 is not constructive. So, it is important to develop a method of finding an element of set X.

Consider bilinear form $(u,v)_{L^2(\Sigma_1)}$ on X. As X is convex closed set, there exists the unique $u_0 \in X$ such that

$$(u_0, v-u_0)_{L^2(\Sigma_1)} \ge 0 \ \ \forall v \in X. \tag{1.16}$$

Assume that

$$J_\varepsilon(u) = J(u) + \varepsilon(u,u)_{L^2(\Sigma_1)}. \tag{1.17}$$

$J_\varepsilon(u)$ is coercive on U_∂. So, there exists the unique $u_\varepsilon \in U_\partial$ such that

$$J_\varepsilon^{'}(u_\varepsilon)(v-u_\varepsilon) \geq 0 \quad \forall v \in U_\partial. \tag{1.18}$$

From inequality (18), taking into account condition (15), it follows that

$$\int\limits_{\Sigma_2} (y(v)-y(u_\varepsilon))(y(u_\varepsilon)-p)d\Sigma + \varepsilon(u_\varepsilon, v-u_\varepsilon) \geq 0 \quad \forall v \in U_\partial. \tag{1.19}$$

Let us replace v by u_0 in (19), then

$$\int\limits_{\Sigma_2} (y(u_0)-y(u_\varepsilon))(y(u_\varepsilon)-p)d\Sigma + \varepsilon(u_\varepsilon, u_0-u_\varepsilon) \geq 0. \tag{1.20}$$

In inequality (15) let us replace u by u_0 and v by u_ε. We will receive:

$$\int\limits_{\Sigma_2} (y(u_\varepsilon)-y(u_0))(y(u_0)-p)d\Sigma \geq 0. \tag{1.21}$$

Sum up inequalities (20) and (21) member by member:

$$\int\limits_{\Sigma_2} (y(u_\varepsilon)-y(u_0))(y(u_0)-p)d\Sigma$$

$$+\int\limits_{\Sigma_2} (y(u_0)-y(u_\varepsilon))(y(u_\varepsilon)-p)d\Sigma + \varepsilon(u_\varepsilon, u_0-u_\varepsilon)_{L^2(\Sigma_1)} \geq 0.$$

From this we receive:

$$-\int\limits_{\Sigma_2} (y(u_0)-y(u_\varepsilon))(y(u_0)-p-y(u_\varepsilon)+p)d\Sigma + \varepsilon(u_\varepsilon, u_0-u_\varepsilon)_{L^2(\Sigma_1)} \geq 0.$$

that is

$$-\int\limits_{\Sigma_2} (y(u_0)-y(u_\varepsilon))^2 d\Sigma + \varepsilon(u_\varepsilon, u_0-u_\varepsilon)_{L^2(\Sigma_1)} \geq 0,$$

so

$$(u_\varepsilon, u_0-u_\varepsilon)_{L^2(\Sigma_1)} \geq 0. \tag{1.22}$$

From this we have

$$(u_\varepsilon, u_0)_{L^2(\Sigma_1)} \geq \|u_\varepsilon\|^2_{L^2(\Sigma_1)},$$

that is

$$\|u_\varepsilon\|_{L^2(\Sigma_1)} \le \|u_0\|_{L^2(\Sigma_1)},$$
$$\|u_\varepsilon\|_{L^2(\Sigma_1)} \le c_3, \quad c_3 = const. \tag{1.23}$$

From inequality (23) it follows that we can pick out from sequence $\{u_\varepsilon\}$ such a subsequence that will be weakly convergent towards an element $w \in L^2(\Sigma_1)$. Denote this sequence by u_ε again. So,

$$u_\varepsilon \rightharpoonup \omega \quad in \quad L^2(\Sigma_1) \ (\rightharpoonup \textit{means weak convergence}).$$

As U_∂ is weakly closed, so $w \in U_\partial$.

From inequality (19) it follows that

$$\int\limits_{\Sigma_2} (y(v) - y(u_\varepsilon))(y(u_\varepsilon) - p)d\Sigma + \varepsilon(u_\varepsilon, v)_{L^2(\Sigma_1)} \ge 0. \tag{1.24}$$

In inequality (24) let us pass to the limit when $\varepsilon \to 0$. As we have already shown, the first summand is weakly lower semi-continuous, and the second summand tends to 0. So

$$\int\limits_{\Sigma_2} (y(v) - y(\omega))(y(\omega) - p)d\Sigma \ge 0 \ \ \forall v \in U_\partial. \tag{1.25}$$

From inequality (25) it follows that $w \in X$.

Now, in inequality (22) let us pass to the limit when $\varepsilon \to 0$. We receive

$$(\omega, u_0 - \omega)_{L^2(\Sigma_1)} \ge 0. \tag{1.26}$$

Replace v by w in inequality (16). We have

$$(u_0, \omega - u_0)_{L^2(\Sigma_1)} \ge 0. \tag{1.27}$$

Sum up inequalities (26) and (27). We receive

$$(u_0 - \omega, u_0 - \omega)_{L^2(\Sigma_1)} \le 0,$$

that is, $u_0 = \omega$.

From this we receive that $u_\varepsilon \to u_0$ in space $L^2(\Sigma_1)$.

Now let us show that $u_\varepsilon \to u_0$ in space $L^2(\Sigma_1)$, that is, show that

$$(u_\varepsilon - u_0, u_\varepsilon - u_0)_{L^2(\Sigma_1)} \to 0,$$

$$(u_\varepsilon - u_0, u_\varepsilon - u_0)_{L^2(\Sigma_1)} = (u_\varepsilon, u_\varepsilon - u_0)_{L^2(\Sigma_1)} - (u_0, u_\varepsilon - u_0)_{L^2(\Sigma_1)}. \tag{1.28}$$

According to (22) and (28) we have

$$(u_\varepsilon - u_0, u_\varepsilon - u_0)_{L^2(\Sigma_1)} \le -(u_0, u_\varepsilon - u_0)_{L^2(\Sigma_1)}. \tag{1.29}$$

As $u_\varepsilon \rightharpoonup u_0$, from inequality (29) it follows that $u_\varepsilon \to u_0$ in space $L^2(\Sigma_1)$.

So, the following theorem holds true:

Theorem 3. Let conditions of theorem 2 be satisfied and X be a non-empty set, then

$$u_\varepsilon \to u_0 \quad \textit{in space} \quad L^2(\Sigma_1) \quad \textit{when} \quad \varepsilon \to 0. \tag{1.30}$$

Remark: If set U_∂ is limited, then $X \neq \oslash$.

Indeed, assume that sequence $\{u_\varepsilon\}$ satisfies condition (13). As $u_\varepsilon \in U_\partial$, so condition (23) automatically holds true.

From this it follows that $X \neq \oslash$.

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