EQUILIBRIUM TYPE SCHEMES FOR SCALAR CONSERVATION LAWS WITH SOURCE TERM

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Abstract

Equilibrium schemes presented in [7] are extended in several space dimensions on unstructured meshes. The scheme maintains only some exact equilibrium states at cell interfaces, it ensures uniform L^∞ bound of approximate solutions and verifies in cell entropy inequalities. Under some regularity requirement on mesh refinement process equilibrium type schemes possessing suitable kinetic interpretation converge towards entropy solution. Numerical tests show high accuracy and efficiency of the developed scheme. We have gathered numerical evidence that its convergence rate increases significantly together with mesh refinement. This ensures exponential wise fast convergence and the advantage of equilibrium type schemes over the schemes with standard cell centered discretization of source term.

Key words and phrases: scalar conservation laws, finite volume schemes, stiff source terms, convergence.

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1. *Introduction*

The aim of the present work is to extend the results of [7] in several space dimensions. Namely, we extend:
(i) the so called equilibrium schemes in several space dimensions;
(ii) the method of proof of convergence of finite volume schemes on unstructured meshes.

Consider the following multidimensional scalar conservation law with source term

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{N}\frac{\partial A_i(u)}{\partial x_i}+\sum_{i=1}^{N}z'_{i,x_i}(x)b(u)=0, \tag{1.1}$$

and the following initial condition

$$u(0,x) = u_0(x), \tag{1.2}$$

where

$$b(0) = 0, \quad |b'(u)| \leq K_b, \tag{1.3}$$

$x \in \mathrm{I\!R}^{\mathrm{N}},\ \mathrm{N} > 1,\ \ A_i, B \in C^1,\ \ z_i, z'_{i,x_i} \in L^\infty, 1 \leq i \leq N,\ u_0(x) \in L^\infty(\mathrm{I\!R}^{\mathrm{N}})$.

The equation (1.1) is endowed with the full family of entropy inequalities and entropy solutions of (1.1) satisfy

$$\frac{\partial S(u)}{\partial t} + \sum_{i=1}^{N} \frac{\partial \eta_i(u)}{\partial x_i} + S'(u) b(u) \sum_{i=1}^{N} \frac{\partial z_i(x)}{\partial x_i} \leq 0, \tag{1.4}$$

for all convex entropy functions $S(\cdot)$ and corresponding entropy fluxes $\eta_i(\cdot)$ that are defined in accordance with the relation

$$\eta_i\prime(u) = S'(u) a_i(u), \qquad a_i(u) = A_i'(u), \tag{1.5}$$

see Kruzkov [20], Lax [22] for more details.
Smooth steady state solutions of (1.1) are defined by the equation

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} (D_i(u) + z_i(x)) = 0, \tag{1.6}$$

where

$$D_i(u) = \int_0^u \frac{a_i(s)}{B(s)} ds < +\infty.$$

A numerical difficulty which arises in connection with the problem (1.1) is to preserve, at a discrete level, the "equilibrium", i.e. steady states. Even in single space dimension relatively small variation of function z can introduce large errors, see e.g. [7], if standard discretization of source term at cell center is used. The question of development of efficient methods for the discretization of source terms has been addressed by several authors during past decades. Upwind methods for the discretization of source term has been introduced by Bermudez and Vazqez [5] and then developed further in the sequel of papers, see [4], [30] for general triangulations in frames of shallow water model. Well balanced schemes have been introduced by Greenberg *et al* [15], and then these schemes have been developed further in the sequel of papers, see Greenberg, LeRoux *et al* [16], Gosse, LeRoux [18], Gosse [17]. The convergence of these schemes is proved in single space dimensions for scalar conservation laws with initial data possessing bounded variation. The method for balancing the source term in frames of

Godunov type schemes has been introduced by LeVeque [24] and applied to Euler equations of ideal gas with source terms [25]. For Saint-Venant system kinetic schemes with equilibrium conservation properties has been introduced by Perthame *et al* [28], [3], see also [31] for other kinetic flux vector splitting type schemes. In Arvantis *et al* [2] some discretization of source terms with relaxation methods and the finite element method is given. In Botchorishvili [6] an imp licit approach for building schemes with equilibrium conservation property was studied.

One of the efficient classes of numerical schemes for scalar conservation laws with stiff sources have been introduced in [7]. These schemes are called *equilibrium schemes* and are claimed to ensure that

equilibrium initial data are maintained;
all the discrete entropy inequalities are valid; (1.7)
approximate solutions are, locally in time, L^∞ bounded.

Notice that in [7] these equilibrium schemes are constructed in single space dimension. In order to preserve the equilibriums and thus to ensure the first property given above equilibrium schemes use the so called discrete equilibrium states, that represent exact solution of a Cauchy problem for suitably selected ordinary differential equations for local steady states. Observe that steady state solution for (1.1) is defined by partial differential equation (1.6). From the computational point of view this equation has almost the same complexity as scalar conservation law (1.1). Furthermore, direct generalization of the equilibrium schemes in several space dimensions is not evident: it is not clear how to correctly formulate corresponding boundary value problem for (1.6) on some part of a cell of a mesh when the values in nodal points are given; even, if this would be possible then, in general, there is no way to solve this equation exactly or efficiently in order to obtain the so called discrete equilibrium states. But this is possible in certain cases. That is why we replace the first requirement in (1.7) as follows

equilibrium initial data are maintained if its variation
is zero in the direction parallel cell interfaces. (1.8)

Numerical schemes possessing the above properties we call equilibrium type schemes, since they coincide with equilibrium schemes in single space dimension. We will see later that condition (1.8) reduces the problem of finding the local steady states to solution of initial value problem for some ordinary differential equations that are consequences of (1.6) in suitably selected directions. Notice that such locally one dimensional approach is in line with finite volume methodology. The present paper is devoted to the building and investigation of one class of equilibrium type schemes on unstructured meshes. Other class of equilibrium type schemes on unstructured meshes was introduced and studied in [8].

The rest of the paper is organized as follows: in section 2 we recall kinetic formulation by Lions, Perthame, Tadmor [26] and the main convergence theorem from [7]. In section 3 we introduce requirements on mesh refinement process that is called regular mesh requirement. On the example of homogenous equation we show that the requirement on mesh regularity is essential for the proof of convergence because of the low regularity of approximate solutions (only uniform L^∞ bound is available) and because of the low accuracy of the finite volume discretization. In section 4 we present equilibrium type schemes on unstructured meshes. The convergence of these schemes is proved in frames of kinetic schemes under regularity requirement on mesh refinement process. In section 5 numerical tests are considered. We have gathered numerical evidence that the convergence rate of equilibrium type schemes increases significantly together with mesh refinement though formally they are only first order accurate. Numerical results demonstrate high accuracy of the developed equilibrium type scheme and its significant advantage over the scheme with standard cell centered discretization of source term.

2. *Kinetic formulation, Convergence theorem*

The equation (1.1) and the family of entropy inequalities (1.3) can be written equivalently as a single kinetic equation with a "density" function $\chi\Big(\xi; u(t,x)\Big)$,

$$\chi(\xi;u) = \begin{cases} +1, & 0 < \xi \le u, \\ -1, & u \le \xi < 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

This approach is the so called kinetic formulation of the problem (Lions, Perthame, Tadmor [26]). It simplifies analysis of the problem, e.g. it allows a very simple uniqueness proof (Perthame [27]). On the basis of kinetic formulation in [7] the notion of kinetic solutions has been introduced that for (1.1) writes:

Definition 2.1. Let the function $f(t,x,\xi)$ belong to $L^\infty\Big(0,T; L^\infty(\mathbb{R}^{N+1}_{x,\xi}) \cap L^1(\mathbb{R}^{N+1}_{x,\xi})\Big)$ for all $T \ge 0$. It is called a generalized kinetic solution to equation (1.1), if

$$\frac{\partial f(t,x,\xi)}{\partial t} + \sum_{i=1}^{N} a_i(\xi)\cdot\frac{\partial f(t,x,\xi)}{\partial x_i} - b(\xi)\sum_{i=1}^{N} z'_{x_i}(x)\frac{\partial f(t,x,\xi)}{\partial \xi} = \frac{\partial m(t,x,\xi)}{\partial \xi}, \tag{2.2}$$

in the sense of distribution for some nonnegative measure $m(t,x,\xi)$ bounded on $[0,T]\times \mathrm{I\!R}^{\mathrm{N}}_{\mathrm{x}}\times \mathrm{I\!R}_{\xi}$ for all $T>0$ which satisfies

$$0 \le \operatorname{sign}(\xi)\ f(t,x,\xi) = |f(t,x,\xi)| \le 1, \tag{2.3}$$

$$\frac{\partial f}{\partial \xi} = \delta(\xi) - \nu(t,x,\xi), \tag{2.4}$$

with $\nu(t,x,\xi)$ some nonnegative measure such that $\int_{\mathrm{I\!R}} \nu(t,x,\xi)\,d\xi = 1$ for all $t,\ x$.

It has been proved in [7] that entropy solutions by Kruzkov [20] and Lax [22] and measure valued solutions by DiPerna [12] can be interpreted in frames of generalized kinetic solutions. Uniqueness theorem of generalized kinetic solutions has been proved and adapted for investigation of numerical schemes. Since here we consider multidimensional equation we recall these theorems and formulate them in several space dimensions.

Theorem 2.2. (Uniqueness of generalized kinetic solutions [7]). Let $f(t,x,\xi)$ be a *generalized kinetic solution* to (1.1), (1.2), such that for a.e. $t>0$,

$$\begin{aligned} &\int_{R^N} f(t,x,\xi)\varphi(x)S'(\xi)\,dx\,d\xi \\ &+\sum_{i=1}^{N}\int_0^t\int_{R^N} a_i(\xi)\varphi'(x)S'(\xi)f(\tau,x,\xi)\,dx\,d\xi\,d\tau \\ &-\int_0^t\int_{R^N}(b(\xi)S'(\xi))_{\xi}\varphi'(x)f(\tau,x,\xi)\,dx\,d\xi\,d\tau \\ &\le \int_{R^N}\chi(\xi;u^0(x))\varphi(x)S'(\xi)\,dx\,d\xi. \end{aligned} \tag{2.5}$$

$$\int_{R^N} u(t,x)\varphi(x)\,dx \longrightarrow \int_{R^N} u^0(x)\varphi(x)\,dx \quad \text{as} \ \ t\to 0, \tag{2.6}$$

for any convex entropy functions $S(\xi)$ and all nonnegative test functions $\varphi\in D(R^N)$. Then $f(t,x,\xi)=\chi(\xi;u)$ where $u(t,x)$ is the entropy solution of (1.1), (1.2) and $f(t,x,\xi)\longrightarrow \chi(\xi;u_0)$ in $L^1(\mathrm{I\!R}^{\mathrm{N+1}})$ as $t\to 0$.

Theorem 2.1. (Main Convergence Theorem [7]). Let the family of approximate solutions $u_{\Delta x}(t,x)\in L^\infty(0,T;L^1(\mathrm{I\!R}^{\mathrm{N}}))$ satisfy, for some constant K_m,K_1, K_∞, some distribution $\Psi_{\Delta x}(t,x,\xi)$, some measure $m_{\Delta x}(t,x,\xi)$ and some function $\Psi_{0,\Delta x}(t)$,

$$\frac{\partial \chi(\xi;u_{\Delta x})}{\partial t} + \sum_{i=1}^{N} a_i(\xi)\frac{\partial \chi(\xi;u_{\Delta x})}{\partial x} -$$

$$\sum_{i=1}^{N}\xi z'_{x_i}(x)\frac{\partial \chi(\xi;u_{\Delta x})}{\partial \xi}\frac{\partial m_{\Delta x}(t,x,\xi)}{\partial \xi} + \Psi_{\Delta x}, \tag{2.7}$$

$$\Psi_{\Delta x}(t,x,\xi) \to 0 \text{ in } D' \text{ as } \Delta x \to 0; \tag{2.8}$$

$$m_{\Delta x}(t,x,\xi) \geq 0, \quad \|m_{\Delta x}(t,x,\xi)\|_{M^1} \leq K_m, \tag{2.9}$$

$$\|u_{\Delta x}\|_{L^1} \leq K_1, \quad \|u_{\Delta x}\|_{L^\infty} \leq K_\infty, \tag{2.10}$$

$$\begin{aligned} &\int_{R^{N+1}} \chi(\xi; u_{\Delta x})\varphi(x)S'(\xi)\,dx\,d\xi \\ &+\sum_{i=1}^{N}\int_0^t \int_{R^{N+1}} a_i(\xi)\varphi'_{x_i}(x)S'(\xi)\chi(\xi;u_{\Delta x})\,dx\,d\xi\,d\tau \\ &\leq \int_{R^{N+1}} \chi(\xi; u^0(x))\varphi(x)S'(\xi)\,dx\,d\xi + \Psi_{0\Delta x}(t). \end{aligned} \tag{2.11}$$

$$\int u_{\Delta x}\varphi(x)\,dx\,d\xi = \int u^0(x)\varphi(x)\,dx\,d\xi + \Psi_{1\Delta x}(t), \tag{2.12}$$

for all nonnegative test functions $\varphi \in D(\mathbb{R}^N)$ and smooth convex entropy functions S, where $\Psi_{i\Delta x}(t)$, are bounded functions such that for $i = 0, 1$,

$$\Psi_{i,\Delta x}(t) \to \Psi_i(t) \text{ in } L^\infty - w*, \quad \Psi_i(t) \text{ is continuous and } \Psi_i(0) = 0. \tag{2.13}$$

Then, as $\Delta x \to 0$, $u_{\Delta x}$ converges strongly in $L^p([0,T] \times \mathbb{R}^N)$, $1 \leq p < \infty$, to the unique entropy solution to (1.1), (1.2).

In order to prove the convergence of numerical schemes it is sufficient to verify that the scheme satisfies the assumptions of the Main Convergence Theorem. We apply this approach in section 4 for studying the convergence of equilibrium type schemes.

3. *Regular mesh refinement*

As is well known quality of a mesh has a direct influence on the quality of numerical solution and suitably selected meshes can accelerate the convergence several times. One of the important properties of a mesh is its regularity. Notice that almost all meshing algorithms contain a step called smoothing or regularization. In general mesh regularity means that characteristics of a mesh vary in a regular, smooth manner. Here we introduce the requirements on mesh regularity that is suitable for the proof of convergence of kinetic schemes with L^∞ initial data. In fact this requirement defines acceptable distortion of a mesh during the refinement process in order to control residuals in the scheme due to low accuracy of finite volume discretization of space derivatives.

3.1. Regular refinement in single space dimension

In order to identify the requirements on the mesh first let us consider standard explicit upwind kinetic scheme for homogenous scalar conservation law in single space dimension on a nonuniform mesh. At kinetic level the scheme writes:

$$\frac{f_j^{n+1}(\xi) - \chi_j^n(\xi)}{\Delta t} \tag{3.1}$$
$$+\frac{a_-(\xi)\chi_{j+1}^n(\xi) + a_+(\xi)\chi_j^n(\xi) - a_-(\xi)\chi_j^n(\xi) - a_+(\xi)\chi_{j-1}^n(\xi)}{\Delta x_j} = 0,$$

where $\Delta x_j = \dfrac{1}{2}(x_{j+1} - x_{j-1})$,

$$a_+(\xi) = max(0, a(\xi)), \quad a_-(\xi) = min(0, a(\xi)).$$

Notice that from (3.1.) one easily arrives at uniform L^∞ estimate and in cell entropy inequality under usual CFL condition:

$$\frac{\Delta t}{\Delta x_j}\max(-\mathrm{a}_-(\xi), \mathrm{a}_+(\xi)) \leq 1. \tag{3.2}$$

We multiply (3.1.) on $S'(\xi)\Delta t\Delta x\varphi(t_n, x_j)$, integrate it in ξ and sum in n and j. After some standard manipulations, see e.g. [7], we have:

$$\int_{\mathbb{R}_t^+}\int_{\mathbb{R}_x}\int_{\mathbb{R}_\xi} S'(\xi)\Big(\chi_{u_{\Delta x}}\varphi_t + a(\xi)\chi_{u_{\Delta x}}\varphi_x\Big)dxdtd\xi \geq \psi_{\Delta x} + \Psi_{\Delta x}(\mathcal{T}, \varphi), \tag{3.3}$$

where

$$u_{\Delta x}(t, x) = \int_{\mathbb{R}_\xi} \chi_j^n(\xi)d\xi \text{ when } (t, x) \in (t_n, t_{n+1}) \times (x_{j-1/2}, x_{j+1/2}),$$

$$|\psi_{\Delta x}| \longrightarrow 0 \quad \text{as} \quad \Delta x \to 0,$$

$$\Psi_{\Delta x}(\mathcal{T}, \varphi) = \sum_{n\geq 0}\sum_j \int_{\mathbb{R}_\xi} S'(\xi)|a(\xi)|\Big(\chi_{j-1}^n - 2\chi_j^n + \chi_{j+1}^n\Big)\varphi_j^n d\xi$$
$$= \sum_{n\geq 0}\sum_j \int_{\mathbb{R}_\xi} S'(\xi)|a(\xi)|\Big(\chi_{j+1}^n - \chi_j^n\Big)(\varphi_j^n - \varphi_{j+1}^n)d\xi \tag{3.4}$$
$$= \sum_{n\geq 0}\sum_j \int_{\mathbb{R}_\xi} S'(\xi)|a(\xi)|\chi_j^n(\varphi_{j-1}^n - 2\varphi_j^n - \varphi_{j+1}^n)d\xi. \tag{3.5}$$

If variation in space of $\chi_{\Delta x}$ is uniformly bounded then from (3.4) one can easily obtain that $\Psi_{\Delta x}$ vanishes together with Δx for arbitrary meshes. If no BV or no similar estimates are available and only uniform L^∞-estimate is known then additional requirement on mesh regularity is needed in order to prove the convergence of the scheme. With account of smoothness of the function φ the residual $\Psi_{\Delta x}(\mathcal{T},\varphi)$ defined by (3.5) writes:

$$\Psi_{\Delta x}(\mathcal{T},\varphi)$$

$$= \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)|a(\xi)|\chi_j^n(x_{j-1}-2x_j+x_{j+1})\varphi_x(t_n,x_j)d\xi+\tilde{\psi}_{\Delta x}$$

$$= \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)|a(\xi)|\chi_j^n\mathcal{R}_{\Delta x}(\mathcal{T},x_j)\varphi_x(t_n,x_j)\Delta x_j d\xi+\tilde{\psi}_{\Delta x}$$

$$= \int_{\mathbb{R}_t}\int_{\mathbb{R}_x}\int_{\mathbb{R}_\xi} S'(\xi)|a(\xi)|\chi_{\Delta x}\mathcal{R}_{\Delta x}(\mathcal{T},x)\varphi_x(t,x)dtdxd\xi+\tilde{\Psi}_{\Delta x}, \tag{3.6}$$

where $\tilde{\Psi}_{\Delta x}\to 0$ as $\Delta x\to 0$,

$$\mathcal{R}_{\Delta x}(\mathcal{T},x)=\frac{x_{j-1}-2x_j+x_{j+1}}{\Delta x_j},\quad \text{when}\quad x\in(x_{j-1/2},x_{j+1/2}). \tag{3.7}$$

Definition 3.1. Mesh $\mathcal{T}$ is γ-regular, if

$$|\mathcal{R}_{\Delta x}(\mathcal{T},x)|\leq K_R|\Delta x|^\gamma,\quad \gamma>0, \tag{3.8}$$

where K_R is some positive constant independent of Δx.

Remark 3.1. If mesh $\mathcal{T}$ is uniform then the indicator of its regularity γ is equal to $+\infty$, of course, if $\Delta x<1$. Thus one can really measure mesh distortion in terms of γ: as small as γ is as distorted is corresponding mesh.

Remark 3.2. For a given mesh $\mathcal{T}$ one can always determine suitable constant K_R in such a way that (3.8) will be satisfied. Thus one can consider (3.8) as a requirement towards mesh refinement process: it does not accept such mesh refinement that will destroy regularity of initial mesh.

Remark 3.3. Another interpretation of the definition given above: mesh refinement should ensure that the second order finite differences of a linear function should vanish faster then first order finite differences of the same linear function do.

3.2. γ regularity in several space dimensions

Let x_j are the nodes of a mesh $x_j\in\mathbb{R}^{\mathrm{N}}$, $\mathrm{j}=0,1,..$; C_j are cells associated with node x_j of a mesh, Γ_{jk} is the interface between cells C_j and C_k, $\Gamma_{jk}=C_j\cap C_k$, $\Gamma_{jk}=\cup_l\Gamma_{jk}^l$, $\vec{n}_{jk}^l$ is normal of Γ_{jk}^l directed to C_k. We suppose that for the mesh minimum angle condition is satisfied. This means that each nodal point x_j is surrounded by maximum K neighbor nodes x_k for which $C_j \cap C_k \neq \emptyset$ holds true. We set

$$\Delta x_{min} = min_j min_{(k,\Gamma_{jk}\neq\emptyset)}\ |C_j|/\ |\Gamma_{jk}|,\ \Delta x = max_j max_{(k,\Gamma_{jk}\neq\emptyset)}\ |C_j|/\ |\Gamma_{jk}|,$$

and suppose that there exists such δ that $\frac{\Delta x}{\Delta x_{min}} \leq \delta$. Notice that this condition ensures $\frac{|C_i|}{|C_j|} \leq \delta$.

Denote by x^l_{jk} the center of each cell interface Γ^l_{jk} ensuring that simple numerical integration formula with one node x^l_{jk} is exact for linear functions, i.e. we have

$$|\Gamma^l_{jk}| < \overrightarrow{y}, x_j - x^l_{jk} >= \int_{\Gamma^l_{jk}} < \overrightarrow{y}, x_j - x^l_{jk} > d\Gamma = \int_{\Gamma^l_{jk}} < \overrightarrow{y}, x_j - x > d\Gamma \tag{3.9}$$

for any constant vector $\overrightarrow{y}$. In (3.9) and throughout below $< \cdot, \cdot >$ is a scalar product in $\mathbb{R}^N$. According to the divergence theorem and with account of (3.9) we have

$$\sum_k \sum_l |\Gamma^l_{jk}| < \overrightarrow{a}, \overrightarrow{n}^l_{jk} >< \overrightarrow{y}, x_j - x^l_{jk} >$$

$$= \sum_k \sum_l \int_{\Gamma^l_{jk}} < \overrightarrow{a}, \overrightarrow{n}^l_{jk} >< \overrightarrow{y}, x_j - x^l_{jk} > d\Gamma$$

$$= \sum_k \sum_l \int_{\Gamma^l_{jk}} < \overrightarrow{a}, \overrightarrow{n}^l_{jk} >< \overrightarrow{y}, x_j - x > d\Gamma$$

$$= \sum_k \sum_l \int_{\Gamma^l_{jk}} < \overrightarrow{a} < \overrightarrow{y}, x_j - x >, \overrightarrow{n}^l_{jk} > d\Gamma = \int_{C_j} < \overrightarrow{a}, \overrightarrow{y} > dx. \tag{3.10}$$

Observe that $\Gamma^l_{jk} = \Gamma^l_{kj}$ and for any vector $\overrightarrow{a} \in \mathbb{R}^N$ we have:

$$< \overrightarrow{a}, \overrightarrow{n}^l_{jk} >= - < \overrightarrow{a}, \overrightarrow{n}^l_{kj} > \tag{3.11}$$

In several space dimensions we can associate the definition of the regularity of a mesh with numerical scheme under consideration. This is natural since different meshes provide different accuracy for the same numerical method, e.g. one can find suitable examples in frames of adaptive triangular meshes, or in frames of finite volume meshes with variety of the cells [8]. Notice that some upwind schemes admit kinetic interpretation, i.e. they can be reformulated as kinetic schemes, e.g. the scheme with Engquist-Osher numerical flux function for homogenous scalar conservation laws in several space dimensions writes:

$$\frac{f_j^{n+1}(\xi)-\chi_j^n(\xi)}{\Delta t}+\frac{1}{|C_j|}\sum_k\sum_l|\Gamma_{jk}^l|\Big(\alpha_{jkl}^+(\xi)\chi_j^n(\xi)+\alpha_{jkl}^-(\xi)\chi_{jk}^n(\xi)\Big)=0, \tag{3.12}$$

where

$$\alpha_{jkl}^+(\xi)=\max(0,\alpha_{jkl}(\xi)),\quad \alpha_{jkl}^-(\xi)=\ \min(0,\alpha_{jkl}(\xi)),$$

$$\alpha_{jkl}^+(\xi)+\alpha_{jkl}^-(\xi)=\alpha_{jkl}(\xi)=\sum_{i=1}^N a_i(\xi)\cdot\overrightarrow{n}_{jk,i}^l,$$

$\overrightarrow{n}_{jk,i}^l$ represents the $i-$th component of the vector $\overrightarrow{n}_{jk}^l$. By analogy with the scheme in single space dimension multiplying of (3.12) on $S'(\xi)\Delta t|C_j|\varphi(t_n,x_j)$, then integrating it in ξ and summing in n,j, after some manipulations yields:

$$\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi}S'(\xi)\varphi_j^n\Delta t|C_j|\frac{f_j^{n+1}(\xi)-\chi_j^n(\xi)}{\Delta t}+\Psi_{1\Delta x}+\Psi_{2\Delta x}\leq 0, \tag{3.13}$$

where

$$\Psi_{1\Delta x}=\sum_{n\geq 0}\sum_j|C_j|\int_{\mathbb{R}_\xi}S'(\xi)\varphi_j^n\frac{1}{|C_j|}\sum_k\sum_l\frac{|\Gamma_{jk}^l|<\overrightarrow{a}(\xi),\overrightarrow{n}_{jk}^l>}{2}(\chi_j^n(\xi)+$$
$$\chi_k^n(\xi))d\xi, \tag{3.14}$$

$$\Psi_{2\Delta x}=\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi}S'(\xi)\varphi_j^n\Delta t\sum_k\sum_l|\Gamma_{jk}^l|\frac{|<\overrightarrow{a}(\xi),\overrightarrow{n}_{jk}^l>|}{2}(\chi_j^n(\xi)-$$
$$\chi_k^n(\xi)))d\xi. \tag{3.15}$$

We set $x_{jk}=\frac{1}{2}(x_j+x_k)$. Observe that $x_j-x_{jk}=\frac{1}{2}(x_j-x_k)$. Then with account of (3.12),(3.11) and the smoothness of φ (3.14) writes:

$$\Psi_{1\Delta x}=\frac{1}{2}\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi}S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l|\Gamma_{jk}^l|<\overrightarrow{a}(\xi),\overrightarrow{n}_{jk}^l>\varphi_j^n d\xi$$

$$+\frac{1}{2}\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi}S'(\xi)\Delta t\chi_k^n(\xi)\sum_k\sum_l|\Gamma_{jk}^l|<\overrightarrow{a}(\xi),\overrightarrow{n}_{jk}^l>\varphi_j^n d\xi$$

$$=\frac{1}{2}\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi}S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l|\Gamma_{jk}^l|<\overrightarrow{a}(\xi),\overrightarrow{n}_{jk}^l>\varphi_j^n d\xi$$

$$-\frac{1}{2}\sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_k^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{kj}^l > \varphi_j^n d\xi$$

$$\frac{1}{2}\sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l > \varphi_j^n d\xi$$

$$-\frac{1}{2}\sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l > \varphi_k^n d\xi$$

$$= \tfrac{1}{2}\textstyle\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >$$
$$\times < grad\varphi_j^n, x_j - x_k > d\xi + \tilde{\Psi}_{1\Delta x}$$

$$= \textstyle\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >$$
$$\times < grad\varphi_j^n, x_j - x_{jk} > d\xi + \tilde{\Psi}_{1\Delta x}$$

$$= \textstyle\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >$$
$$\times < grad\varphi_j^n, x_j - \tilde{x}_{jk}^l > d\xi + \tilde{\Psi}_{1\Delta x}$$

$$+ \textstyle\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >$$
$$\times < grad\varphi_j^n, \tilde{x}_{jk}^l - x_{jk} > d\xi$$

$$= \textstyle\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l \int_{\Gamma_{jk}^l} < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >$$
$$\times < grad\varphi_j^n, x_j - \tilde{x}_{jk}^l > d\Gamma d\xi$$

$$+\tilde{\Psi}_{1\Delta x} + \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\psi_{j\Delta x}(\mathcal{T},\xi)d\xi$$

$$= \textstyle\sum_{n\geq 0}\sum_j\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\sum_k\sum_l \int_{\Gamma_{jk}^l} < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >$$
$$\times < grad\varphi_j^n, x_j - x > d\Gamma d\xi$$

$$+\tilde{\Psi}_{1\Delta x} + \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\psi_{j\Delta x}(\mathcal{T},\xi)d\xi$$

$$= \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\int_{C_j} < \overrightarrow{a}(\xi), grad\varphi_j^n > dxd\xi$$

$$+\tilde{\Psi}_{1\Delta x} + \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t\chi_j^n(\xi)\psi_{j\Delta x}(\mathcal{T},\xi)d\xi,$$

where $\tilde{\Psi}_{1\Delta x} = O(\Delta x \cdot |supp(\varphi)|)$,

$$\begin{aligned}\psi_{j\Delta x}(\mathcal{T},\xi) &= \textstyle\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >< grad\varphi_j^n, \tilde{x}_{jk}^l - x_{jk} > \\ &=< grad\varphi_j^n, \textstyle\sum_k\sum_l |\Gamma_{jk}^l| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l > \tilde{x}_{jk}^l - x_{jk} > .\end{aligned} \tag{3.16}$$

With account of similar manipulations (3.15) writes

$$\Psi_{2\Delta x} = \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t \chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l|| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l > |$$

$$\frac{1}{2}(\varphi_j^n - \varphi_k^n)d\xi = \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t \chi_j^n(\xi)\sum_k\sum_l |\Gamma_{jk}^l|\frac{| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l| > |}{2}$$

$$\times < \mathrm{grad}_x\varphi_j^n, x_j - x_k > d\xi + \tilde{\Psi}_{2\Delta x}, \tag{3.17}$$

where $\tilde{\Psi}_{2\Delta x} = O(\Delta x \cdot |supp(\varphi)|)$.

Thus finally we can rewrite (3.13) in the following form:

$$-\sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t \mathrm{area}(C_j)\Big(\frac{\varphi_j^n - \varphi_j^{n-1}(\xi)}{\Delta t} + < \overrightarrow{a}(\xi), \mathrm{grad}_x\varphi_j^n >\Big)$$

$$\times\chi_j^n(\xi)d\xi + \tilde{\Psi}_{\Delta x} + \Psi_{\Delta x} \leq 0 \tag{3.18}$$

where $\tilde{\Psi}_{\Delta x} = O(\Delta x \cdot |supp(\varphi)|)$ and

$$\Psi_{\Delta x} = \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t \chi_j^n(\xi)\Big(\sum_k\sum_l |\Gamma_{jk}^l|$$

$$\times < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l >< grad\varphi_j^n, \tilde{x}_{jk}^l - x_{jk} >$$

$$+\sum_k\sum_l |\Gamma_{jk}^l|\frac{| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l| > |}{2} < \mathrm{grad}_x\varphi_j^n, x_j - x_k >\Big)d\xi$$

$$= \sum_{n\geq 0}\sum_{j}\int_{\mathbb{R}_\xi} S'(\xi)\Delta t \chi_j^n(\xi) < \mathrm{grad}_x\varphi_j^n, \mathcal{R}_{j\Delta x}(\mathcal{T}, \xi) > d\xi, \tag{3.19}$$

$$\mathcal{R}_{j\Delta x}(\mathcal{T}, \xi) = \sum_k\sum_l |\Gamma_{jk}^l|\Big(< \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l > (\tilde{x} + \frac{| < \overrightarrow{a}(\xi), \overrightarrow{n}_{jk}^l > |}{2}(x_j - x_k)\Big). \tag{3.20}$$

Observe that $\mathcal{R}_{j\Delta x}(\mathcal{T}, \xi)$ defined according to (3.20) is N dimensional vector.

Definition 3.2. Mesh $\mathcal{T}$ is γ-regular, if

$$\|\frac{1}{|C_j|}\mathcal{R}_{j\Delta x}(\mathcal{T}, \xi)\| \leq K_R \Delta x^\gamma, \tag{3.21}$$

where K_R is some positive constant independent of Δx.

3.3. Examples and interpretation of γ-regularity

We need the regularity requirement on mesh refinement process in order to ensure that $\Psi_{\Delta x}$ defined by (3.19),(3.20) vanishes together with Δx. In this subsection we show that such kind meshes exist and they comprise quite wide range of different meshes. In two space dimensions uniform rectangular meshes with quadrangular cells and uniform triangular meshes with standard hexagonal cell definition are the simplest examples of meshes admitting regular refinement in the sense of definition 2, see fig.1. It is easy to verify that $\mathcal{R}_{\Delta x}(\mathcal{T}, \xi)$ defined according to (3.20) is equal to zero for these meshes. For cartesian meshes smooth deformation of order $\Delta x^{1+\gamma},\ \gamma > 0$, in each coordinate direction ensures the vanishing of $\Psi_{\Delta x}$, thus providing examples of nonuniform meshes with γ regularity property. Observe that cells corresponding to cartesian meshes are defined by parallelepiped. In (3.19) due to symmetry property of the cell we can couple the terms corresponding to opposite faces. These opposite faces have the same surface area and possess the normal vectors of opposite signs. Thus the first term in (3.20) is zero for cartesian meshes and the second one is exactly the same in each coordinate direction as corresponding one in single space dimension. E.g. to have γ-regularity of nonuniform cartesian meshes we can set

$$x^i_{j+1} - x^i_j = x^i_j - x^i_{j-1} + \beta_{j-1/2}|x^i_j - x^i_{j-1}|^{1+\gamma},\ 1 \le i \le N,\ |\beta_{j-1/2}| \le K.$$

Observe that $\mathcal{R}_{\Delta x}(\mathcal{T}, \xi)$ is a Lipshietz continuous function with respect to nodal points. Thus the suitable displacement of the nodes of any uniform mesh at the distance $\kappa\Delta x^{1+\gamma}$ ensures that resulting nonuniform mesh possesses γ regularity property with some constant $K_{\kappa\gamma}$, see fig.2 for examples of such nonuniform meshes. Furthermore, in order to control the residuals $\Psi_{\Delta x}$ we can respect no mesh regularity requirement on sub domains that have $N-$dimensional Lebesgue measure zero in the limit $\Delta x \to 0$. Observe that in this case $\Psi_{\Delta x}$ defined by (3.19),(3.20) vanishes together with Δx. Thus we are allowed to perform local mesh refinement in frames of γ regularity, see example on fig.3.

Figure 1: Uniform rectangular and triangular meshes

Lemma 3.1.(sufficient condition of γ regularity) If

$$|a(\xi) - \frac{1}{2|C_j|} \sum_k \sum_l |\Gamma^l_{jk}| < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > x_k| \leq K_1 \Delta x^{\gamma}, \tag{3.22}$$

Figure 2: Nonuniform smooth rectangular and triangular meshes

Figure 3: Composite mesh with local refinement and derefinement

$$|x_j - \frac{\sum_k \sum_l |\Gamma^l_{jk}| \cdot | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > |x_k}{\sum_k \sum_l |\Gamma^l_{jk}| \cdot | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > |}| \leq K_2 \Delta x^{1+\gamma}, \tag{3.23}$$

then mesh $\mathcal{T}$ is γ regular with $K_R = K_1 + K_3 \cdot K_2 \cdot \delta$,

$$K_3 = \frac{1}{2} max_{|\xi| \leq |u_0|} \Big(\sum_k \sum_l | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > | \Big).$$

Proof: In fact (3.22),(3.23) correspond to the first and second terms in the right hand side of (3.20). Indeed for the first term in right hand side of (3.20) we have:

$$\begin{aligned}
&\tfrac{1}{2|C_j|} \textstyle\sum_k \sum_l |\Gamma^l_{jk}| < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > (\tilde{x}^l_{jk} - x_{jk}) \\
&= \tfrac{1}{2|C_j|} \Big(\textstyle\sum_k \sum_l \int_{\Gamma^l_{jk}} < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > x \; d\Gamma - \tfrac{1}{2}{}^l_{jk} > (x_j + x_k) \Big) \\
&= \tfrac{1}{2|C_j|} \Big(\textstyle\int_{C_j} \overrightarrow{a}(\xi) \; dx - \tfrac{1}{2} \sum_k \sum_l |\Gamma^l_{jk}| < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > x_k \Big) \\
&= \tfrac{1}{2} \Big(\overrightarrow{a}(\xi) - \tfrac{1}{2|C_j|} \textstyle\sum_k \sum_l |\Gamma^l_{jk}| < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > x_k \Big).
\end{aligned} \tag{3.24}$$

The second term in the right hand side of (3.20) writes:

$$\begin{array}{l} \sum_k \sum_l |\Gamma^l_{jk}| \frac{|<\overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk}>|}{2}(x_j - x_k) \\ = \frac{1}{2} x_j \sum_k \sum_l |\Gamma^l_{jk}| \cdot | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > | \\ - \frac{1}{2} \sum_k \sum_l |\Gamma^l_{jk}| \cdot | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > | x_k \\ = \frac{1}{2} \sum_k \sum_l |\Gamma^l_{jk}| \cdot | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > | \\ \times \Big(x_j - \frac{\sum_k \sum_l |\Gamma^l_{jk}| \cdot |<\overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk}>| x_k}{\sum_k \sum_l |\Gamma^l_{jk}| \cdot |<\overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk}>|} \Big). \end{array} \tag{3.25}$$

Observe that

$$\tag{3.26}$$

$$\frac{|\Gamma^l_{jk}| \cdot \Delta x}{|C_j|} \quad \leq \quad \delta. \tag{3.27}$$

Thus we can replace the terms in the right hand side of (3.20) by equivalent expressions (3.24),(3.25). Application of triangles inequality in (3.21) with account of (3.26) accomplishes the proof.

Remark 3.4. (3.24) means that for the integration at cell interfaces simple one point quadrature formula with the node defined as midpoint between x_j and x_k is good enough to ensure suitable order of accuracy of the approximation of the integral from constant function on the cell. Thus this is very natural requirement.

Remark 3.5. (3.25) means that we can perform the displacement of the cell center within the circle of radius $K_2 \Delta x^{1+\gamma}$. Observe that the center of the circle in which we can move x_j, i.e. the circle defined by (3.25), is a linear combination of nodes surrounding x_j. This radius can be even large in comparison with the cell size in highly refined regions, see e.g. fig.4. In practise this means that we should have uniform and good meshes where mesh size is large and for sufficiently refined domains it can be almost arbitrary.

Remark 3.6. Some examples on how regularity of finite volume cells affect accuracy of computations are given in [8].

Figure 4: Cells of a mesh with the same acceptable radius of cell center displacement

3.4. Convergence of standard finite volume scheme for homogeneous scalar conservation laws

Theorem 3.2. Assume that $u_0 \in L^\infty(\mathbb{R}^{\mathrm{N}})$ and CFL condition

$$\frac{\Delta t}{\Delta x_{min}} max_j \; max_{|\xi| \le \|u_0\|_{L^\infty}} \sum_k \sum_l | < \overrightarrow{a}(\xi), \overrightarrow{n}^l_{jk} > | \le 1 \tag{3.28}$$

is satisfied. Then approximate solution $u_{\Delta x}(t,x)$,

$$u_{\Delta x}(t,x) = u_j^n \text{ for } t \in [t_n, t_{n+1}) \text{ and } x \in C_j,$$
$$u_j^n = \int_{\mathbb{R}_\xi} f_j^n(\xi) d\xi, \; f_j^n(\xi) \text{ are defined by the scheme (3.12),(4.2),}$$

converges in $L^p_{loc}([0,T] \times \mathbb{R}^{\mathrm{N}})$ for all p, $1 \le p < \infty$, and all $T > 0$, towards the unique entropy solution to (1.1) with $z = 0$, as $\Delta x \to 0$ under γ-regularity requirement on mesh refinement process.
Proof: Let assume that $u_0(x)$ is compactly supported. Then with account of finite speed of propagation of perturbations in scalar conservation laws, it is sufficient to obtain uniform L^∞ estimate in order to have uniform boundedness in L^1. Notice that after the proof of the convergence on bounded domain in L^p we can relax this requirement by using standard diagonalization process that will result in convergence in $L^p_{loc}([0,T] \times \mathbb{R}^{\mathrm{N}})$. Following the method of proof formulated in the previous subsections, we verify the requirements of the main convergence theorem departing from the kinetic formulation of the scheme. Proof is decomposed into four steps.

(i) *Derivation of (2.10).*
The scheme (3.12) equivalently writes:

$$\begin{aligned} f_j^{n+1}(\xi) &= \chi_j^n(\xi) - \frac{\Delta t}{|C_j|} \sum_k \sum_l |\Gamma^l_{jk}| \Big(\alpha^+_{jkl}(\xi) \chi^n_j(\xi) + \alpha^-_{jkl}(\xi) \chi^n_{jk}(\xi) \Big) \\ &= \Big(1 - \frac{\Delta t}{|C_j|} \sum_k \sum_l |\Gamma^l_{jk}| \alpha^+_{jkl}(\xi) \Big) \chi^n_j(\xi) - \frac{\Delta t}{|C_j|} \sum_k \sum_l |\Gamma^l_{jk}| \alpha^-_{jkl}(\xi) \chi^n_{jk}(\xi). \end{aligned} \tag{3.29}$$

With account of the CFL-condition it follows from (3.29) that $f_j^{n+1}(\xi)$ is a linear and convex combination of $\chi^n_{jk}(\xi)$. Observe that

$$0 \le \chi^n_j(\xi) \cdot sign(\xi) \le 1,$$

see (2.1). Then multiplying (3.29) on $sign(\xi)$ yields

$$0 \le \mathrm{sign}(\xi) f_j^{n+1}(\xi) = |f_j^{n+1}| \le 1, \tag{3.30}$$

$$\begin{aligned} &\mathrm{sign}(\xi) f_j^{n+1}(\xi) \leq \Big(1 - \tfrac{\Delta t}{|C_j|} \textstyle\sum_k \sum_l |\Gamma_{jk}^l| \alpha_{jkl}^+(\xi)\Big) |\chi_j^n(\xi)| \\ &- \tfrac{\Delta t}{|C_j|} \textstyle\sum_k \sum_l |\Gamma_{jk}^l| \alpha_{jkl}^-(\xi) |\chi_{jk}^n(\xi)|. \end{aligned} \tag{3.31}$$

Integration of (3.31) in ξ yields: $|u_j^{n+1}| \leq max_k |u_k^n|$ that results in uniform L^∞ estimate (2.10).

(ii) *Derivation of (2.7),(2.8).*
We set: $\chi_{\Delta x} := \chi(\xi; u_{\Delta x})$, $\varphi_j^n(\xi) = \varphi(t_n, x_j, \xi)$, with $\varphi(t, x, \xi)$ test function. Then these estimates are direct consequences of the supposition of the theorem on γ-regularity of mesh refinement, see (3.13)-(3.20).

(iii) *Derivation of (2.9).*
As a consequence of (3.30) and Brenier's lemma [10] we have

$$\frac{\chi_j^{n+1}(\xi) - f_j^{n+1}(\xi)}{\Delta t} = \frac{\partial m_j^{n+1}(\xi)}{\partial \xi}, \quad m_j^{n+1}(\xi) \geq 0. \tag{3.32}$$

Uniform boundedness of the support of the measure $m_j^{n+1}(\xi)$ can be obtained from uniform L^∞ boundedness of approximate solutions. Observe that the scheme written in terms of $m_j^{n+1}(\xi)$ and $\chi_j^{n+1}(\xi)$ is conservative in the left hand side. We multiply (3.32) on $\Delta t \cdot C_j$ and then we sum it in j and n that results in (2.9) with $K_m = 2 \cdot |u_0|_{L^1}$.

(iv) *Derivation of (2.11)- (2.13).*
The technique of derivation of these estimates is almost the same as in single space dimension [7]. We multiply (3.12) on $\varphi_j |C_j|$, $\varphi_j := \varphi(x_j, \xi)$, $\varphi \in D(\mathrm{I\!R}^{\mathrm{N}})$. Then we integrate it in ξ over R_ξ and we do sum of it in j, $j = 0, 1, \ldots + \infty$. The obtained expression we sum in n until any $k, k\Delta t \leq T$. With account of nonnegativity of the measure m, uniform bound of approximate solutions, conservativeness property of the scheme and γ-regularity requirement on mesh refinement process we obtain the following expression:

$$\begin{aligned} &\Delta x \textstyle\sum_j |C_j| \int_{R_\xi} \chi^{k+1} \varphi_j S'(\xi) d\xi \\ &= \Delta x \textstyle\sum_j |C_j| \int_{R_\xi} \chi^0 \varphi_j S'(\xi) d\xi + \psi_{0\Delta x}(t_{k+1}, \varphi, S), \end{aligned}$$

where

$$|\Psi_{0\Delta x}| \leq t_{k+1} |\varphi|_{W^{1,1}} \max_{|u| \leq |u_0|_{L^\infty}} |a(u)|.$$

Clearly $\Psi_{0\Delta x}(t_{k+1}, \varphi)$ vanishes together with t_{k+1} for any $\varphi \in D(\mathrm{I\!R}^{\mathrm{N}})$. Evidently, with $S'(\xi) = 1$ and by use of the same technique as above one can easily recover the weak continuity requirements (2.12) of approximate solutions $u_{\Delta x}$ at $t = 0$.

Finally, applying the main convergence theorem results in the strong convergence of the equilibrium scheme that concludes the proof.

4. *Schemes with equilibrium type discretization of source term*

4.1. Monotone finite volume schemes in several space dimensions, Cell centered discretization of source term

Monotone finite volume scheme with cell centered discretization of source term writes:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \sum_k \sum_l \frac{|\Gamma_{jk}^l|}{|C_j|} A(u_j^n, u_k^n, \overrightarrow{n}_{jk}^l)$$

$$+\frac{b(u_j^n)}{|C_j|} \sum_k \sum_l |\Gamma_{jk}^l| < z(x_k), \overrightarrow{n}_{jk}^l >= 0, \tag{4.1}$$

$$u_j^0 = \frac{1}{|C_j|} \int_{C_j} u_0(x) dx, \tag{4.2}$$

where $< \cdot, \cdot >$ is a scalar product in $\mathbb{R}^{\mathrm{N}}$, $A(u_j^n, u_k^n, \overrightarrow{n}_{jk})$ is a monotone numerical flux function [29] satisfying usual requirements on consistency :

$A(u, u, \overrightarrow{n}) =< A(u), \overrightarrow{n} >$,
$A(u, v, \overrightarrow{n})$ is Liepshitz continuous with respect to u, v,

and on monotonicity :

$A(u, v, \overrightarrow{n})$ is nondecreasing in u and nonincreasing in v.

The example of such monotone flux function is Engquist-Osher [13] numerical flux function which writes for unstructured meshes as follows

$$A(u, v, \overrightarrow{n}) = \int_0^u max(0, \sum_{i=1}^N a_i(\xi) n_i)\ d\xi$$

$$+\int_0^v min(0, \sum_{i=1}^N a_i(\xi) n_i)\, d\xi. \tag{4.3}$$

4.2. Equilibrium type schemes

Formally we can extend equilibrium schemes from [7] on arbitrary meshes in multidimension, e.g. corresponding finite volume scheme writes:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{1}{|C_j|}\sum_k\sum_l|\Gamma^l_{jk}|A(u_j^n,u^n_{k,l-},\overrightarrow{n}^l_{jk})=0, \tag{4.4}$$

where $u^n_{k,l-}$ is some steady state solution of (1.1), e.g. defined according to the following equation

$$< D(u^n_{k,l-})+z(x_j),\overrightarrow{n}^l_{jk}>=< D(u^n_k)+z(x_k),\overrightarrow{n}^l_{jk}> . \tag{4.5}$$

Lemma 4.1. (4.4) is equilibrium type scheme under suppositions that (1.3) is satisfied and the following CFL condition holds true:

$$\frac{\Delta t}{\Delta x_{min}}max_j\ max_{|\xi|\le K_\infty}\sum_k\sum_l|<\overrightarrow{a}(\xi),\overrightarrow{n}^l_{jk}>|\le 1, \tag{4.6}$$

$$K_\infty=exp(T\cdot K\cdot l_{max}\cdot K_b\cdot\delta|z|_{W^{1,1}}T)\cdot|u_0|_{L^\infty}.$$

Proof: We have to show that the scheme verifies (1.7) where the first requirement is replaced by (1.8). Thus we prove the lemma in three steps:
(i)*Equilibrium property.*
Observe that (4.5) defines exact solution to (1.6) with z and u having nonzero variation only in direction normal to cell interfaces. Thus if u_j and u_k are in the local equilibriums then $u_{k,l-}=u_j$. Putting the latter into (4.4) we obtain the validity of the equilibrium property as it is defined in the section 1.
(ii)*Uniform L^∞-bound.*

$$\begin{aligned}&|A(u_j^n,u^n_{k,l-},\overrightarrow{n}^l_{jk})-A(u_j^n,u_k^n,\overrightarrow{n}^l_{jk})|\\&=\tfrac{|A(u_j^n,u^n_{k,l-},\overrightarrow{n}^l_{jk})-A(u_j^n,u_k^n,\overrightarrow{n}^l_{jk})|}{<D(u^n_{k,l-})-D(u^n_k),\overrightarrow{n}^l_{jk}>|}\cdot|< D(u^n_{k,l-})-D(u^n_k),\overrightarrow{n}^l_{jk}>|\\&\le K_b\cdot|z|_{W^{1,1}}\cdot|u|_{L^\infty}\Delta x,\end{aligned}$$

$$\begin{aligned}&\|u_j^{n+1}|=|u_j^n+\tfrac{\Delta t}{|C_j|}\textstyle\sum_k\sum_l|\Gamma^l_{jk}|A(u_j^n,u_k^n,\overrightarrow{n}^l_{jk})\\&+\tfrac{\Delta t}{|C_j|}\textstyle\sum_k\sum_l|\Gamma^l_{jk}|A(u_j^n,u^n_{k,l-},\overrightarrow{n}^l_{jk})-A(u_j^n,u_k^n,\overrightarrow{n}^l_{jk})|\\&\le(1+\Delta t\cdot K\cdot l_{max}\cdot K_b\cdot\delta|z|_{W^{1,1}})\cdot|u|_{L^\infty}.\end{aligned}$$

The latter inequality results in uniform L^∞ boundedness of approximate solutions with K_∞ as a bound on $[O,T]$.
(iii)*Entropy inequality*
Notice that monotone numerical flux functions ensure the validity of cell

entropy inequality for homogenous scalar conservation laws, see e.g. [29]. Thus it is easy to see that for (4.4) the following in cell entropy inequality is valid:

$$\frac{S(u_j^{n+1}) - S(u_j^n)}{\Delta t} + \frac{1}{|C_j|}\sum_k \sum_l |\Gamma^l_{jk}| \eta(u_j^n, u_{k,l-}^n, \overrightarrow{n}^l_{jk}) \leq 0. \tag{4.7}$$

Thus the scheme (4.4) is equilibrium type in the sense of definition introduced in the introduction.

Observe that in (4.7) discretization of source term can be explicitized in the usual way by means of adding and substracting finite volume approximation of space derivatives of entropy fluxes that writes

$$\frac{1}{|C_j|}\sum_k \sum_l |\Gamma^l_{jk}| \eta(u_j^n, u_{k,l}^n, \overrightarrow{n}^l_{jk}).$$

As an approximation of $S'(u)b(u)\sum_{i=1}^N z'_{i,x_i}(x)$ this results in the following expression

$$\frac{1}{|C_j|}\sum_k \sum_l |\Gamma^l_{jk}| \Big(\eta(u_j^n, u_{k,l-}^n, \overrightarrow{n}^l_{jk}) - \eta(u_j^n, u_{k,l}^n, \overrightarrow{n}^l_{jk})\Big).$$

The drawback of the equilibrium type scheme (4.4) is that the equations (4.5) can have multiple solutions or do not have solutions at all. In those cases when unique solution of (4.5) exists it is computationally expensive to be solved due to the nonlinearity. The good feature of the scheme in the form (4.4) is that it allows to control L^∞-norm of approximate solutions due to monotonicity of numerical flux function. More practical equilibrium type scheme that is less expensive from computational points of view and that incorporates into the features of the standard and equilibrium type discretizations is the following:

$$\begin{aligned} &\tfrac{u_j^{n+1} - u_j^n}{\Delta t} + \textstyle\sum_k \sum_l \tfrac{|\Gamma^l_{jk}|}{|C_j|} A(u_j^n, u_{k,l-}^n, \overrightarrow{n}^l_{jk}) \\ &+ \tfrac{b_s(u_j^n)}{|C_j|} \textstyle\sum_k \sum_l |\Gamma^l_{jk}| < z(x_k), \overrightarrow{n}^l_{jk} >= 0, \end{aligned} \tag{4.8}$$

where the function b is splitted into two parts, $b(u) = b_s(u) + b_e(u)$, in such a way that the equation

$$< D_e(u_{k,l-}^n) + z(x_j), \overrightarrow{n}^l_{jk} >=< D_e(u_k^n) + z(x_k), \overrightarrow{n}^l_{jk} >,$$

$$D_{ei}(u) = \int_0^u \frac{a_i(s)}{b_e(s)} ds < +\infty, \ 1 \leq i \leq N,$$

has unique solution. In other words, if equilibrium type discretization is possible at cell interfaces, then source term is discretized in accordance with equilibrium approach; standard discretization is applied if no equilibrium type discretization is possible. Practically it is more convenient to perform the splitting of the source term for each cell interface separately. We propose the following scheme:

$$a^n_{jk,i} = \{ \begin{array}{c} \frac{A_i(u^n_j)-A_i(u^n_k)}{u^n_j-u^n_k}, \text{if } u^n_j \neq u^n_k, \\ a_i(u^n_j), \text{ otherwise }, \end{array} \tag{4.9}$$

$$d^n_{jk,i} = \{ \begin{array}{c} \frac{D_i(u^n_j)-D_i(u^n_k)}{u^n_j-u^n_k}, \text{if } u^n_j \neq u^n_k, \\ a_i(u^n_j)/b_i(u^n_j), \text{ otherwise }, \end{array} \tag{4.10}$$

$$\bar{d}^n_{jk,i} = \{ \begin{array}{c} d^n_{jk,i}, \text{ if } |\frac{a^n_{jk,i}}{d^n_{jk,i}}| \leq K_b, \\ 0, \text{ otherwise }, \end{array} \tag{4.11}$$

$$\overrightarrow{d}^n_{jk} = \big(\bar{d}^n_{jk,1}, \bar{d}^n_{jk,2}, \ldots, \bar{d}^n_{jk,N}),$$

$$\beta^n_{jk,l} =< \overrightarrow{d}^n_{jk}, \overrightarrow{n}^l_{jk} >, \tag{4.12}$$

$$\beta^n_{jk,l} \cdot (u^n_{jk,l-} - u^n_k) =< z(x_k) - z(x_j), \overrightarrow{n}^l_{jk} >, \tag{4.13}$$

$$b^n_{jk,l} =\Big(1 - \frac{\sum_{i=1}^N |\bar{d}^n_{jk,i}|}{\sum_{i=1}^N |d^n_{jk,i}|}\Big) \cdot \frac{b(u^n_j) + b(u^n_k)}{2}, \tag{4.14}$$

$$\begin{aligned} &\frac{u^{n+1}_j - u^n_j}{\Delta t} + \sum_k \sum_l \frac{|\Gamma^l_{jk}|}{|C_j|} A(u^n_j, u^n_{k,l-}, \overrightarrow{n}^l_{jk}) \\ &+ \frac{1}{|C_j|} \sum_k \sum_l |\Gamma^l_{jk}| b^n_{jk,l} < z(x_k), \overrightarrow{n}^l_{jk} >= 0. \end{aligned} \tag{4.15}$$

It is easy to see that the schemes (4.4) and (4.9)-(4.15) coincide at equilibrium states of the scheme (4.4). Observe that the proposed splitting of the source term enables to control L^∞ bound of approximate solutions. Another good property of the latter scheme is that the equation for the definition of the discrete equilibrium states is linear.

Remark 4.1. Traditionally simple and useful approach for the treatment of source terms in partial differential equations is provided by operator splitting method. Then fractional step scheme treats the PDE part of the equation by some numerical scheme at one fractional step and the source term is treated at another fractional step, usually implicitly if the source is stiff. For conservation laws with stiff sources this traditional approach can lead to wrong solution, see e.g. [19], and special modifications are needed. With the above splitting of the source term we can develop fractional step scheme as well. Namely, instead of (4.15) we can use the following scheme:

$$\frac{u_j^{n+1/2}-u_j^n}{\Delta t}+\sum_k\sum_l\frac{|\Gamma_{jk}^l|}{|C_j|}A(u_j^n,u_{k,l-}^n,\overrightarrow{n}_{jk}^l)=0 \tag{4.16}$$

$$\frac{u_j^{n+1}-u_j^{n+1/2}}{\Delta t}+\sum_k\sum_l\frac{|\Gamma_{jk}^l|}{|C_j|}\Big(1-\frac{\sum_{i=1}^N|\bar{d}_{jk,i}^n|}{\sum_{i=1}^N|d_{jk,i}^n|}\Big)\cdot\frac{b(u_j^{n+1})+b(u_k^{n+1})}{2}<z(x_k),\overrightarrow{n}_{jk}^l>=0. \tag{4.17}$$

The investigation of the fractional step scheme (4.16),(4.17) is beyond of the scope of the present paper. It is considered in details in [9]. Here we notice only that the scheme (4.16),(4.17) has the same discrete equilibrium state as the scheme (4.15) and, as in modified fractional step variant of [19] it allows fractional steps to work only out of equilibrium states.

4.3. Convergence theorem

In this subsection we prove the convergence of equilibrium type scheme (4.8) under suppositions that

numerical flux function of the scheme admits interpretation at kinetic level, (4.18)

and

$$|b_e'(u)|\le K_e,\quad |b_s'(u)|\le K_s. \tag{4.19}$$

In particular, we have the following theorem:

Theorem 4.2. Assume that $u_0\in L^\infty(\mathbb{R}^N)$, the following CFL condition

$$\frac{\Delta t}{\Delta x_{min}}max_j\ max_{|\xi|\le\|u_0\|_{K_\infty}\sum_k\sum_l|<\overrightarrow{a}(\xi),\overrightarrow{n}_{jk}^l>|}\le 1, \tag{4.20}$$

$$K_\infty=exp(T\cdot K\cdot l_{max}\cdot(K_e+K_s)\cdot\delta|z|_{W^{1,1}})\cdot|u_0|_{L^\infty},$$

and (1.3),(4.18),(4.19) are satisfied. Then approximate solution $u_{\Delta x}(t,x)=u_j^n$ for $t\in[t_n,t_{n+1})$ and $x\in C_j$, u_j^n are defined by the scheme (4.8),(4.2), converges in $L_{loc}^p([0,T]\times\mathbb{R}^N)$, for all $1\le p<\infty$, and all $T>0$, towards the unique entropy solution to (1.1), as $\Delta x\to 0$ under γ-regularity requirement on mesh refinement process.

Proof: Observe that in fact uniform L^∞-bound of approximate solutions is obtained in lemma 4.2. The rest of the proof is exactly the same as for homogenous conservation laws, see the theorem 3.2, if in cell entropy inequality is known. Notice that in the proof of the theorem 3.2 cell entropy inequality is obtained with the help of Brenier's lemma [10] in conjunction with the suitable maximum principle at kinetic level, see (3.32). We can not apply the same approach to the scheme (4.8) since it contains explicit approximation of the splitted part of the source term and therefore no maximum principle like (3.32) is available at kinetic level. That is why we first obtain in cell entropy inequality at macroscopic level and then we do kinetic interpretation in order to have possibility to apply main convergence theorem.

The scheme (4.8) equivalently writes:

$$\frac{v_{1j}^{n+1} - u_j^n}{2\Delta t} + \sum_k \sum_l \frac{|\Gamma_{jk}^l|}{|C_j|} A(u_j^n, u_{k,l-}^n, \overrightarrow{n}_{jk}^l) = 0, \tag{4.21}$$

$$\frac{v_{2j}^{n+1} - u_j^n}{2\Delta t} + \frac{b_s(u_j^n)}{|C_j|} \sum_k \sum_l |\Gamma_{jk}^l| < z(x_k), \overrightarrow{n}_{jk}^l >= 0, \tag{4.22}$$

$$u_j^{n+1} = 0.5 \cdot (v_{1j}^{n+1} + v_{2j}^{n+1}).$$

From (4.22) we have:

$$|v_{2j}^{n+1} - u_j^n| \le 2K_s |z|_{W^{1,1}} \Delta t, \tag{4.23}$$

$$\begin{gathered}\frac{S(v_{2j}^{n+1}) - S(u_j^n)}{2\Delta t} = \frac{S'(\theta_j^{n+1/2} v_{2j}^{n+1} + (1-\theta_j^{n+1/2}) u_j^n)}{2\Delta t} (v_{2j}^{n+1} - u_j^n) = \\ -S'((1-\theta_j^{n+1/2}) v_{2j}^{n+1} + \theta_j^{n+1/2} u_j^n) \frac{b_s(u_j^n)}{|C_j|} \sum_k \sum_l |\Gamma_{jk}^l| < z(x_k), \overrightarrow{n}_{jk}^l >= \\ -S'(u_j^n) \frac{b_s(u_j^n)}{|C_j|} \sum_k \sum_l |\Gamma_{jk}^l| < z(x_k), \overrightarrow{n}_{jk}^l > + \psi_{sj}^n,\end{gathered} \tag{4.24}$$

where $0 \le \theta_j^{n+1/2} \le 1$,

$$|\psi_{sj}^n| \le 2 \; max_{|u|_{L^\infty} \le K_\infty} S''(u) |b_s(u)| \cdot |z|_{W^{1,1}} \cdot K_s \cdot \Delta t. \tag{4.25}$$

Observe that in cell entropy inequality for (4.21) is already obtained in lemma 4.1. Thus we have the validity of in cell entropy inequalities for (4.21),(4.22), see (4.24),(4.25). Then with account of these entropy inequalities and the convexity of S we obtain the following in cell entropy inequality at macroscopic level:

$$\begin{aligned}&\frac{S(u_j^{n+1})-S(u_j^n)}{\Delta t}+\frac{1}{|C_j|}\sum_k\sum_l|\Gamma_{jk}^l|\eta(u_j^n,u_{k,l-}^n,\overrightarrow{n}_{jk}^l)+\\&+S'(u_j^n)\frac{b_s(u_j^n)}{|C_j|}\sum_k\sum_l|\Gamma_{jk}^l|<z(x_k),\overrightarrow{n}_{jk}^l>-\psi_{sj}^n\le 0.\end{aligned}\tag{4.26}$$

In order to be able to apply main convergence theorem and to accomplish the proof it remains to rewrite (4.8) equivalently at kinetic level with non-negative bounded measure in right hand side. Namely, we have:

$$\begin{aligned}&\frac{\chi_j^{n+1}(\xi)-\chi_j^n(\xi)}{\Delta t}+\frac{1}{|C_j|}\sum_k\sum_l\Big(|\Gamma_{jk}^l|\alpha_{jkl}^+(\xi)\chi_j^n(\xi)+|\Gamma_{jk}^l|\alpha_{jkl}^-(\xi)\chi_{u_{jk,l-}^n}(\xi)\Big)\\&+b_s(\xi)\cdot\frac{\partial\chi_j^n(\xi)}{\partial\xi}\cdot\frac{1}{|C_j|}\sum_k\sum_l|\Gamma_{jk}^l|<z(x_k),\overrightarrow{n}_{jk}^l>=\frac{\partial\tilde{m}_j^{n+1}(\xi)}{\partial\xi}+\tilde{\psi}_{sj}^n(\xi),\end{aligned}\tag{4.27}$$

where

$$\frac{\partial\tilde{m}_j^{n+1}(\xi)}{\partial\xi}=\frac{\chi_j^{n+1}(\xi)-f_j^{n+1}(\xi)}{\Delta t}.\tag{4.28}$$

We multiply (4.27) over $S'(\xi)$ and then we integrate it in ξ. With account of in cell entropy inequality at macroscopic level, see (4.26), we have:

$$\int_\xi S''(\xi)\cdot\tilde{m}_j^{n+1}(\xi)d\xi+\tilde{\psi_{sj}^n}\ge 0,\tag{4.29}$$

where $\tilde{\psi_{sj}^n}$ satisfies (4.25). Observe that (4.29) does not allow to identify nonnegativity of $\tilde{m}_j^{n+1}(\xi)$ defined by (4.28). That is why we introduce new residual $\psi_{sj}^n(\xi)$ and we define the measure $m_j^{n+1}(\xi)$ as follows:

$$\frac{\partial m_j^{n+1}(\xi)}{\partial\xi}=\frac{\chi_j^{n+1}(\xi)-f_j^{n+1}(\xi)}{\Delta t}+K_j^n.\frac{\partial(\chi_{u_j^n}(\xi)-\chi_{w_j^{n+1/2}}(\xi))}{\partial\xi},\tag{4.30}$$

$$\psi_{sj}^n(\xi)=\tilde{\psi_{sj}^n}(\xi)-K_j^n.\frac{\partial(\chi_{u_j^n}(\xi)-\chi_{w_j^{n+1/2}}(\xi))}{\partial\xi},\tag{4.31}$$

where

$$w_j^{n+1/2}=(1-\theta_j^{n+1/2})v_{2j}^{n+1}+\theta_j^{n+1/2}u_j^n,\text{see (4.24)},$$

$$K_j^n=\frac{b_s(u_j^n)}{|C_j|}\sum_k\sum_l|\Gamma_{jk}^l|<z(x_k),\overrightarrow{n}_{jk}^l>.$$

Observe that the second term in the right hand side of (4.31) vanishes together with Δt, see (4.25). For the measure $m_j^{n+1}(\xi)$ with account of (4.28)-(4.30) we obtain the following inequality:

$$\int_\xi S''(\xi) \cdot m_j^{n+1}(\xi) d\xi = \int_\xi S''(\xi) \cdot \tilde{m}_j^{n+1}(\xi) d\xi + \tilde{\psi_{sj}^n} \geq 0. \tag{4.32}$$

Since S is arbitrary convex function the nonnegativity of the measure $m_j^{n+1}(\xi)$ follows from (4.32). Thus the equation can be written equivalently in the form (2.7) with $\Psi_{\Delta x}$ and the nonnegative measure m defined by means of (4.31) and (4.30) respectively.
Other suppositions of the main convergence theorem can be easily verified exactly in the same way as we did for the homogenous equation in theorem 3.2, subsection 3.4, and therefore we omit these verifications.
Thus all the requirements of the main convergence theorem are valid and proof is accomplished.

5. *Numerical tests*

In this section we perform numerical investigation of our equilibrium type scheme by means of computation of test problems for inviscid Burgers equation. In all computations given below Engquist-Osher numerical flux function is used for discretization of space derivatives.

5.1. Inviscid Burgers equation with source term

5.1.1. Test problem

We consider inviscid Burgers equation with source term in two space dimensions

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} + \frac{\partial}{\partial y}\frac{u^2}{2} + z'_x(x,y)u + z'_y(x,y)u = 0, \tag{5..1}$$

where $(x,y) \in \Omega,\ \Omega \in \mathbb{R}$, $t > 0$, the function $z(x,y)$ is defined as

$$z(x,y) = \begin{cases} \cos(\pi(x+y)), & 4.5 \leq x+y \leq 5.5, \\ 0, & \text{otherwise}, \end{cases} \tag{5..2}$$

or

$$z(x,y) = 4sin((x-0.25)^2 + (y-0.2)^2 - 0.2) \cdot {}_{\{((x-0.25)^2+(y-0.2)^2<0.2)\}}$$
$$+4sin((x+0.25)^2 + (y+0.2)^2 - 0.2) \cdot {}_{\{((x+0.25)^2+(y+0.2)^2<0.2)\}}. \tag{5..3}$$

Together with the equation (5..1) we consider two different sets of initial and boundary conditions. Namely, we use the following set of initial and boundary conditions:

$$
\begin{aligned}
&u(0,x,y) = \begin{cases} u_b, & 0 < x+y < 1, \\ 0, & \text{otherwise}, \end{cases} \\
&\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \qquad \text{on the border of } \Omega,
\end{aligned} \tag{5..4}
$$

or

$$u(0,x,y) = 0, (x,y) \in \Omega,\ \Omega \subset \mathbb{R}^2, \tag{5..5}$$

$$u(t,s) = u_b(s), \tag{5..6}$$

where s belongs to the border of the domain under consideration, i.e. $s \in \partial\Omega$, and u_b is a given L^∞ function.

Steady state solution of the above test problems is defined by the following formula

$$u(x,y) = u_b + z_b - z(x,y). \tag{5..7}$$

5.1.2. Unit square test

In the table 1 numerical results are given for the test problem (5..1),(5..2),(5..4). In the second column of the table 1 the term "standard" stands for the scheme with standard, i.e. cell centered discretization of source term, and the term "Equilibrium" corresponds to the equilibrium type scheme (4.4).

Table 1. Comparison of numerical schemes, uniform rectangular mesh

Grid	*Method*	*CFL − number*	L^∞ − *error*	L^1 − *error*
5000×5000	*Standard*	0.7	$3.69 \cdot 10^{-3}$	$5.56 \cdot 10^{-3}$
50×50	*Standard*	0.7	0.1650527	0.4880051
40×40	*Equilibrium type*	0.7	$6.4 \cdot 10^{-5}$	$1.3 \cdot 10^{-5}$

Observe that by means of the change of variables we can reformulate test problem (5..1),(5..4) as one dimensional problem along the diagonal of the square. Notice also that 2-D schemes, both with standard and equilibrium type discretization of the source term, are equivalent to 1-D schemes on the lines parallel to diagonal of the square. In order to avoid huge computations and obtain the results for table 1 corresponding equivalent 1-D problems is solved, i.e. instead of 5001×5001 nodes we have used 10

001 nodes for the equivalent scheme along the diagonal of the square, i.e. in single space dimension. Numerical results on computation of the same test problem on triangular meshes are given in table 2. Pointing out the results given in the tables 1 and 2 we can conclude that our equilibrium type scheme is far more accurate then the standard one.

Table 2. Comparison of numerical schemes, triangular mesh, 191 nodal points, 336 triangles.

$Method$	$CFL - number$	$L^{\infty} - error$	$L^1 - error$
$Standard$	0.35	0.448974	0.3979535
$Equilibrium\ type$	0.7	$3.57216 \cdot 10^{-10}$	$7.3524 \cdot 10^{-11}$

5.1.3. Unit circle test

We consider test problem (5..1), (5..5), (5..6), where the function $z(x, y)$ is defined according to (5..3), see fig.5. Computational domain Ω for this test problem is the unit circle. We consider several different triangulations, see fig.6. left. Boundary condition (5..6) is approximated according to numerical model for Dirichlet conditions developed in [8], consult [14] for other approaches on numerical implementation of Dirichlet boundary conditions for hyperbolic conservation laws. Computations are performed by equilibrium type and standard schemes.

Convergence history. Numerical solutions computed by these two schemes on the fine mesh are given on fig.7 for different consecutive time moments. We can observe that level lines of numerical solution corresponding to our equilibrium type scheme has the same shape as the level lines of function z as it should be according to formulae (5..7). The same plot due to the standard scheme does not allow to conclude any reasonable similarity with z.

Mesh refinement history. On fig.8 mesh refinement history in terms of number of nodes and L^1 and L^{∞} errors is given. Advantage of equilibrium type discretization over the cell centered one is evident: together with mesh refinement the errors decrease slowly for standard scheme while for our scheme it is of order 10^{-4} on the rough mesh and of order 10^{-11} on the fine mesh.

Studying convergence rate. Notice that meshes on fig.6 are nonuniform unstructured and they are not nested. For such meshes a rough definition of the rate of the scheme can be done in accordance with the following formulae:

$$(\text{char.length})^{\text{rate}} = \text{error}, \tag{5..8}$$

where *char.length* is some characteristic length of finite volume cells. We introduce two characteristic lengths for finite volume cells:

$$\text{char.length} = \sqrt{min_j |C_j|}$$

and

$$\text{char.length} = \sqrt{max_j |C_j|}.$$

Then we can conjecture that a real rate of the scheme is in between of those corresponding to these two different characteristic lengths. The rates are computed in L^1 and L^∞ norms. The results are given on fig.9. We can observe that for standard scheme the rates are almost constant functions of the characteristic lengths. The values are in between 0 and 1 for both norms and for all definitions of characteristic lengths. Our equilibrium type scheme exhibits interesting behavior:

(i) though the scheme is formally first order accurate the rates computed are above 4 for all meshes and for all definitions;

(ii) the rate of the scheme increases when characteristic length of finite volume mesh decreases. On the fine mesh in L^∞ norm the rate is approximately 5.5 and 9.5 respectively for two different characteristic lengths. In L^1 norm the rates are higher, approximately 6.4 and 10.5 respectively.

Notice that similar behavior of other equilibrium type scheme has been observed in [8].

Computational domain and triangular meshes.

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Inviscid burgers equation with source term. Convergence history: equilibrium type scheme-left; standard scheme-right.

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Inviscid Burgers equation with source term on unit circle. Mesh refinement history; max error for equilibrium type scheme is of order 10^{-4}, min error is of order 10^{-11}.

Convergence rates in L^{∞} and L^{1} norms. Inviscid Burgers equation with source term on unit circle.

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