ON A CUSPED ELASTIC SOLID-FLUID INTERACTION PROBLEM

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Abstract

Admissible static and dynamical problems are investigated for a cusped plate. The setting of boundary conditions at the plates ends depends on the geometry of sharpenings of plates ends, while the setting of initial conditions is independent of them. Interaction problem between an elastic cusped plate and ideal incompressible fluid is studied.

Key words and phrases: solid-fluid interaction, cusped plate, degenerate ordinary differential equation, degenerate hyperbolic equation, boundary value problems, vibration.

AMS subject classification: 74F10, 74K20.

1 *Cylindrical Bending of Cusped Plates*

In 1955 I.N. Vekua [10-12] raised the problem of investigation of cusped elastic plates, i.e., such ones whose thickness vanishes on some part of the plate boundary or on the whole boundary.

In this chapter we will consider a plate, whose projection on $x_3 = 0$ occupies the domain Ω

$$\Omega = \{(x_1, x_2, x_3) : \ -\infty < x_1 < \infty, \ \ 0 < x_2 < l, \ \ x_3 = 0\}.$$

The equation of the cylindrical bending of plates has the following form (see, e.g., [9])

$$(D(x_2)w,_{22}(x_2)),_{22} = q(x_2), \qquad 0 < x_2 < l, \tag{1.1}$$

where $w(x_2)$ is a deflection of the plate, $q(x_2)$ is an intensity of a lateral load, $D(x_2)$ is a flexural rigidity,

$$D(x_2) := \frac{2Eh^3(x_2)}{3(1-\nu^2)}, \tag{1.2}$$

where E is the Young's modulus, ν is the Poison's ratio, and $2h(x_2)$ is the thickness of the shell. Let E =const, ν =const, and

$$D(x_2) = D_0 x_2^\alpha (l - x_2)^\beta, \quad D_0,\, \alpha,\, \beta = \text{const}, \quad D_0 > 0, \quad \alpha,\, \beta \geq 0. \quad (1.3)$$

Then

$$2h(x_2) = h_0 x_2^{\alpha/3} (l - x_2)^{\beta/3}, \quad h_0 = \text{const} > 0.$$

In the case $\alpha^2 + \beta^2 > 0$ equation (1.1) becomes degenerate one. Such plates are called cusped plates.

The profile of the plate under consideration has one of the following forms (see Appendix A).

In the case under consideration (see [9])

$$M_2(x_2) := -D(x_2) w_{,22}(x_2), \quad (1.4)$$

$$Q_2(x_2) := M_{2,2}(x_2), \quad (1.5)$$

where $M_2(x_2)$ is a bending moment, $Q_2(x_2)$ is an intersecting force.

Obviously, if we suppose $q(x_2) \in C([0, l])$, $w(x_2) \in C^4(]0, l[)$, for $Q_{2,2}(x_2)$, $M_{2,2}(x_2)$, $w_{,2}(x_2)$, and $w(x_2)$ we have

$$Q_2(x_2) := -\int_{x_2^0}^{x_2} q(\xi) d\xi - c_1, \quad (1.6)$$

$$M_2(x_2) := -\int_{x_2^0}^{x_2} (x_2 - \xi) q(\xi) d\xi - c_1 x_2 - c_2, \quad (1.7)$$

$$w_{,2}(x_2) := \int_{x_2^0}^{x_2} \left\{ -\left[\int_{x_2^0}^{\xi} \eta q(\eta) d\eta + c_2 \right] + \xi \left[\int_{x_2^0}^{\xi} q(\eta) d\eta + c_1 \right] D^{-1}(\xi) \right\} d\xi + c_3, \quad (1.8)$$

$$w(x_2) := \int_{x_2^0}^{x_2} (x_2 - \xi) \left\{ \left[-\int_{x_2^0}^{\xi} \eta q(\eta) d\eta + c_2 \right] + \xi \left[\int_{x_2^0}^{\xi} q(\eta) d\eta + c_1 \right] D^{-1}(\xi) \right\} d\xi + c_3 x_2 + c_4, \quad x_2^0 \in]0, l[. \quad (1.9)$$

At points $0, l$ all above quantities are defined as the corresponding limits when $x_2 \to 0_+$ and $x_2 \to l_-$.

Obviously,

$$Q_2(x_2),\ \ M_2(x_2) \in C([0,l]),$$
$$w(x_2),\ \ w_{,2}(x_2) \in C(]0,l[),$$

and the behaviour of the $w_{,2}(x_2)$ and $w(x_2)$ when $x_2 \to 0_+$ and $x_2 \to l_-$ depends, in view of (1.8), (1.9), on α and β.

Let us consider the following problems.

Problem 1 *Let $\alpha < 1$, $\beta < 1$. Find $w \in C^4(]0,l[) \cap C^1([0,l])$ satisfying (1.1) and the following boundary conditions (BCs):*

$$w(0) = g_{11},\ \ w_{,2}(0) = g_{21},\ \ w(l) = g_{12},\ \ w_{,2}(l) = g_{22}; \tag{1.10}$$

Problem 2 *Let $\alpha < 1$, $\beta < 1$. Find $w \in C^4(]0,l[) \cap C^1([0,l])$ satisfying (1.1) and BCs:*

$$w(0) = g_{11},\ \ w_{,2}(0) = g_{21},\ \ w_{,2}(l) = g_{22},\ \ Q_2(l) = h_{22};$$

Problem 3 *Let $0 \le \alpha < 1$, $\quad 0 \le \beta < 2$. Find $w \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l])$ satisfying (1.1) and BCs:*

$$w(0) = g_{11},\ \ w_{,2}(0) = g_{21},\ \ w(l) = g_{12},\ \ M_2(l) = h_{12};$$

Problem 4 *Let $0 \le \alpha < 1$, $\beta \ge 0$. Find $w \in C^4(]0,l[) \cap C^1([0,l[)$ satisfying (1.1) and the following BCs:*

$$w(0) = g_{11},\ \ w_{,2}(0) = g_{21},\ \ M_2(l) = h_{12},\ \ Q_2(l) = h_{22};$$

Problem 5 *Let $0 \le \alpha$, $\beta < 1$. Find $w \in C^4(]0,l[) \cap C^1([0,l])$ satisfying (1.1) and the following BCs:*

$$w_{,2}(0) = g_{21}\ \ Q_2(0) = h_{21},\ \ w(l) = g_{12},\ \ w_{,2}(l) = g_{22};$$

Problem 6 *Let $0 \le \alpha < 1$, $0 \le \beta < 2$. Find $w \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l])$ satisfying (1.1) and the following BCs:*

$$w_{,2}(0) = g_{21},\ \ Q_2(0) = h_{21},\ \ w(l) = g_{12},\ \ M_2(l) = h_{12};$$

Problem 7 *Let $0 \le \alpha < 2$, $\quad 0 \le \beta < 1$. Find $w \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l])$ satisfying (1.1) and the following BCs:*

$$w(0) = g_{11},\ \ M_2(0) = h_{11},\ \ w(l) = g_{12},\ \ w_{,2}(l) = g_{22};$$

Problem 8 *Let* $0 \leq \alpha < 2$, $0 \leq \beta < 1$. *Find* $w \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l])$ *satisfying (1.1) and the following BCs:*

$$w(0) = g_{11}, \ \ M_2(0) = h_{11}, \ \ w,_2(l) = g_{22}, \ \ Q_2(l) = h_{22}; \tag{1.11}$$

Problem 9 *Let* $0 \leq \alpha, \ \beta < 2$. *Find* $w \in C^4(]0,l[) \cap C([0,l])$ *satisfying (1.1) and the following BCs:*

$$w(0) = g_{11}, \ \ M_2(0) = h_{11} \ \ w(l) = g_{12}, \ \ M_2(l) = h_{12};$$

Problem 10 *Let* $\alpha \geq 0, \quad 0 \leq \beta < 1$. *Find* $w \in C^4(]0,l[) \cap C^1(]0,l])$ *satisfying (1.1) and the following BCs:*

$$M_2(0) = h_{11}, \ \ Q_2(0) = h_{22} \ \ w(l) = g_{12}, \ \ w,_2(l) = g_{22}.$$

In all these problems $g_{i,j}$, h_{ij} $(i,j = 1,2)$ are given constants.

All the problems above are solved in the explicit forms. Let us solve typical ones. For the sake of simplicity we consider homogeneous BCs.

Solution of Problem 1:

By virtue of (1.8) and homogeneous boundary conditions for $w,_2$ we have

$$c_3 = \int\limits_0^{x_2^0} \left[\int\limits_{x_2^0}^{\xi} (\eta) q(\eta) d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi) d\xi, \tag{1.12}$$

$$c_3 = \int\limits_l^{x_2^0} \left[\int\limits_{x_2^0}^{\xi} (\xi - \eta) q(\eta) d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi) d\xi. \tag{1.13}$$

Taking into account of (1.9) and homogeneous conditions (1.10) for w, we obtain

$$c_4 = -\int\limits_0^{x_2^0} \xi \left[\int\limits_{x_2^0}^{\xi} (\xi - \eta) q(\eta) d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi) d\xi, \tag{1.14}$$

$$c_4 = -\int\limits_l^{x_2^0} \xi \left[\int\limits_{x_2^0}^{\xi} (\xi - \eta) q(\eta) d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi) d\xi. \tag{1.15}$$

Obviously, from (1.12)-(1.15), for c_1 and $c_2$we have the following system

$$c_1 \int\limits_0^l \xi D^{-1}(\xi) d\xi \ \ + \ \ c_2 \int\limits_0^l D^{-1}(\xi) d\xi$$

$$
\begin{aligned}
&= -\int_{x_2^0}^{l} q(\eta)d\eta \int_{\eta}^{l} (\xi-\eta)D^{-1}(\xi)d\xi \\
&\quad + \int_{0}^{x_2^0} q(\eta)d\eta \int_{0}^{\eta} (\xi-\eta)D^{-1}(\xi)d\xi =: d_1, \quad (1.16)
\end{aligned}
$$

$$
\begin{aligned}
c_1\int_0^l \xi^2 D^{-1}(\xi)d\xi \quad &+ \quad c_2\int_0^l \xi D^{-1}(\xi)d\xi \\
&= -\int_{x_2^0}^{l} q(\eta)d\eta \int_{\eta}^{l} \xi(\xi-\eta)D^{-1}(\xi)+d\xi \\
&\quad \int_{0}^{x_2^0} q(\eta)d\eta \int_{0}^{\eta} \xi(\xi-\eta)D^{-1}(\xi)d\xi =: d_2. \quad (1.17)
\end{aligned}
$$

The determinant of this system is equal to

$$
\Delta = \left\{ \int_0^l \xi D^{-1}(\xi)d\xi \right\}^2
$$

$$
-\int_0^l D^{-1}(\xi)d\xi \int_0^l \xi^2 D^{-1}(\xi)d\xi < 0, \quad (1.18)
$$

The last assertion follows from the Hölder inequality which is strong since $\xi D^{-\frac{1}{2}}(\xi)$ and $D^{-\frac{1}{2}}(\xi)$ are positive on $]0, l[$, and $\xi^2 D^{-1}(\xi)$ and $D^{-1}(\xi)$ differ from each other by a nonconstant factor ξ^2.

Further,

$$
\begin{aligned}
c_1 &= \frac{d_1 \int_0^l \frac{\xi}{\xi^\alpha (l-\xi)^\beta} d\xi - d_2 \int_0^l \frac{1}{\xi^\alpha (l-\xi)^\beta} d\xi}{\Delta}, \\
c_2 &= \frac{d_2 \int_0^l \frac{\xi}{\xi^\alpha (l-\xi)^\beta} d\xi - d_1 \int_0^l \frac{\xi^2}{\xi^\alpha (l-\xi)^\beta} d\xi}{\Delta}.
\end{aligned}
$$

After substitution c_1 and c_2 into (1.12) and (1.14) we get expressions for c_3 and c_4. It is obvious, that the last integral of the expression c_1 exists if and only if $\alpha < 1$, $\beta < 1$.

Solution of Problem 8: From (1.6), (1.7) and BC we get

$$c_1 = -\int\limits_{x_2^0}^{l} q(\xi)d\xi, \quad c_2 = -\int\limits_{0}^{x_2^0} \xi q(\xi)d\xi, \tag{1.19}$$

and hence,

$$Q_2(x_2) = \int\limits_{x_2}^{l} q(\xi)d\xi, \quad M_2(x_2) = \int\limits_{0}^{x_2} \xi q(\xi)d\xi + x_2 \int\limits_{x_2}^{l} q(\xi)d\xi,$$

Substituting (1.19) in (1.8) and (1.9) and taking into account BCs, after using Dirichlet formula we have

$$c_3 = \int\limits_{x_2^0}^{l} \left[\int\limits_{0}^{\xi} \eta q(\eta)d\eta + \xi \int\limits_{\xi}^{l} q(\eta)d\eta \right] D^{-1}(\xi)d\xi$$

$$+ \int\limits_{x_2^0}^{l} \xi q(\xi) \int\limits_{x_2^0}^{l} \eta D^{-1}(\eta)d\eta d\xi$$

$$- \int\limits_{x_2^0}^{l} q(\xi) \int\limits_{\xi}^{l} (\eta - \xi)D^{-1}(\eta)d\eta d\xi + \int\limits_{0}^{x_2^0} q(\xi) \int\limits_{x_2^0}^{l} D^{-1}(\eta)d\eta d\xi,$$

$$c_4 = \int\limits_{0}^{x_2^0} q(\xi) \int\limits_{0}^{\xi} \eta(\eta - \xi)D^{-1}(\eta)d\eta d\xi$$

$$+ \int\limits_{x_2^0}^{l} q(\xi) \int\limits_{0}^{x_2^0} \eta^2 D^{-1}(\eta)d\eta d\xi + \int\limits_{0}^{x_2^0} \xi q(\xi) \int\limits_{0}^{x_2^0} \eta D^{-1}(\eta)d\eta d\xi,$$

and

$$w_{,2}(x_2) = \int\limits_{x_2}^{l} \frac{\int\limits_{0}^{\xi} \eta q(\eta)d\eta + \xi \int\limits_{\xi}^{l} q(\eta)d\eta}{D_0 \xi^\alpha (l - \xi)^\beta} d\xi,$$

$$w(x_2) = \int\limits_{x_2}^{l} \frac{q(\xi)}{D_0}$$

$$\times \quad \left[-x_2 \int\limits_{\xi}^{x_2} \frac{d\eta}{\eta^{\alpha-1}(l-\eta)^{\beta}} + \int\limits_{0}^{x_2} \frac{d\eta}{\eta^{\alpha-2}(l-\eta)^{\beta}} + x_2\xi \int\limits_{\xi}^{l} \frac{d\eta}{\eta^{\alpha}(l-x)^{\beta}} \right] d\xi$$

$$+ \quad \int\limits_{0}^{x_2} \frac{q(\xi)}{D_0} \left[\xi \int\limits_{\xi}^{x_2} \frac{d\eta}{\eta^{\alpha-1}(l-\eta)^{\beta}} \right.$$

$$+ \quad \left. \int\limits_{0}^{\xi} \frac{d\eta}{\eta^{\alpha-2}(l-\eta)^{\beta}} + x_2\xi \int\limits_{\xi}^{l} \frac{d\eta}{\eta^{\alpha}(l-x)^{\beta}} \right] d\xi. \tag{1.20}$$

It is easy to see that $w(x_2)$ and $w_{,2}(x_2)$ belong to $C([0,l])$, since

$$\begin{aligned} \lim_{\xi\to 0+} \frac{\int\limits_0^{\xi} \eta q(\eta)d\eta}{\xi^{\alpha}(l-\xi)^{\beta}} &= \lim_{\xi\to 0+} \frac{\xi q(\xi)}{\alpha\xi^{\alpha-1}(l-\xi)^{\beta} - \beta\xi^{\alpha}(l-\xi)^{\beta-1}} \\ &= \lim_{\xi\to 0+} \frac{q(\xi)}{\xi^{\alpha-2}\left[\alpha(l-\xi)^{\beta} - \beta\xi(l-\xi)^{\beta-1}\right]}. \end{aligned}$$

Solution of Problem 9:

$$Q_2(x_2) = \int\limits_{x_2}^{l} q(\xi)d\xi - \frac{1}{l}\int\limits_{0}^{l} \xi q(\xi)d\xi,$$

$$M_2(x_2) = x_2\int\limits_{x_2}^{l} q(\xi)d\xi + \int\limits_{0}^{x_2} \xi q(\xi)d\xi - \frac{x_2}{l}\int\limits_{0}^{l} \xi q(\xi)d\xi,$$

$$w_{,2}(x_2) = -\int\limits_{x_2}^{l} \frac{R_1(\xi)}{\xi^{\alpha}(l-\xi)^{\beta}}d\xi + \frac{1}{l}\int\limits_{0}^{l} \frac{R_1(\xi)}{\xi^{\alpha-1}(l-\xi)^{\beta}}d\xi,$$

$$\begin{aligned} w(x_2) \quad = \quad & -x_2\int\limits_{x_2}^{l} \frac{R_1(\xi)}{\xi^{\alpha}(l-\xi)^{\beta}}d\xi - \int\limits_{0}^{x_2} \frac{R_1(\xi)}{\xi^{\alpha-1}(l-\xi)^{\beta}}d\xi \\ & + \frac{x_2}{l}\int\limits_{0}^{l} \frac{R_1(\xi)}{\xi^{\alpha-1}(l-\xi)^{\beta}}d\xi, \end{aligned}$$

where

$$R_1(\xi) = -\frac{1}{l}(l-\xi)\int\limits_{0}^{\xi} \eta q(\eta)d\eta - \frac{\xi}{l}\int\limits_{\xi}^{l} (l-\eta)q(\eta)d\eta.$$

Solution of Problem 10:

$$Q_2(x_2) = -\int\limits_0^{x_2} q(\xi)d\xi, \quad M_2(x_2) = -\int\limits_0^{x_2} (x_2-\xi)q(\xi)d\xi,$$

$$w_{,2}(x_2) = \int\limits_{x_2}^{l} \frac{\int\limits_0^{\xi}(\xi-\eta)q(\eta)d\eta}{\xi^\alpha(l-\xi)^\beta}d\xi,$$

$$w(x_2) = -\int\limits_{x_2}^{l}(x_2-\xi)\frac{\int\limits_0^{\xi}(\xi-\eta)q(\eta)d\eta}{\xi^\alpha(l-\xi)^\beta}d\xi,$$

$w_{,2}$ and w are bounded as $x_2 \to 0_+$ if

$$\exists q^{([\alpha]-2)}, \ \text{such that} \ \lim_{\xi\to 0_+} q^{([\alpha]-2)}(\xi) \neq \infty,$$

$$\lim_{\xi\to 0_+} q^{(i-3)} = 0, \ \ i=\overline{3,[\alpha]}, \ \ \alpha \geq 3,$$

is fulfilled, since

$$\lim_{\xi\to 0+}\frac{\int\limits_0^{\xi}\eta q(\eta)d\eta}{\xi^\alpha(l-\xi)^\beta} = \lim_{\xi\to 0+}\frac{q(\xi)}{\xi^{\alpha-2}\left[\alpha(l-\xi)^\beta - \beta\xi(l-\xi)^{\beta-1}\right]}.$$

Using finite difference method we can solve the above problems numerically. In Appendix B on Figures 11-14 these results are graphically given. On figures 15-18 the numerical results are given by means of direct calculation of the integrals in the corresponding explicit analytical solutions.

Let us make some remarks concerning well-posedness of the BVPs (boundary value problems) above.

Remark 1.1 *Problems 1-10 are not correct for the different values of α and β indicated in Problems 1-10. It is evident from the fact that in the above cases, in general, the limits of w and $w_{,2}$ as $x_2 \to 0_+,\ l_-$ do not exist. the last assertions easily follow from the general representations (1.9) and (1.8) of w and $w_{,2}$ with (13).*

Remark 1.2 *Let us consider the cylindrical bending of the cusped plate under consideration on the basis of the classical geometrically non-linear* *bending theory [3, 9]. In the case of the cylindrical bending some of the non-linear relations will be linearized and we get the following relations:*

$$e_{22} = u_{2},_{2} + \frac{1}{2}(w,_{2})^2, \; e_{12} = u_{1},_{2}, \; e_{11} = e_{33} = e_{13} = e_{23} = 0;$$
$$N_1 = \sigma N_2, \;\; N_2 = C_1 = \text{const}, \;\; N_{12} = C_2 = \text{const},$$
$$M_1 = \sigma M_2, \;\; M_2 = -Dw,_{22}, \;\; Q_1 = 0, \;\; Q_2 = M_{2,2}, \tag{1.21}$$
$$(-Dw,_{22}),_{22} = -(q + N_2 w,_{22}), \tag{1.22}$$

where u_i, $i = 1,2,3$ ($u_3 = w$), are the components of the displacement vector; e_{ij}, $i = 1,2,3$, are components of the deformation tensor, N_γ, $\gamma = 1,2$, are the normal forces in the middle plane, N_{12} is the shearing force parallel to the axis x_2. From (1.22), (1.21) we have

$$M_{2,22} - N_2 D^{-1} M_2 = -q, \;\; \textit{i.e.}, \;\; DM_{2,22} - N_2 M_2 = -qD, \tag{1.23}$$

and

$$Q_{2,2} = -q + N_2 D^{-1} M_2, \;\; \textit{i.e.}, \;\; (DQ_{2,2}),_2 - N_2 Q_2 = -(qD),_2, \tag{1.24}$$

Therefore,

$$DQ_{2,22} + D,_2 Q_{2,2} - N_2 Q_2 = -(qD),_2 .$$

If $\beta = 0$, then we can rewrite (1.23) and (1.24) in the following forms:

$$x_2^\alpha M_{2,22} - N_2 D_0^{-1} M_2 = -q x_2^\alpha,$$

and

$$x_2 Q_{2,22} + \alpha Q_{2,2} - N_2 D_0^{-1} x_2^{1-\alpha} Q_2 = -(q x_2^\alpha),_2 .$$

Let constant $N_2 > 0$ be given. Then, according to well-known results, for M_2 and Q_2 we can prescribe the Dirichlet condition at $x_2 = 0$ if $0 \leq \alpha < 2$ and $0 \leq \alpha < 1$, respectively, while by $\alpha \geq 2$ and $\alpha \geq 1$ it is not the case and the Keldish conditions (boundedness of M_2 and Q_2 as $x_2 \to 0_+$, respectively) are correct. This is a new effect in comparison with the linear theory when also in the cases $\alpha \geq 2$ and $\alpha \geq 1$ we can arbitrarily prescribe the Dirichlet conditions at $x_2 = 0$ for M_2 and Q_2, respectively (see Problem 10). It is obvious that this peculiarity of the geometrically non-linear cylindrical bending will be preserved also in the case of the general (i.e., non-cylindrical) geometrically non-linear bending.

Remark 1.3 *Let $\beta = 0$. Homogeneous Problem 1 corresponds to the three-dimensional problem when the upper and lower surfaces are loaded by surface forces, the edge $x_2 = l$ ($x_1 \in]-\infty, +\infty[$) is fixed and the edge $x_2 = 0$ ($x_1 \in]-\infty, +\infty[$) is glued to absolutely rigid tangent plane. In the case* *of homogeneous Problem 7 the above mentioned plane is rigid parallel to the axis x_3. Problem 10 corresponds to the three-dimensional problem when along the edge $x_2 = 0$ ($x_1 \in]-\infty, +\infty[$) the concentrated along the above edge force [4] and moment are applied and they are equal to h_{22} and h_{11} respectively (see Fig.1):*

$$h_{22} = Q_2(0) = Q_2(x_1^0, 0) \tag{1.25}$$

$$= \lim_{x_2 \to 0_+} Q_2(x_1^0, x_2) := \lim_{x_2 \to 0_+} \int_{-h}^{+h} X_{23}(x_1^0, x_2, x_3) dx_3,$$

$$h_{11} = M_2(0) = M_2(x_1^0, 0)$$

$$= \lim_{x_2 \to 0_+} M_2(x_1^0, x_2) := \lim_{x_2 \to 0_+} \int_{-h}^{+h} x_3 X_{23}(x_1^0, x_2, x_3) dx_3.$$

The above integrals are taken on the striped area. $\{X_{ij}\}$, $i, j = 1, 2, 3$, is stress tensor.

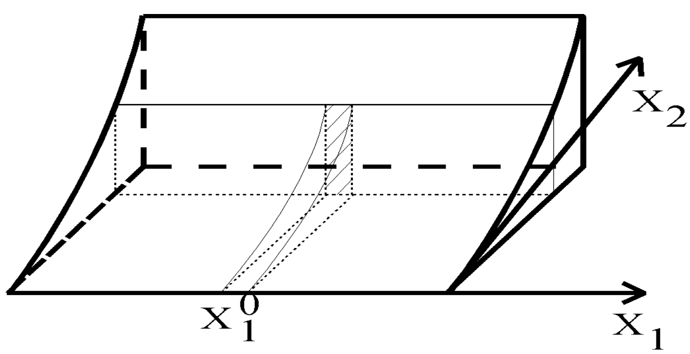

Fig. 1

2 *Vibration of the Plate with Two Cusped Edges*

The equation of bending vibration has the following form

$$(D(x_2)w,_{22}(x_2, t)),_{22} = q(x_2, t) - 2\rho h(x_2) \frac{\partial^2 w(x_2, t)}{\partial t^2}, \quad 0 < x_2 < l, \tag{2.1}$$

where ρ is a density of the shell.

In this case we have to add to the BCs of Problems 1-10 the initial conditions

$$w(x_2, 0) = \varphi_1(x_2), \quad w,_t(x_2, 0) = \varphi_2(x_2), \quad x_2 \in]0, l[, \tag{2.2}$$

where $\varphi_i(x_2) \in C^4(]0,l[), \quad i = 1,2$ are given functions.

Let us consider the following initial boundary value problem (IBVP):

Problem 11 *Let* $0 \le \alpha < 2$, $0 \le \beta < 1$. *Find*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]) \\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t > 0), \;\; w(x_2,t) \in C(0 \le x_2 \le l, \; t \ge 0) \end{array} \tag{2.3}$$

satisfying equation (2.1), the BCs

$$w(0,t) = M_2(0,t) = w,_2(l,t) = Q_2(l,t) = 0, \tag{2.4}$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]), \;\; i = 1,2. \tag{2.5}$$

$$\begin{aligned} \varphi_i(0) &= -D(x_2)\varphi_i''(x_2)|_{x_2=0_+} = \varphi_i'(l) \\ &= \left(-D(x_2)\varphi_i''(x_2)\right)'|_{x_2=l_-} = 0, i = 1,2. \end{aligned} \tag{2.6}$$

Solution. In this section all quantities, particularly those in (1.4), (1.5), depend on x_2 and t.

Using the Fourier method, we look for $w(x_2,t)$ in the following form

$$w(x_2,t) = X(x_2)T(t). \tag{2.7}$$

Let firstly $q(x_2,t) \equiv 0$. Then from (2.1) we get

$$\frac{(D(x_2)X''(x_2))''}{g(x_2)X(x_2)} = -\frac{T''(t)}{T(t)} = \lambda = \text{const}.$$

Hence,

$$T''(t) + \lambda T(t) = 0, \tag{2.8}$$

and

$$\left(D(x_2)X''(x_2)\right)'' = \lambda g(x_2)X(x_2), \tag{2.9}$$

where $g(x_2) := 2\rho h(x_2)$.

From (2.4) for $X(x_2)$ we obtain the following BCs

$$X(0) = -D(x_2)X''(x_2)|_{x_2=0} = X'(l) = (-D(x_2)X''(x_2))'|_{x_2=l} = 0. \tag{2.10}$$

Now, in view of (2.3), we have to solve the following BVP:

Find

$$X(x_2) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]), \tag{2.11}$$

which satisfies equation (2.9) and BCs (2.10).

If in (1.20) we replace $w(x_2)$ and $q(x_2)$ by $X(x_2)$ and $\lambda g(x_2)X(x_2)$, respectively, then, similarly to Section 1, for $X(x_2)$ we obtain

$$X(x_2) = \lambda \int_0^l g(\xi)K(x_2,\xi)X(\xi))d\xi, \tag{2.12}$$

where

$$K(x_2,\xi) = \begin{cases} K_3(\xi,x_2), & 0 \le \xi \le x_2, \\ K_3(x_2,\xi), & x_2 \le \xi \le l. \end{cases}$$

$$K_3(x_2,\xi) := -x_2 \int_\xi^{x_2} \eta D^{-1}(\eta)d\eta + \int_0^{x_2} \eta^2 D^{-1}(\eta)d\eta + x_2\xi \int_\xi^l D^{-1}(\eta)d\eta \tag{2.13}$$

Proposition 2.1 *$K(x_2,\xi)$ is symmetric with respect to x_2 and ξ.*

Proof. For z_1 and z_2, such that $0 \le z_1$, $z_2 \le l$ we get

$$K(z_1,z_2) = \begin{cases} K_3(z_2,z_1), & 0 \le z_2 \le z_1, \\ K_3(z_1,z_2), & z_1 \le z_2 \le l, \end{cases}$$

$$K(z_2,z_1) = \begin{cases} K_3(z_1,z_2), & z_1 \le z_2 \le l, \\ K_3(z_2,z_1), & 0 \le z_2 \le z_1, \end{cases}$$

i.e.,

$$K(z_1,z_2) = K(z_2,z_1), \quad \text{for any } z_1,z_2 \in [0,l].$$

□

(2.12) can be rewritten as follows

$$Y(x_2) = \lambda \int_0^l R(x_2,\xi)Y(\xi)d\xi, \tag{2.14}$$

where

$$Y(x_2) = \sqrt{g(x_2)}X(x_2), \quad R(x_2,\xi) = \sqrt{g(x_2)}K(x_2,\xi)\sqrt{g(\xi)}. \tag{2.15}$$

(2.14) is an integral equation with a symmetric kernel.

Recall the following three Hilbert-Schmidt theorems

Theorem 2.2 *If $u(x_2) \in C([0,l])$ has the form*

$$u(x_2) = \lambda \int_0^l R(x_2,\xi)u(\xi)d\xi,$$

then

$$u(x_2) = \sum_{n=1}^{\infty} (u, Y_n) Y_n(x_2), \tag{2.16}$$

where

$$(u, Y_n) := \int_0^l u(x_2) Y_n(x_2) dx_2,$$

Y_n is an eigenfunction of $R(x_2, \xi)$, and the series on the right hand side of (2.16) is convergent absolutely and uniformly on $[0, l]$.

Theorem 2.3 *If the number of eigenvalues λ_n of the symmetric and continuous kernel is finite then*

$$R(x_2, \xi) = \sum_{n=1}^{N} \frac{Y_n(x_2) Y_n(\xi)}{\lambda_n}.$$

Theorem 2.4 *If $f(x_2) \in C([0, l])$, then*

$$\int_0^l R(x_2, \xi) f(\xi) d\xi = \sum_{n=1}^{\infty} \frac{(f, Y_n)}{\lambda_n} Y_n,$$

and the series is convergence absolutely and uniformly. Here $R(x_2, \xi)$ is a symmetric and continuous kernel with respect to x_2; ξ, Y_n are eigenfunctions of R corresponding to the eigenvalues λ_n.

Proposition 2.5 *Number of eigenvalues λ_n of (2.14) is not finite.*

Proof. Let it be finite, and $n = \overline{1, m}$. Then we can express $R(x_2, \xi)$ as follows (see Theorem 2.3)

$$R(x_2, \xi) = \sum_{n=1}^{m} \frac{Y_n(x_2) Y_n(\xi)}{\lambda_n},$$

where $Y_n(x_2) \in C^4(]0, l[)$, i.e.,

$$R(x_2, \xi) \in C^4\left(]0, l[\times]0, l[\right). \tag{2.17}$$

On the other hand, by virtue of (2.13),

$$K'''_{x_2}(x_2, \xi)|_{\xi \to x_2 -} - K'''_{x_2}(x_2, \xi)|_{\xi \to x_2 +} = \frac{1}{D(x_2)},$$

then kernel

$$R(x_2, \xi) \notin C^4\left(]0, l[\times]0, l[\right). \tag{2.18}$$

But, (2.17) and (2.18) contradict each other, thus the number of λ_n is not finite. $\square$

Proposition 2.6 *All of λ_n are positive.*

Proof. Obviously, if we denote by Y_n orthonormalized eigenfunctions (it can be assumed without loss of generality) of (2.14), then

$$X_n(x_2) = \frac{Y_n(x_2)}{\sqrt{g(x_2)}}$$

are eigenfunctions of (2.12) (i.e., of (2.9)). Let us multiply both sides of the following equation

$$(D(x_2)X_n''(x_2))'' = \lambda_n g(x_2)X_n(x_2), \tag{2.19}$$

by $X_n(x_2)$ and integrate it from 0 to l. Taking into account the first expression of (2.15), we obtain

$$\begin{aligned}\int_0^l X_n(x_2)(D(x_2)X_n''(x_2))''dx_2 &= \lambda_n \int_0^l g(x_2)X_n(x_2)X_n(x_2)dx_2 \\ &= \lambda_n \int_0^l Y_n(x_2)Y_n(x_2)dx_2 = \lambda_n.\end{aligned}$$

Further,

$$\begin{aligned}\lambda_n &= \int_0^l X_n(x_2)(D(x_2)X_n''(x_2))''dx_2 = X_n(x_2)(D(x_2)X''(x_2))'\Big|_0^l \\ &- \int_0^l X_n'(x_2)(D(x_2)X_n''(x_2))'dx_2 \\ &\quad \text{(by virtue of the BCs (2.10))} \\ &= -\int_0^l X_n'(x_2)(D(x_2)X_n''(x_2))'dx_2 = X_n'(x_2)(D(x_2)X''(x_2))\Big|_0^l \\ &+ \int_0^l D(x_2)(X_n'')^2(x_2)dx_2 = \int_0^l D(x_2)(X_n'')^2(x_2)dx_2 \geq 0.\end{aligned}$$

Hence, $\lambda_n > 0$ for any n, since in non trivial case $X_n \not\equiv 0$. □

We can write the solution of (2.8) as follows

$$T_n(t) = b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right), \quad b_i^n = \text{const}, \quad i = 1, 2.$$

Now, we can find a solution of the Problem 11 in the form as follows

$$w(x_2,t)=\sum_{n=1}^{\infty}\frac{Y_n(x_2)}{\sqrt{g(x_2)}}\left(b_1^n\sin\left(\sqrt{\lambda_n}t\right)+b_2^n\cos\left(\sqrt{\lambda_n}t\right)\right) \tag{2.20}$$

or taking into account (2.15) in the following form

$$w(x_2,t)=\sum_{n=1}^{\infty}X_n(x_2)\left(b_1^n\sin\left(\sqrt{\lambda_n}t\right)+b_2^n\cos\left(\sqrt{\lambda_n}t\right)\right). \tag{2.21}$$

In view of initial conditions (2.2), we formally have

$$\sum_{n=1}^{\infty}Y_n(x_2)b_2^n=\varphi_1(x_2)\sqrt{g(x_2)},\ \ \sum_{n=1}^{\infty}\sqrt{\lambda_n}Y_n(x_2)b_1^n=\varphi_2(x_2)\sqrt{g(x_2)}. \tag{2.22}$$

If $\psi_i(x_2):=\frac{(D\varphi_i'')''}{\sqrt{g(x_2)}}\in C[0,l]$, $(i=1,2)$, then after integration of the last expression, $\sqrt{g(x_2)}\varphi_i(x_2)$ can be expressed as follows

$$\sqrt{g(x_2)}\varphi_i(x_2)=\int_0^l\sqrt{g(x_2)g(\xi)}K(x_2,\xi)\psi_i(\xi)d\xi$$

i.e.,

$$\sqrt{g(x_2)}\varphi_i(x_2)=\int_0^l R(x_2,\xi)\psi_i(\xi)d\xi.$$

Evidently, (2.22) series will be absolutely and uniformly convergent on $]0,l[$. Since there exists positive minimum of eigenvalues, from the convergence of the second series follows absolute and uniform convergence on $]0,l[$ of the series $\sum\limits_{n=1}^{N}X_n(x_2)b_1^n$

Hence, by virtue of Theorem 2.2, since $\psi_i(\xi)\in C([0,l])$ and symmetric $R(x_2,\xi)\in C(]0,l[\times]0,l[)$, we get absolutely and uniformly convergence of the series (2.22) on $[0,l]$, and

$$b_1^n=\frac{1}{\sqrt{\lambda_n}}\int_0^l g(x_2)X_n(x_2)\varphi_2(x_2)dx_2,\ \ b_2^n=\int_0^l g(x_2)X_n(x_2)\varphi_1(x_2)dx_2. \tag{2.23}$$

Further, taking into account (2.12) $X(x_2)\in C([0,l])$, then, by virtue of (2.15), we can rewrite (2.20) as follows

$$\varphi_i(x_2)=\sum_{n=1}^{\infty}X_n(x_2)b_j^n,\quad i,j=1,2\ \ i\neq j.$$

Therefore, the series (2.20) is absolutely and uniformly convergent on $]0,l[$.

After formal differentiation of (2.20) with respect to t we get

$$w_{,t}(x_2,t)=\sum_{n=1}^{\infty}X_n(x_2)\sqrt{\lambda_n}\left(b_1^n\cos(\sqrt{\lambda_n}t)-b_2^n\sin(\sqrt{\lambda_n}t)\right), \qquad (2.24)$$

$$w_{,tt}(x_2,t)=-\sum_{n=1}^{\infty}X_n(x_2)\lambda_n\left(b_1^n\sin(\sqrt{\lambda_n}t)+b_2^n\cos(\sqrt{\lambda_n}t)\right). \qquad (2.25)$$

Theorem 2.7 *(2.24) and (2.25) converge absolutely and uniformly on* $]0,l[$ *if*

$$\Psi_i(x_2):=\frac{\psi_i(x_2)}{\sqrt{g(x_2)}},\quad i=1,2, \qquad (2.26)$$

are satisfying conditions (2.6) and

$$\chi_i(x_2)\sqrt{g(x_2)}:=\left(D(x_2)\Psi_i''(x_2)\right)'',\quad i=1,2, \qquad (2.27)$$

are integrable functions on $]0,l[$ *(for this, e.g., it is sufficient that* $\varphi_i^{(j)}(x_2)=O(x_2^{\gamma_{ij}})$, $x_2\to 0_+$, $\gamma_{ij}=\text{const}>7-j-\frac{5\alpha}{3}$, $\varphi_i^{(j)}(x_2)=O((l-x_2)^{\delta_{ij}})$, $x_2\to l_-$, $\delta_{ij}=\text{const}>7-j-\frac{5\beta}{3}$, $i=1,2;\ j=\overline{2,8}$*).*

Proof. Substituting in (2.23) the function $g(x_2)X_n(x_2)$ found from (2.19), we get

$$b_1^n = \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l (D(x_2)X_n''(x_2))''\varphi_2(x_2)dx_2$$

(after integrating by parts 4-times, taking into account BCs (2.4), (2.6), (2.10) and (2.15))

$$=\frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{(D(x_2)X_n''(x_2))'\varphi_2(x_2)\Big|_0^l-\int_0^l (D(x_2)X_n''(x_2))'\varphi_2'(x_2)dx_2\right\}$$

$$=\frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{-D(x_2)X_n''(x_2)\varphi_2'(x_2)\Big|_0^l+\int_0^l D(x_2)X_n''(x_2)\varphi_2''(x_2)dx_2\right\}$$

$$\frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l X_n''(x_2)(D(x_2)\varphi_2''(x_2))dx_2$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\Bigg\{ X_n'(x_2)(D(x_2)\varphi''(x_2))|_0^l$$

$$- \int_0^l X_n'(x_2)(D(x_2)\varphi_2''(x_2))'dx_2 \Bigg\}$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\Bigg\{ - X_n(x_2)(D(x_2)\varphi_2''(x_2))'|_0^l$$

$$+ \int_0^l X_n(x_2)(D(x_2)\varphi_2''(x_2))''dx_2 \Bigg\}$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}} \int_0^l X_n(x_2)(D(x_2)\varphi_2''(x_2))''dx_2$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}} \int_0^l Y_n(x_2)\psi_2(x_2)dx_2. \qquad (2.28)$$

Analogously,

$$b_2^n = \frac{1}{\lambda_n}\int_0^l Y_n(x_2)\psi_1(x_2)dx_2.$$

In view of (2.27), $\Psi_i(x_2)$ can be expressed as follows

$$\Psi_i(x_2) = \int_0^l K(x_2,\xi)\sqrt{g(\xi)}\chi_i(\xi)d\xi, \ \ i = 1,2,$$

and by virtue of (2.26), (2.15) we obtain

$$\psi_i(x_2) = \int_0^l R(x_2,\xi)\chi_i(\xi)d\xi, \ \ i = 1,2.$$

According to the Theorem 2.2, the following series

$$\sum_{n=1}^{\infty} \beta_i^n Y_n(x_2),$$

where

$$\beta_i^n = \int_0^l Y_n(x_2)\psi_i(x_2)dx_2, \ \ i = 1,2, \qquad (2.29)$$

is convergent absolutely and uniformly on $]0,l[$, i.e.,

$$\sum_{n=1}^{\infty}|\beta_i^n||Y_n(x_2)|<+\infty, \tag{2.30}$$

Further, from (2.24)

$$\begin{aligned}
|w_{,t}(x_2,t)| &= \left|\sum_{n=1}^{\infty}X_n(x_2)\sqrt{\lambda_n}\left(b_1^n\cos(\sqrt{\lambda_n}t)-b_2^n\sin(\sqrt{\lambda_n}t)\right)\right| \\
&\leq \left|\sum_{n=1}^{\infty}X_n(x_2)\sqrt{\lambda_n}b_1^n\cos(\sqrt{\lambda_n}t)\right| \\
&\quad+\left|\sum_{n=1}^{\infty}X_n(x_2)\sqrt{\lambda_n}b_2^n\sin(\sqrt{\lambda_n}t)\right| \\
&\leq \sum_{n=1}^{\infty}\left|X_n(x_2)\sqrt{\lambda_n}b_1^n\right|+\sum_{n=1}^{\infty}\left|X_n(x_2)\sqrt{\lambda_n}b_2^n\right|. \qquad (2.31)
\end{aligned}$$

According to Proposition 2.6, all of λ_n are positive. Therefore, we can find λ_0 such that $\lambda_0\leq\min\limits_{1\leq i\leq\infty}\{\lambda_i\}$, and by virtue of (2.15), (2.28)-(2.30), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|X_n(x_2)\sqrt{\lambda_n}b_2^n\right| &= \frac{1}{\sqrt{g(x_2)}}\sum_{n=1}^{\infty}\left|Y_n\sqrt{\lambda_n}\frac{1}{\lambda_n}\beta_1^n\right| \\
&\leq \frac{1}{\sqrt{\lambda_0}}\frac{1}{\sqrt{g(x_2)}}\sum_{n=1}^{\infty}|Y_n||\beta_1^n|<\infty, \\
\sum_{n=1}^{\infty}\left|X_n(x_2)\sqrt{\lambda_n}b_1^n\right| &= \frac{1}{\sqrt{g(x_2)}}\sum_{n=1}^{\infty}\left|Y_n\sqrt{\lambda_n}\frac{1}{\lambda_n\sqrt{\lambda_n}}\beta_2^n\right| \\
&\leq \frac{1}{\lambda_0}\frac{1}{\sqrt{g(x_2)}}\sum_{n=1}^{\infty}|Y_n||\beta_2^n|<\infty.
\end{aligned}$$

Hence, the series in (2.31) are convergent. Thus, (2.24) is convergent absolutely and uniformly on $]0,l[$. Similarly, we get the absolute and uniform convergence of (2.25) on $]0,l[$. $\square$

Let us now differentiate formally (2.20) i-times and consider the following expressions

$$w_{x_2}^{(i)}(x_2,t)=\sum_{n=1}^{\infty}X_n^{(i)}(x_2)\left(b_1^n\sin(\sqrt{\lambda_n}t)+b_2^n\cos(\sqrt{\lambda_n}t)\right), \quad i=1,2,3,4, \tag{2.32$_\mathrm{i}$}$$

$$(D(x_2)w,_{x_2x_2}(x_2,t))^{(i-1)}_{x_2} = \sum_{n=1}^{\infty}(D(x_2)X''_n(x_2))^{(i-1)}\left(b_1^n \sin(\sqrt{\lambda_n}t)\right.$$

$$\left.+b_2^n \cos(\sqrt{\lambda_n}t)\right),\ i=1,2 \tag{2.33$_\mathrm{i}$}$$

Theorem 2.8 *The series* (2.32_i) $(i = 1, ..., 4)$ *are convergent absolutely and uniformly on* $]0, l[$. *The series* (2.33_i) $(i = 1, 2)$ *are convergent absolutely and uniformly on* $[0, l]$.

Proof. Obviously, in view of (2.10), after integration of (2.19), we get

$$X'_n(x_2) = \lambda_n \int\limits_0^l R_1(x_2,\xi)X_n(\xi)d\xi, \tag{2.34}$$

where

$$R_1(x_2,\xi) = \begin{cases} \xi \int\limits_{x_2}^{l} D^{-1}(\eta)d\eta, \ \ 0 \le \xi \le x_2, \\ -\int\limits_{\xi}^{x_2} \eta D^{-1}(\eta)d\eta + \xi \int\limits_{\xi}^{l} D^{-1}(\eta)d\eta, \ \ x_2 \le \xi \le l, \end{cases}$$

and

$$R_1(x_2,\xi) \in C([0,l]\times[0,l]), \tag{2.35}$$

because of $0 \le \alpha < 2$, $0 \le \beta < 1$.

Substituting (2.34) into (2.32_1), we obtain

$$w'_{x_2}(x_2,t) = \sum_{n=1}^{\infty}\lambda_n \int\limits_0^l R_1(x_2,\xi)X_n(\xi)d\xi\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right) =$$

$$= \int\limits_0^l R_1(x_2,\xi)\left[\sum_{n=1}^{\infty}X_n(\xi)\lambda_n\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right)\right]d\xi, \tag{2.36}$$

since (2.25) is absolutely and uniformly convergent on $]0, l[$ and in view of (2.35) and $X_n(x_2) \in C([0, l])$ we conclude that the corresponding integral in (2.36) is absolutely convergent on $]0, l[$. Similarly, we can prove the convergence of the series (2.32_2), (2.32_3), (2.32_4), on $]0, l[$ and (2.33_i) $(i = 1, 2)$ on $[0, l]$. □

After substituting (2.12) into (2.21) we have

$$w(x_2,t)=\int\limits_0^l K(x_2,\xi)\left[\sum_{n=1}^{\infty}\lambda_n X_n(\xi)\left(b_1^n\sin(\sqrt{\lambda_n}t)+b_2^n\cos(\sqrt{\lambda_n}t)\right)\right]d\xi. \tag{2.37}$$

Since the integrand series is convergent absolutely and uniformly on $]0,l[$, $X_n(\xi)\in C([0,l])$ and $K(x_2,\xi)\in C([0,l]\times[0,l])$, we obtain that the right hand side of (2.37) is convergent absolutely and uniformly on $[0,l]$.

Thus, (2.20) is the solution of the Problem 11 in the case $q(x_2,t)\equiv 0$.

Now, let us consider Problem 11 when $q(x_2,t)\not\equiv 0$, $\varphi_i=0$, and let $\frac{q}{\sqrt{g}}(\cdot,t)\in L_2(0,l)$. Then $q(x_2,t)$ can be represented as convergent series in $L_2(0,l)$:

$$q(x_2,t)=\sum_{n=1}^{\infty}g(x_2)X_n(x_2)q_n(t),\quad q_n(t):=\int\limits_0^l q(x_2,t)X_n(x_2)dx_2.$$

Further, we look for the solution in the form

$$w(x_2,t)=\sum_{n=1}^{\infty}w_n(x_2,t),$$

where $w_n(x_2,t)$ is a solution of the Problem 11 with $q(x_2,t)$ replaced by $g(x_2)X_n(x_2)q_n(t)$. Now, we will look for w by the method of separation of variables

$$w_n(x_2,t)=X_n(x_2)T_{1n}(t),$$

where

$$T_{1n}''(t)+\lambda_n T_{1n}(t)=q_n(t)$$

and $X_n(x_2)$ satisfies (2.12).

Therefore, $w(x_2,t)$ can be expressed as follows

$$w(x_2,t)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}X_n\int\limits_0^t\sin(\sqrt{\lambda_n}(t-\tau))q_n(\tau)d\tau. \tag{2.38}$$

Now, similarly to the proofs of Theorems 2.9 and 2.10, if the following conditions are fulfilled

$$\tau(x_2,t):=\frac{1}{\sqrt{g(x_2)}}\left(D(x_2)\left(\frac{q(x_2,t)}{g(x_2)}\right)_{,x_2x_2}\right)_{,x_2x_2}\in C[0,l],$$

$$\frac{\tau}{\sqrt{g}}(0,t) = -D(x_2)\left(\frac{\tau(x_2,t)}{\sqrt{g(x_2)}}\right)_{,x_2x_2}\Bigg|_{x_2=0_+} = \left(\frac{\tau(x_2,t)}{\sqrt{g(x_2)}}\right)_{,x_2}\Bigg|_{x_2=l} \tag{2.39}$$

$$= \left(-D(x_2)\left(\frac{\tau(x_2,t)}{\sqrt{g(x_2)}}\right)_{,x_2x_2}\right)_{,x_2}\Bigg|_{x_2=l_-} = 0,$$

(for this, e.g., it is sufficient that $q^{(j)}(\cdot,t) = O(x_2^{\gamma_j})\ x_2 \to 0_+$, $\gamma_j > 7-j-\frac{2\alpha}{3}$, $q^{(j)}(\cdot,t) = O((l-x_2)^{\delta_j})\ x_2 \to l_-$, $\gamma_j > 7-j-\frac{2\beta}{3}$, $j=\overline{0,8}$) we have the absolute and uniform convergence of the series (2.38) and

$$(D(x_2)w_{,x_2x_2}(x_2,t))^{(i)}_{x_2} = \sum_{n=1}^{\infty}(D(x_2)X_n'')^{(i)}(x_2)T_{1n}(t), \quad i=0,1,$$

on $[0,l]$, and the absolute and uniform convergence of the series

$$w^{(i)}_{x_2}(x_2,t) = \sum_{n=1}^{\infty}X_n^{(i)}(x_2)T_{1n}(t), \quad i=1,...,4,$$

$$w^{(i)}_{t}(x_2,t) = \sum_{n=1}^{\infty}X_n(x_2)T_{1n}^{(i)}(t), \quad i=1,2,$$

on $]0,l[$.

Remark 2.9 *Solution of the Problem 11 in case $q(x_2,t)$, $\varphi_i \not\equiv 0$ can be expressed as follows*

$$w(x_2,t) = \sum_{n=1}^{\infty}w_n(x_2,t),$$

where

$$w_n(x_2,t) = X_n(x_2)(T_{1n}(t)+T_n(t)).$$

Remark 2.10 *Similarly, we can solve the following initial boundary value problems which correspond to the Problems 1-7, 9, 10.*

Problem 12 *Let $0 \le \alpha,\ \beta < 1$. Find*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]),$$

satisfying equation (2.1), the BCs

$$w(0,t) = w_{,2}(0,t) = w(l,t) = w_{,2}(l,t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l])$$

$$\varphi_i(0) = \varphi_i'(x_2)|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(x_2) = 0, \quad i=1,2.$$

Problem 13 *Let* $0 \le \alpha,\ \beta < 1$. *Find*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]),$$

satisfying equation (2.1), the BCs

$$w(0,t) = w,_2(0,t) = w,_2(l,t) = Q_2(l,t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l])$$

$$\varphi_i(0) = \varphi_i'(x_2)|_{x_2=0_+} = \varphi_i'(l) = \big(-D(x_2)\varphi_i''(x_2)\big)'|_{x_2=l_-} = 0,\ i=1,2.$$

Problem 14 *Let* $0 \le \alpha, < 1,\ 0 \le \beta < 2$. *Find*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

satisfying equation (2.1), the BCs

$$w(0,t) = w,_2(0,t) = w(l,t) = M_2(l,t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

$$\varphi_i(0) = \varphi_i'(x_2)|_{x_2=0_+} = \varphi_i(l) = \big(-D(x_2)\varphi_i''(x_2)\big)|_{x_2=l_-} = 0,\ i=1,2.$$

Problem 15 *Let* $0 \le \alpha < 1,\ \beta \ge 0$. *Find*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[),$$

satisfying equation (2.1), the BCs

$$w(0,t) = w,_2(0,t) = M,_2(l,t) = Q_2(l,t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[),$$

$$\begin{aligned}\varphi_i(0) &= \varphi_i'(x_2)|_{x_2=0_+} = \big(-D(x_2)\varphi_i''(x_2)\big)|_{x_2=l_-} \\ &= \big(-D(x_2)\varphi_i''(x_2)\big)'|_{x_2=l_-} = 0,\ i=1,2.\end{aligned}$$

Problem 16 *Let* $0 \leq \alpha,\ \beta < 1$. *Find*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]),$$

satisfying equation (2.1), the BCs

$$w,_2(0, t) = Q_2(0, t) = w(l, t) = w,_2(l, t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$

$$\varphi_i'(x_2)|_{x_2=0_+} = \left(-D(x_2)\varphi_i''(x_2)\right)'|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i = 1, 2.$$

Problem 17 *Let* $0 \leq \alpha < 1,\ 0 \leq \beta < 2$. *Find*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l[) \cap C([0, l]),$$

satisfying equation (2.1), the BCs

$$w,_2(0, t) = Q_2(0, t) = w(l, t) = M_2(l, t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l[) \cap C([0, l]),$$

$$\begin{aligned} \varphi_i'(x_2)|_{x_2=0_+} &= \left(-D(x_2)\varphi_i''(x_2)\right)'|_{x_2=0_+} = \varphi_i(l) \\ &= \left(-D(x_2)\varphi_i''(x_2)\right)|_{x_2=l_-} = 0,\ i = 1, 2. \end{aligned}$$

Problem 18 *Let* $0 \leq \alpha < 2,\ 0 \leq \beta < 1$. *Find*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1(]0, l]) \cap C([0, l]),$$

satisfying equation (2.1), the BCs

$$w(0, t) = M_2(0, t) = w(l, t) = w,_2(l, t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l]) \cap C([0, l]),$$

$$\varphi_i(0) = \left(-D(x_2)\varphi_i''(x_2)\right)|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i = 1, 2.$$

Problem 19 *Let $0 \le \alpha,\ \beta < 2$. Find*

$$w(\cdot, t) \in C^4(]0, l[) \cap C([0, l]),$$

satisfying equation (2.1), the BCs

$$w(0, t) = M_2(0, t) = w(l, t) = M_2(l, t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C([0, l]),$$

$$\begin{aligned}\varphi_i(0) &= \left(-D(x_2)\varphi_i''(x_2)\right)|_{x_2=0_+} = \varphi_i(l)\\ &= \left(-D(x_2)\varphi_i''(x_2)\right)|_{x_2=l_-} = 0,\ i = 1, 2.\end{aligned}$$

Problem 20 *Let $\alpha \ge 0,\ 0 < \beta < 1$. Find*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1(]0, l]),$$

satisfying equation (2.1), the BCs

$$M_2(0, t) = Q_2(0, t) = w(l, t) = w,_2(l, t) = 0,$$

and ICs (2.2), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l]),$$

$$\begin{aligned}\left(-D(x_2)\varphi_i''(x_2)\right) &= \left(-D(x_2)\varphi_i''(x_2)\right)'|_{x_2=0_+}\\ &= \varphi_i(l) = \varphi_i'(l) = 0,\ i = 1, 2.\end{aligned}$$

In all these cases we get integral equations with symmetric kernels.

We can avoid the restrictions (2.39) on $q(x_2, t)$ if we consider harmonic vibration. In this case

$$w(x_2, t) = e^{i\omega t} w_0(x_2),\quad q(x_2, t) = e^{i\omega t} q_0(x_2),$$

where $\omega = const$ is an oscillation frequency, $q_0(x_2) \in C([0, l])$ is a given function. Now, for $w_0(x_2)$ from (2.1), (1.11) we get the following problem

$$\begin{aligned}\left(D(x_2)w_0''(x_2)\right)'' &= q_0(x_2) + 2\omega^2 \rho h(x_2) w_0(x_2),\\ w_0(0) = M_2(0) &= w'(l) = Q_2(l) = 0,\quad 0 \le \alpha < 2,\ 0 \le \beta < 1,\end{aligned}$$

$$w_0(x_2) \in C^4(]0, l[) \cap C([0, l]) \cap C^1(]0, l]).$$

This problem is equivalent to the integral equation

$$w_0(x_2) - \omega^2 \int_0^l K(x_2,\xi)\, g(\xi)\, w_0(\xi) d\xi = F(x_2), \tag{2.40}$$

where

$$F(x_2) := \int_0^l K(x_2,\xi)\, q_0(\xi) d\xi.$$

Introducing a new unknown function

$$w_1(x_2) = w_0(x_2)\sqrt{g(x_2)}$$

we can reduce (2.40) to the following integral equation

$$w_1(x_2) - \omega^2 \int_0^l R(x_2,\xi)\, w_1(\xi) d\xi = F(x_2)\sqrt{g(x_2)} \tag{2.41}$$

with $R(x_2,\xi)$ given by (2.15). If $\omega^2 \neq \lambda_n$, the unique solution of (2.41) can be written as follows (see, e.g., [6], Theorem XVIII, p.140)

$$\begin{aligned} w_1(x_2) &= F(x_2)\sqrt{g(x_2)} \\ &+\omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi)\, \sqrt{g(x_2)}\, Y_n(\xi) d\xi \right] Y_n(x_2), \end{aligned} \tag{2.42}$$

where the series in the right hand side of (2.42) is absolutely and uniformly convergent on $[0,l]$.

3 *A Cusped Elastic Plate-Fluid Interaction Problem*

Let us consider the problem of the interaction of a plate whose variable flexural rigidity is given by the equation (1.3) and of a flow of the fluid.

Let the flow of the fluid be independent of x_1, parallel to the plane $0x_2x_3$, i.e. $v_1 \equiv 0$, and generating bending of the plate. Let at infinity,

$$v_2(x_2,x_3,t) = O(1), \quad v_3(x_2,x_3,t) \to v_{3\infty}(t), \tag{3.1}$$

$$p(x_2,x_3,t) \to p_\infty(t), \quad \text{when } |x| \to \infty, \tag{3.2}$$

where $v := (v_2, v_3)$ is a velocity vector of the fluid, $p(x_2,x_3,t)$ is a pressure, and $v_{3\infty}(t)$, $p_\infty(t)$ are given functions.

Further, if the plate is thin, we can assume that:

– the fluid occupies the whole space R^3 but the middle plane Ω of the plate, i.e., $\Omega^f = R^3 \backslash \Omega$.

– transmission conditions for $v_3(x_2, x_3, t)$ has the following form (see [1], [2], [8], [13])

$$v_3(x_2, 0, t) = \frac{\partial w(x_2, t)}{\partial t}, \quad x_2 \in [0, l], \quad t \geq 0. \tag{3.3}$$

For an ideal fluid we have (see e.g., [2], p.7; [5])

$$\sigma_{ij}^f = -p\delta_{ij}.$$

Therefore, the transmission condition for p has the following form

$$-p(x_2, \overset{(+)}{h}(x_2), t) + p(x_2, \overset{(-)}{h}(x_2), t) = q(x_2, t), \quad x_2 \in [0, l]. \tag{3.4}$$

In case of the potential motion of the flow there exists a complex function $\Phi = \psi + i\varphi$ such that

$$\begin{aligned} \frac{\partial \varphi(x_2, x_3, t)}{\partial x_2} &= \frac{\partial \psi(x_2, x_3, t)}{\partial x_3} = v_2(x_2, x_3, t), \\ \frac{\partial \varphi(x_2, x_3, t)}{\partial x_3} &= -\frac{\partial \psi(x_2, x_3, t)}{\partial x_2} = v_3(x_2, x_3, t). \end{aligned} \tag{3.5}$$

The pressure is given by the formula

$$p(x_2, x_3, t) = \rho^f \left[\frac{v_\infty^2}{2} + \frac{p_\infty}{\rho^f} + \frac{\partial \varphi_\infty}{\partial t} - \frac{\partial \varphi}{\partial t} - \frac{1}{2}(v_2^2 + v_3^2) \right]. \tag{3.6}$$

In case under consideration $w(x_2, t)$ is given by the equation (2.1).

Taking into account transmission condition (3.4), we have

$$\left(x_2^\alpha (l - x_2)^\beta w,_{22}(x_2, t) \right),_{22} = -\frac{2h(x_2)\rho^s}{D_0} w,_{tt}(x_2, t) \tag{3.7}$$

$$+ \frac{-p\left(x_2, \overset{(+)}{h}(x_2), t\right) + p\left(x_2, \overset{(-)}{h}(x_2), t\right)}{D_0}.$$

For $\Phi,_2(x_2, x_3, t) = -v_3 + iv_2$, in view of (3.3) and (3.1), we get the following expression (see [11])

$$\Phi_{,2} = -\frac{1}{\pi i\sqrt{(x_2+ix_3)(x_2+ix_3-l)}}$$
$$\times\int\limits_0^l \tfrac{\sqrt{(\xi_2+ix_3)(\xi_2+ix_3-l)}}{(\xi_2-x_2)-ix_3} w_{,t}(\xi_2,t)d\xi_2$$

$$+v_{3\infty}\frac{x_2+ix_3-l/2}{\sqrt{(x_2+ix_3)(x_2+ix_3-l)}}. \qquad (3.8)$$

Let

$$w(x_2,t) = e^{i\omega t}w_0(x_2),\ \ q(x_2,t) = e^{i\omega t}q_0(x_2), \qquad (3.9)$$

$$\begin{gathered} p(x_2,x_3,t) = e^{i\omega t}p_0(x_2,x_3), \\ u_2(x_2,x_3,t) = e^{i\omega t}u_2^0(x_2,x_3),\ \ u_3(x_2,x_3,t) = e^{i\omega t}u_3^0(x_2,x_3), \end{gathered} \qquad (3.10)$$

where $\omega = \text{const} > 0$, $v_2 = u_{2,t}$ $(v_3 = u_{3,t})$. Further,

$$\varphi(x_2,x_3,t) = ie^{i\omega t}\varphi_0(x_2,x_3),\ \ \ \psi(x_2,x_3,t) = ie^{i\omega t}\psi_0(x_2,x_3),$$

$$v_2(x_2,x_3,t) = ie^{i\omega t}v_2^0(x_2,x_3),\ \ v_3(x_2,x_3,t) = ie^{i\omega t}v_3^0(x_2,x_3),$$

$$p_\infty(t) = e^{i\omega t}p_\infty^0,\ \ \ v_{3\infty}(t) = ie^{i\omega t}v_{3\infty}^0,\ \ p_\infty^0,\ v_{3\infty}^0 = \text{const}.$$

From (3.8) we have expression for v_3. By means of the latter, in view of (3.5), we can calculate φ which we have to substitute in (3.6). Then substituting the obtained expression of $p(x_2,x_3,t)$ in (3.4), by virtue of (3.9), we get the following expression for $q_0(x_2)$

$$q_0(x_2) = \frac{\omega^2\rho^f}{\pi}\int\limits_0^l w_0(\xi)\int\limits_{-\overset{(+)}{h}(x_2)}^{\overset{(+)}{h}(x_2)}\frac{\sqrt{r(\xi,x_3)}}{\sqrt{r(x_2,x_3)}} \qquad (3.11)$$

$$\times\frac{(x_2-\xi)cos[(\phi(\xi,x_3)-\phi(x_2,x_3))/2]+x_3sin[(\phi(\xi,x_3)-\phi(x_2,x_3))/2]}{(\xi-x_2)^2+x_3^2}dx_3d\xi$$

$$-\omega^2\rho^f\int\limits_{-\overset{(+)}{h}(x_2)}^{\overset{(+)}{h}(x_2)}\left\{(x_2-l/2)cos\frac{\phi(x_2,x_3)}{2}+x_3sin\frac{\phi(x_2,x_3)}{2}\right\}\frac{v_{3\infty}^0dx_3}{\sqrt{r(x_2,x_3)}},$$

where $\phi(x_2, x_3)$ is defined either by

$$cos\phi(x_2, x_3) = (x_2^2 - x_3^2 - lx_2)/r(x_2, x_3)$$

or by

$$sin\phi(x_2, x_3) = (2x_2 - l)x_3/r(x_2, x_3)$$

and

$$r(x_2, x_3) = \sqrt{(x_2^2 - x_3^2 - lx_2)^2 + ((2x_2 - l)x_3)^2}.$$

Taking into account (3.9), (3.10), (3.11), from (3.7) after integrating four times with respect to x_2 we get the following relation

$$\begin{aligned} w_0(x_2) &- 2\rho^s\omega^2\int_{x_2^0}^{x_2} h(\xi)K(x_2,\xi)w_0(\xi)d\xi = \int_{x_2^0}^{x_2} (c_1\xi + c_2)(x_2-\xi)D^{-1}(\xi)d\xi \\ &- c_3x_2 + c_4 + \int_{x_2^0}^{x_2} K(x_2,\xi)q_0(\xi)d\xi, \end{aligned} \tag{3.12}$$

where

$$x_2^0 \in]0, l[, \quad K(x_2,\xi) = -\int_{\xi}^{x_2} (x_2-\eta)(\xi-\eta)D^{-1}(\eta)d\eta.$$

Constants c_i $(i = 1, ..., 4)$ should be defined from the admissible boundary value conditions (see in Section 1 Problems 1-10).

Let us consider, e.g., boundary conditions (1.11). Then for $w_0(x_2)$ we get the following equation

$$\begin{aligned} w_0(x_2) &- \omega^2\int_0^l K_1(x_2,\xi)w_0(\xi)d\xi \\ &- 2\rho^s\omega^2\left\{\int_{x_2^0}^{x_2} h(\xi)K(x_2,\xi)w_0(\xi)d\xi + \int_{x_2^0}^{l} h(\xi)K_l(x_2,\xi)w_0(\xi)d\xi\right. \\ &+ \left.\int_0^{x_2^0} h(\xi)K_0(x_2,\xi)w_0(\xi)d\xi\right\} \\ &= f(x_2), \end{aligned} \tag{3.13}$$

where

$$K_0(x_2,\xi) = \xi\left\{\int\limits_l^{x_2} x_2 D^{-1}(\eta)d\eta - \int\limits_0^{x_2} \eta D^{-1}(\eta)d\eta\right\} - K(0,\xi),$$

$$K_l(x_2,\xi) = x_2\int\limits_l^{x_2} \eta D^{-1}(\eta)d\eta$$

$$-\int\limits_0^{x_2} \eta^2 D^{-1}(\eta)d\eta + x_2\int\limits_\xi^l (\eta-\xi)D^{-1}(\eta)d\eta,$$

$$K_1(x_2,\xi) = \frac{\rho^f}{\pi}\left\{\int\limits_{x_2^0}^{l} K_l(x_2,\zeta)\int\limits_{-\overset{(+)}{h}(\zeta)}^{\overset{(+)}{h}(\zeta)} \frac{\sqrt{r(\xi,x_3)}}{\sqrt{r(\zeta,x_3)}}\right.$$

$$\times\frac{(\zeta-\xi)cos[(\phi(\xi,x_3)-\phi(\zeta,x_3))/2] + x_3 sin[(\phi(\xi,x_3)-\phi(\zeta,x_3))/2]}{(\xi-\zeta)^2+x_3^2}dx_3 d\zeta$$

$$+\int\limits_{x_2^0}^{0} K_0(x_2,\zeta)\int\limits_{-\overset{(+)}{h}(\zeta)}^{\overset{(+)}{h}(\zeta)} \frac{\sqrt{r(\xi,x_3)}}{\sqrt{r(\zeta,x_3)}}$$

$$\times \quad \frac{(\zeta-\xi)cos[(\phi(\xi,x_3)-\phi(\zeta,x_3))/2] + x_3 sin[(\phi(\xi,x_3)-\phi(\zeta,x_3))/2]}{(\xi-\zeta)^2+x_3^2}dx_3 d\zeta$$

$$+\int\limits_{x_2^0}^{x_2} K(x_2,\zeta)\int\limits_{-\overset{(+)}{h}(\zeta)}^{\overset{(+)}{h}(\zeta)} \frac{\sqrt{r(\xi,x_3)}}{\sqrt{r(\zeta,x_3)}}$$

$$\left.\times \quad \frac{(\zeta-\xi)cos[(\phi(\xi,x_3)-\phi(\zeta,x_3))/2] + x_3 sin[(\phi(\xi,x_3)-\phi(\zeta,x_3))/2]}{(\xi-\zeta)^2+x_3^2}dx_3 d\zeta\right\};$$

$$f(x_2) = x_2\left(g_{22} + h_{22}\int\limits_{x_2^0}^{l} \xi D^{-1}(\xi)d\xi + h_{11}\int\limits_{x_2^0}^{l} D^{-1}(\xi)d\xi\right) + g_{11} + h_{22}\int\limits_0^{x_2^0} \xi^2 D^{-1}(\xi)d\xi$$

$$-h_{11}\int\limits_0^{x_2^0} \xi D^{-1}(\xi)d\xi - \int\limits_{x_2^0}^{x_2}(h_{22}\xi + h_{11})(x_2-\xi)D^{-1}(\xi)d\xi - \omega^2\rho^f\left\{\int\limits_{x_2^0}^{l} K_l(x_2,\xi)\right.$$

$$\times \int\limits_{-\overset{(+)}{h}(\xi)}^{\overset{(+)}{h}(\xi)} \left\{(\xi - l/2)cos\frac{\phi(\xi,x_3)}{2} + x_3 sin\frac{\phi(\xi,x_3)}{2}\right\} \frac{v^0_{3\infty}dx_3}{\sqrt{r(\xi,x_3)}}d\xi$$

$$- \int\limits_{x_2^0}^{0} K_0(x_2,\xi) \int\limits_{-\overset{(+)}{h}(\xi)}^{\overset{(+)}{h}(\xi)} \left\{(\xi - l/2)cos\frac{\phi(\xi,x_3)}{2}\right.$$

$$+ \left. x_3 sin\frac{\phi(\xi,x_3)}{2}\right\} \frac{v^0_{3\infty}dx_3}{\sqrt{r(\xi,x_3)}}d\xi$$

$$\left. - \int\limits_{x_2^0}^{x_2} K_0(x_2,\xi) \int\limits_{-\overset{(+)}{h}(x_2)}^{\overset{(+)}{h}(x_2)} \left\{(\xi - l/2)cos\frac{\phi(\xi,x_3)}{2} + x_3 sin\frac{\phi(\xi,x_3)}{2}\right\} \frac{v^0_{3\infty}dx_3}{\sqrt{r(\xi,x_3)}}d\xi \right\}.$$

It is easy to show that $2\rho^s h(\xi)K(x_2,\xi)$, $2\rho^s h(\xi)K_0(x_2,\xi)$, $2\rho^s h(\xi)K_l(x_2,\xi)$, $K_1(x_2,\xi) \in C([0,l])$ (in our case $0 \leq \alpha < 2$, $0 \leq \beta < 1$).

The integral equation (3.13) can be solved by method of successive approximations.

Remark 3.1 *In case of other boundary conditions above (see problems 1-7, 9, 10), the problem under consideration is solved analogously and in all cases we get integral equations of (3.13) type.*

Below we give expressions for kernels K_0 and K_l under BCs of Problem 1-7, 9, 10.

Problem 1.

$$K_0(x_2,\xi) = -K(0,\xi) + \left\{ \frac{\left(-K(0,\xi) + l\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta\right)\int\limits_0^l \eta^2 D^{-1}(\eta)d\eta}{\int\limits_0^l \eta^2 D^{-1}(\eta)d\eta \int\limits_0^l D^{-1}(\eta)d\eta - \left(\int\limits_0^l \eta D^{-1}(\eta)d\eta\right)^2} \right.$$

$$\left. - \frac{\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta \int\limits_0^l \eta D^{-1}(\eta)d\eta}{\int\limits_0^l \eta^2 D^{-1}(\eta)d\eta \int\limits_0^l D^{-1}(\eta)d\eta - \left(\int\limits_0^l \eta D^{-1}(\eta)d\eta\right)^2} \right\} \int\limits_0^l \eta(x_2-\eta)D^{-1}(\eta)d\eta$$

$$
+\left\{-\frac{\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta\int\limits_{0}^{l}\eta^2D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2D^{-1}(\eta)d\eta\int\limits_{0}^{l}D^{-1}(\eta)d\eta-\left(\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta\right)^2}\right.
$$

$$
\left.+\frac{K(0,\xi)\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2D^{-1}(\eta)d\eta\int\limits_{0}^{l}D^{-1}(\eta)d\eta-\left(\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta\right)^2}\right\}\int\limits_{0}^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta,
$$

$$
\begin{aligned}
K_l(x_2,\xi) \quad = \quad & -\left\{\frac{\left(K(l,\xi)+l\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta\right)\int\limits_{0}^{l}D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2D^{-1}(\eta)d\eta\int\limits_{0}^{l}D^{-1}(\eta)d\eta-\left(\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta\right)^2}\right. \\
& \left.-\frac{\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2D^{-1}(\eta)d\eta\int\limits_{0}^{l}D^{-1}(\eta)d\eta-\left(\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta\right)^2}\right\} \\
& \times\int\limits_{0}^{l}\eta(x_2-\eta)D^{-1}(\eta)d\eta \\
& +\left\{K(l,\xi)+l\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta\right\}\int\limits_{0}^{l}(x_2-\eta)D^{-1}(\eta)d\eta.
\end{aligned}
$$

Problem 2.

$$
\begin{aligned}
K_0(x_2,\xi) \quad = \quad & -K(0,\xi)+\int\limits_{\xi}^{0}x_2(\xi-\eta)D^{-1}(\eta)d\eta \\
& -\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta\times\frac{\int\limits_{0}^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}D^{-1}(\eta)d\eta},
\end{aligned}
$$

$$
\begin{aligned}
K_l(x_2,\xi) &= \int\limits_0^{x_2} \eta(x_2-\eta)D^{-1}(\eta)d\eta \\
&-\frac{\int\limits_0^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_0^{l}D^{-1}(\eta)d\eta}\int\limits_\xi^{l}(\xi-\eta)D^{-1}(\eta)d\eta \\
&-\frac{\int\limits_0^{x_2}\eta D^{-1}(\eta)d\eta\int\limits_0^{l}\eta D^{-1}(\eta)d\eta}{\int\limits_0^{l}D^{-1}(\eta)d\eta}.
\end{aligned}
$$

Problem 3.

$$
\begin{aligned}
K_0(x_2,\xi) &= x_2\int\limits_\xi^{0}(\xi-\eta)D^{-1}(\eta)d\eta \\
&-\frac{\int\limits_{x_2}^{0}\eta(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_0^{l}\eta^2D^{-1}(\eta)d\eta}\left\{-K(0,\xi)+l\int\limits_\xi^{0}(\xi-\eta)D^{-1}(\eta)d\eta\right\},
\end{aligned}
$$

$$
\begin{aligned}
K_l(x_2,\xi) &= -K(l,\xi)+\xi\int\limits_{x_2}^{0}\eta D^{-1}(\eta)d\eta-\frac{K(l,\xi)\int\limits_{x_2}^{0}\eta(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_0^{l}\eta^2D^{-1}(\eta)d\eta} \\
&-\frac{(l-\xi)\int\limits_{x_2}^{0}\eta D^{-1}(\eta)d\eta}{\int\limits_0^{l}\eta^2D^{-1}(\eta)d\eta}\int\limits_0^{l}\eta(x_2-\eta)D^{-1}(\eta)d\eta.
\end{aligned}
$$

Problem 4.

$$K_0(x_2,\xi)=-K(0,\xi)+x_2\int\limits_\xi^{0}(\xi-\eta)D^{-1}(\eta)d\eta,$$

$$K_l(x_2,\xi)=\int\limits_{x_2}^{0}(x_2-\eta)(\xi-\eta)D^{-1}(\eta)d\eta.$$

Problem 5.

$$K_0(x_2,\xi) = \int\limits_{x_2}^{l} \eta(x_2-\eta)D^{-1}(\eta)d\eta - \frac{\int\limits_{x_2}^{l}(x_2-\eta)D^{-1}(\eta)d\eta \int\limits_{x_2}^{l} D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} D^{-1}(\eta)d\eta}$$

$$-\frac{\int\limits_{x_2}^{l}(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} D^{-1}(\eta)d\eta}\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta,$$

$$K_l(x_2,\xi) = K(l,\xi) - (x_2-l)\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta$$

$$-\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta \times \frac{\int\limits_{x_2}^{l}(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} D^{-1}(\eta)d\eta}.$$

Problem 6.

$$K_0(x_2,\xi) = (x_2-l)\int\limits_{0}^{x_2}\eta D^{-1}(\eta)d\eta - \int\limits_{l}^{x_2}\eta^2 D^{-1}(\eta)d\eta + (x_2-l)\int\limits_{\xi}^{0}(\eta-\xi)D^{-1}(\eta)d\eta,$$

$$K_l(x_2,\xi) = \xi\left\{\int\limits_{0}^{x_2}(x_2-l)D^{-1}(\eta)d\eta - \int\limits_{l}^{x_2}\eta D^{-1}(\eta)d\eta\right\} - K(l,\xi)$$

Problem 7.

$$K_0(x_2,\xi) = x_2\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta -$$

$$\frac{\int\limits_{x_2}^{0}\eta(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2 D^{-1}(\eta)d\eta}\left\{-K(0,\xi) + l\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta\right\},$$

$$K_l(x_2,\xi) = -K(l,\xi) + \xi\int\limits_{x_2}^{0}\eta D^{-1}(\eta)d\eta - \frac{K(l,\xi)\int\limits_{x_2}^{0}\eta(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2 D^{-1}(\eta)d\eta}$$

$$-\frac{(l-\xi)\int\limits_{x_2}^{0}\eta D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^2 D^{-1}(\eta)d\eta}\int\limits_{x_2}^{0}\eta(x_2-\eta)D^{-1}(\eta)d\eta.$$

Problem 9.

$$K_0(x_2,\xi)=\frac{\xi}{l}\int\limits_{x_2^0}^{x_2}(l-\eta)(x_2-\eta)D^{-1}d\eta+\frac{\xi(l-x_2)}{l^2}\int\limits_{x_2^0}^{0}\eta(l-\eta)D^{-1}(\eta)d\eta$$

$$-\frac{x_2\xi}{l}\int\limits_{x_2^0}^{l}(l-\eta)D^{-1}(\eta)d\eta-\frac{x_2}{l}K(0,\xi),$$

$$K_l(x_2,\xi) = \frac{\xi-l}{l}\int\limits_{x_2^0}^{x_2}\eta(x_2-\eta)D^{-1}d\eta-\frac{(\xi-l)(l-x_2)}{l^2}\int\limits_{x_2^0}^{0}\eta^2 D^{-1}(\eta)d\eta$$

$$-\frac{x_2(\xi-l)}{l}\int\limits_{x_2^0}^{l}\eta D^{-1}(\eta)d\eta+\frac{x_2}{l}K(l,\xi).$$

Problem 10.

$$K_0(x_2,\xi)=-\int\limits_{x_2}^{l}(x_2-\eta)(\xi-\eta)D^{-1}(\eta)d\eta,$$

$$K_l(x_2,\xi)=K(l,\xi)-(x_2-\xi)\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta.$$

Thus, the following Proposition is valid.

Proposition 3.2 *Problem of the harmonic vibration of the plate with two cusped edges under action of the incompressible ideal fluid (i.e., equations (3.5), (3.6), (3.7), under transmission conditions (3.3), (3.4) and under conditions at infinity (3.1), (3.2) and BCs have) has a unique solution when*

$$\omega^2<\frac{1}{Ml},$$

where

$$M := \max_{x_2,\xi\in[0,l]} \{|2\rho^s h(\xi)K(x_2,\xi)|, \\ |2\rho^s h(\xi)K_0(x_2,\xi)|,\ |2\rho^s h(\xi)K_l(x_2,\xi)|,\ |K_1(x_2,\xi)|\}.$$

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Appendix A.

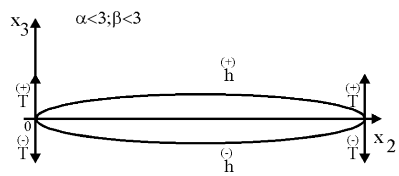

fig.2

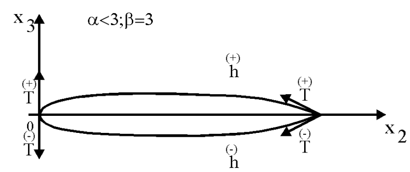

fig.3

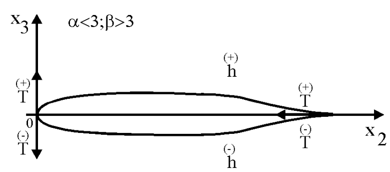

fig.4

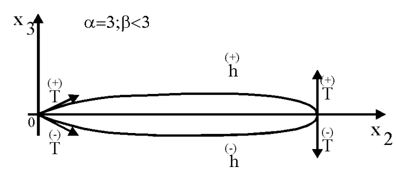

fig.5

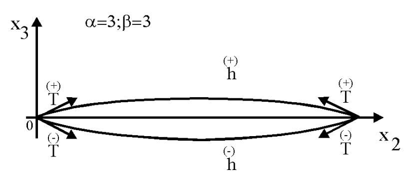

fig.6

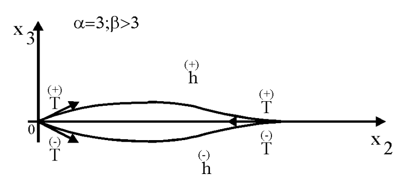

fig.7

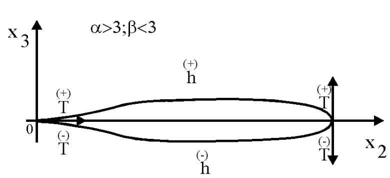

fig.8

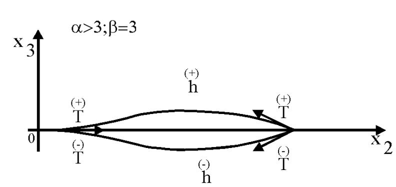

fig.9

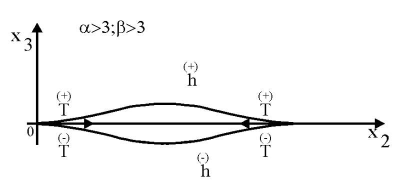

fig.10

Appendix B

Problem 1, Aluminium
α=0.3, β=0.3;
l=3*π; q=sin(x)

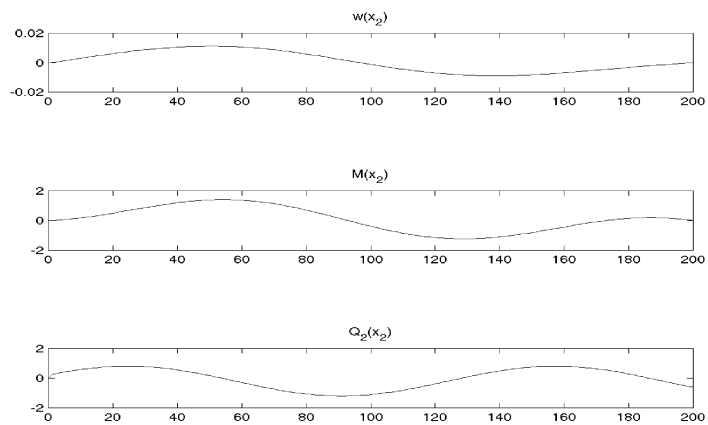

Fig. 11

Problem 8, Iron
α=1.5, β=0.1;
l=1; q=const

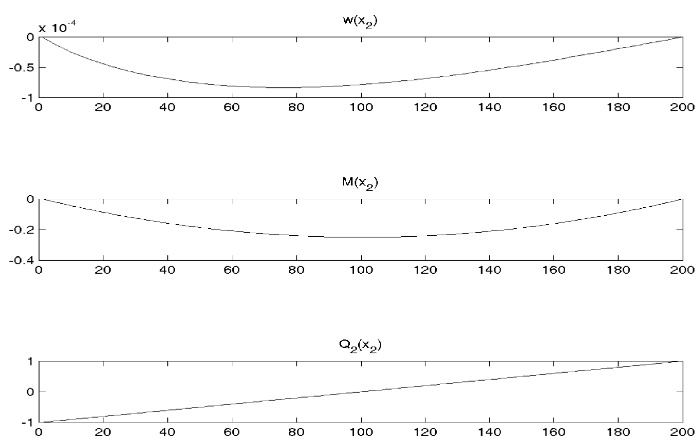

Fig. 12

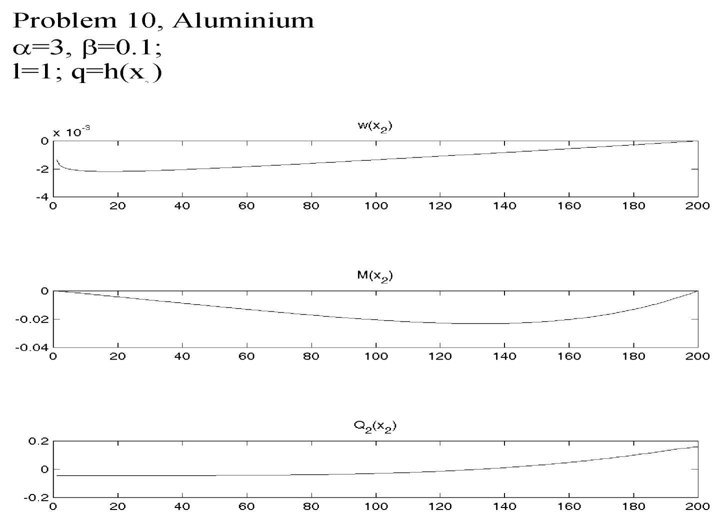
Problem 10, Aluminium
α=3, β=0.1;
l=1; q=h(x)
w(x2)
x 10-3
M(x2)
Q2(x2)

Fig. 13

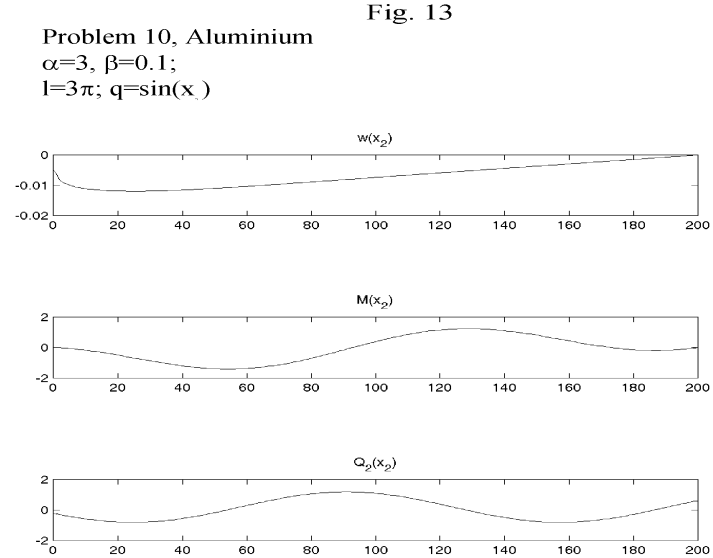
Problem 10, Aluminium
α=3, β=0.1;
l=3π; q=sin(x)
w(x2)
M(x2)
Q2(x2)

Fig. 14

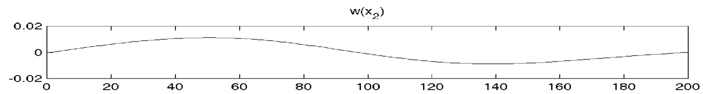

Fig. 15

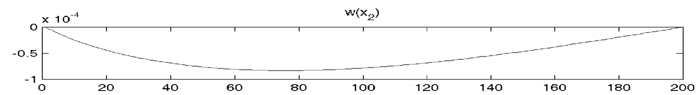

Fig. 16

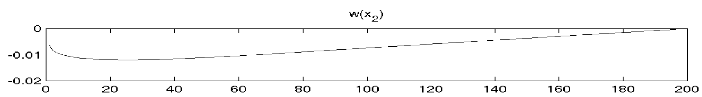

Fig. 17

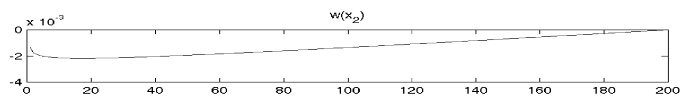

Fig. 18

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