THE JACOBI NONLINEAR ITERATION METHOD FOR A DISCRETE KIRCHHOFF SYSTEM

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Abstract

An initial boundary value problem for an integro-differential equation describing string vibration is considered. Using the Galerkin method and a Crank-Nicholson type scheme, the solution is discretized with respect to a spatial and a time variable. Thus the problem is reduced to a system of nonlinear algebraic equations which is solved by the iteration method. The convergence of the method is proved.

Key words and phrases: nonlinear equation, Kirchhoff string, iteration method, convergence.

AMS subject classification: 65M06,65M12.

1. *Introduction*

As it is well-known, the equation $w_{tt} = w_{xx}$ models string vibration. This equation should be considered as the first approximation of the vibration process. The equation

$$w_{tt} - \left(\lambda + \frac{2}{\pi}\int_0^\pi w_x^2 dx\right) w_{xx} = 0, \tag{1.1}$$

$$0 < x < \pi, \quad 0 < t \leq T, \quad \lambda > 0,$$

proposed by Kirchhoff [5] in 1883 is regarded as the best model as it takes into account an increase in tension resulting from the extension of the string. Equation (1.1) was first studied by S. Bernstein [2] in 1940. In the subsequent years, many authors (see, for example, [7], [8] and the references cited therein) devoted their studies to this equation and its natural generalization

$$w_{tt} - g\left(\int_\Omega |\nabla w|^2 dx\right) \triangle w = 0,$$

$$x \in R^n, \quad g(u) > 0, \quad n > 1.$$

In most cases these studies were concerned with the problem of the existence of a solution. Problems of controllability and stabilization were also considered, and computations were carried out (see, for example, [1], [10] and the references cited therein). We know only of one work [3] in which an approximate algorithm is applied to an initial boundary value problem for equation (1.1) and the convergence of the algorithm is proved. Speaking more exactly, in [3] equation (1.1) is reduced to a system of equations, a numerical algorithm of the solution is suggested and one of it's parts - the finite element method - is studied.

Here we consider equation (1.1) when

$$w(x,0) = w^{(0)}(x), \quad w_t(x,0) = w^{(1)}(x), \tag{1.2}$$

$$w(0,t) = w(\pi,t) = 0,$$

where $w^{(p)}(x)$ are the given functions, $p = 0, 1$.

2. *Discretization with respect to x*

An approximate solution of (1.1),(1.2) is written in the form

$$w_n = \sum_{i=1}^{n} w_{ni}(t) sin\, ix,$$

where the coefficients w_{ni} are determined by the Galerkin method from the system of ordinary differential equations

$$\underline{w}_{ntt} + (\lambda + ||\underline{w}_n||_2^2) P_n \underline{w}_n = 0, \tag{2.1}$$

where the vector function $\underline{w}_n(t) = (w_{ni})_{i=1}^n$, the matrix $P_n = [diag(1,2,\dots,n)]^2$ and the norm $||\underline{w}_n||_2 = \left(\sum_{i=1}^{n} i^2 w_{ni}^2\right)^{1/2}$, provided that at the initial point we have the conditions

$$w_{ni}(0) = \frac{2}{\pi}\int_0^\pi w^{(0)}(x) sin\, ixdx, \quad w_{nit}(0) = \frac{2}{\pi}\int_0^\pi w^{(1)}(x) sin\, ixdx, \tag{2.2}$$

$$i = 1, 2, \dots, n.$$

3. *Discretization with respect to t*

To solve the Cauchy problem (2.1),(2.2), let us introduce, on the time segment $[0,T]$, a grid with pitch $\tau = T/M$ and nodes $t_m = mr$, $m =$

$0, 1, \ldots, M$. An approximate value of $\underline{w}_n(t_m)$ denoted by $\underline{w}_n^m$ is determined by a Crank-Nicholson type scheme

$$\underline{w}_{n\bar{t}t}^{m-1} + \frac{1}{2}\sum_{p=0}^{1}\left(\lambda + \frac{||\underline{w}_n^{m-p}||_2^2 + ||\underline{w}_n^{m-p-1}||_2^2}{2}\right) P_n \frac{\underline{w}_n^{m-p} + \underline{w}_n^{m-p-1}}{2} = 0, \tag{3.1}$$

$$m = 2, 3, \ldots, M,$$

under the condition

$$\underline{w}_n^0 = \underline{w}_n(0),$$

$$\underline{w}_n^1 = \underline{w}_n^0 + \tau\underline{w}_{nt}(0) - \frac{\tau^2}{2}\left(\lambda + \frac{||\underline{w}_n^1||_2^2 + ||\underline{w}_n^0||_2^2}{2}\right) P_n \frac{\underline{w}_n^1 + \underline{w}_n^0}{2}. \tag{3.2}$$

4. *Solution of a nonlinear system*

Now we shall consider an approximate solution of system (3.1),(3.2). If the calculation is performed from layer to layer, then, knowing the results for the preceding layers, on the m-th time layer, i.e., for $t = t_m$, we have to solve a nonlinear equation with respect to the vector $\underline{w}_n^m$, which has the form

$$\underline{w}_n^m + \frac{\tau^2}{2}\left(\lambda + \frac{||\underline{w}_n^m||_2^2 + ||\underline{w}_n^{m-1}||_2^2}{2}\right) P_n \frac{\underline{w}_n^m + \underline{w}_n^{m-1}}{2} = \underline{f}_n^m, \tag{4.1}$$

$$m = 1, 2, \ldots, M,$$

where

$$\underline{f}_n^m = \begin{cases} \underline{w}_n^0 + \tau\underline{w}_n(0), \quad m = 1, \\ 2\underline{w}_n^{m-1} - \underline{w}_n^{m-2} - \frac{\tau^2}{2}\Bigg(\lambda \\ +\dfrac{||\underline{w}_n^{m-1}||_2^2 + ||\underline{w}_n^{m-2}||_2^2}{2}\Bigg) P_n \dfrac{\underline{w}_n^{m-1} + \underline{w}_n^{m-2}}{2}, \ m = 2, 3, \ldots, M. \end{cases}$$

Let us introduce the notation $\underline{w}_n^m = (w_{ni}^m)_{i=1}^n$, $\underline{f}_n^m = (f_{ni}^m)_{i=1}^n$. As it follows from (4.1), on the m-th time layer we have a system of nonlinear algebraic equations

$$\frac{8}{\tau^2 i^2} w_{ni}^m + \left[2\lambda + \sum_{j=1}^{n} j^2\left((w_{nj}^m)^2 + (w_{nj}^{m-1})^2\right)\right](w_{ni}^m + w_{ni}^{m-1}) = \frac{8}{\tau^2 i^2} f_{ni}^m, \tag{4.2}$$

$$i = 1, 2, \ldots, n.$$

Equation (4.2) is solved by the iteration method consisting in calculating successive approximations by Jacobi's rule

$$\frac{8}{\tau^2 i^2} w^m_{ni,k+1} + [2\lambda + i^2((w^m_{ni,k+1})^2 + (w^{m-1}_{ni})^2)$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{n} j^2((w^m_{nj,k})^2 + (w^{m-1}_{nj})^2)](w^m_{ni,k+1} + w^{m-1}_{ni}) = \frac{8}{\tau^2 i^2} f^m_{ni}, \tag{4.3}$$

$$i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots,$$

where $w^m_{ni,l}$ is the l-th approximation of w^m_{ni}, $l = 0, 1, \ldots$. The Cardano formula [6] given below allows us to determine $w^m_{ni,k+1}$ from (4.3) in an explicit form as follows

$$w^m_{ni,k+1} = \varphi_i(w^m_{n1,k}, w^m_{n2,k}, \ldots, w^m_{nn,k}), \tag{4.4}$$

$$i = 1, 2, \ldots, n.$$

After denoting $\underline{w}^m_{n,l} = (w^m_{ni,l})^n_{i=1}$, $\varphi = (\varphi_i)^n_{i=1}$, the iteration process (4.4) can be written as a vector equality

$$\underline{w}^m_{n,k+1} = \varphi(\underline{w}^m_{n,k}), \tag{4.5}$$

$$k = 0, 1, \ldots.$$

5. *Convergence of iterations*

To realize algorithm (4.3), we have to solve a cubic equation with respect to $w^m_{ni,k+1}$ (or $iw^m_{ni,k+1}$) on the $(k+1)$-th iteration pitch for each i. In this context recall the Cardano formula for the equation

$$y^3 + Ay^2 + By + C = 0, \tag{5.1}$$

whose a priori real root is equal to

$$y = -\frac{A}{3} + [-\frac{S}{2} + (\frac{S^2}{4} + \frac{R^3}{27})^{1/2}]^{1/3} - [\frac{S}{2} + (\frac{S^2}{4} + \frac{R^3}{27})^{1/2}]^{1/3}, \tag{5.2}$$

where

$$R = -\frac{A^2}{3} + B, \quad S = \frac{2A^3}{27} - \frac{AB}{3} + C. \tag{5.3}$$

Multiply (4.3) by i and write the obtained equality in form (5.1) as follows

$$(iw^m_{ni,k+1})^3 + a_i(iw^m_{ni,k+1})^2 + b_i(iw^m_{ni,k+1}) + c_i = 0, \tag{5.4}$$

where

$$a_i = iw_{ni}^{m-1}, \quad b_i = d_i + (iw_{ni}^{m-1})^2 + \frac{8}{\tau^2 i^2},$$

$$c_i = iw_{ni}^{m-1}(d_i + (iw_{ni}^{m-1})^2) - \frac{8}{\tau^2 i^2}\, if_{ni}^m, \tag{5.5}$$

while

$$d_i = 2\lambda + \sum_{\substack{j=1 \\ j\neq i}}^{n} j^2((w_{nj,k}^m)^2 + (w_{nj}^{m-1})^2). \tag{5.6}$$

By analogy with (5.3) we introduce the values

$$r_i = -\frac{a_i^2}{3} + b_i, \quad s_i = \frac{2a_i^3}{27} - \frac{a_i b_i}{3} + c_i.$$

By virtue of (5.5)

$$r_i = d_i + \frac{2}{3}(iw_{ni}^{m-1})^2 + \frac{8}{\tau^2 i^2}, \tag{5.7}$$

$$s_i = \frac{2}{3} iw_{ni}^{m-1}(d_i + \frac{10}{9}(iw_{ni}^{m-1})^2) - \frac{8}{\tau^2 i^2}(\frac{iw_{ni}^{m-1}}{3} + if_{ni}^m). \tag{5.8}$$

Taking into account (5.2), for the a priori real root of equation (5.4) we can write

$$iw_{ni,k+1}^m = -\frac{a_i}{3} + \sigma_{i,1} - \sigma_{i,2}\,, \tag{5.9}$$

$$i = 1, 2, \ldots, n,$$

where the notation

$$\sigma_{i,p} = [(-1)^p \frac{s_i}{2} + (\frac{s_i^2}{4} + \frac{r_i^3}{27})^{1/2}]^{1/3}, \tag{5.10}$$

$$p = 1, 2,$$

is used. System (5.9) can be represented as

$$iw_{ni,k+1}^m = \psi_i(1w_{n1,k}^m, 2w_{n2,k}^m, \ldots, nw_{nn,k}^m)\,, \tag{5.11}$$

$$i = 1, 2, \ldots, n.$$

To establish a condition for the convergence of process (5.11) as $k \to \infty$, we have to estimate the norm of the Jacobian

$$J = (\frac{\partial\psi_i}{\partial(jw_{nj,k}^m)})_{i,j=1}^n = (\frac{1}{j}\frac{\partial\psi_i}{\partial w_{nj,k}^m})_{i,j=1}^n \tag{5.12}$$

(in this paper this is the second notion connected with the name of C. Jacobi, 1804-1851). Using (5.6)-(5.11) and the first equality from (5.5), we see that on the principal diagonal of the matrix J we have zeros. As to nondiagonal elements, $i \neq j$, we have for them the formula

$$\frac{1}{j}\frac{\partial\psi_i}{\partial w^m_{nj,k}} = -\frac{jw^m_{nj,k}}{9}\sum_{p=1}^{2}\frac{1}{\sigma^2_{i,p}}[2iw^{m-1}_{ni}$$

$$+(-1)^p(is_iw^{m-1}_{ni} + \frac{1}{3}r_i^2)(\frac{s_i^2}{4} + \frac{r_i^3}{27})^{-1/2}]. \tag{5.13}$$

After some transformations (5.13) takes the form

$$\frac{1}{j}\frac{\partial\psi_i}{\partial w^m_{nj,k}} = \psi^{(1)}_{ij} + \psi^{(2)}_{ij}\,, \tag{5.14}$$

where

$$\psi^{(1)}_{ij} = -\frac{4}{9}ijw^m_{nj,k}w^{m-1}_{ni}(\sigma^2_{i,1} - \frac{r_i}{3} + \sigma^2_{i,2})^{-1},$$

$$\psi^{(2)}_{ij} = \frac{2}{3}jw^m_{nj,k}s_i(\sigma^4_{i,1} + \frac{r_i^2}{9} + \sigma^4_{i,2})^{-1}. \tag{5.15}$$

Now we have to estimate the modules of $\psi^{(p)}_{ij}$, $p = 1, 2$. To this end, we introduce the functions

$$\psi^{(p)}(\xi) = [\xi - (\xi^2 + r^3)^{1/2}]^{2p/3} + (-r)^p + [\xi + (\xi^2 + r^3)^{1/2}]^{2p/3},$$

$$-\infty < \xi < \infty, \quad r = const > 0, \quad p = 1, 2.$$

They possess the properties $\psi^{(p)}(-\xi) = \psi^{(p)}(\xi) > 0$ and, as follows from the formula

$$(\psi^{(p)}(\xi))' = \frac{2p}{3(\xi^2 + r^3)^{1/2}}\{[\xi + (\xi^2 + r^3)^{1/2}]^{2p/3} - [\xi - (\xi^2 + r^3)^{1/2}]^{2p/3}\}\,,$$

the inequality $(\psi^{(p)}(\xi))' \geq 0$ is fulfilled at $\xi \geq 0$. Therefore

$$\min|\psi^{(p)}(\xi)| = \psi^{(p)}(0) = (2p - 1)r^p. \tag{5.16}$$

Applying the foregoing arguments to functions (5.15) and using (5.16), (5.10), (5.7) and (5.6), we obtain

$$|\psi^{(1)}_{ij}| \leq \frac{4ij|w^m_{nj,k}|\,|w^{m-1}_{ni}|}{3r_i} \leq \frac{\tau^2i^3j|w^m_{nj,k}|\,|w^{m-1}_{ni}|}{6},$$

$$|\psi_{ij}^{(2)}| \leq \frac{2j|w_{nj,k}^m|\,|s_i|}{r_i^2} \leq \frac{r^4 i^4 j|w_{nj,k}^m|\,|s_i|}{32}.$$

This and (5.14), (5.8), (5.6) imply

$$\left|\frac{1}{j}\frac{\partial\psi_i}{\partial w_{nj,k}^m}\right| \leq \frac{1}{4}\tau^2 i^3 j|w_{nj,k}^m|\left\{\frac{1}{6}\tau^2 i^2|w_{ni}^{m-1}|\left[\lambda + \sum_{p=1}^{n} p^2(\frac{1}{2}(w_{np,k}^m)^2\right.\right.$$

$$\left.\left.+\frac{5}{9}(w_{np}^{m-1})^2)\right] + |w_{ni}^{m-1}| + |f_{ni}^m|\right\}. \qquad (5.17)$$

Further we will need a vector norm equal to $||\underline{v}||_1 = \sum_{i=1}^{n}|v_i|$ and the corresponding norm for the matrices $||U||_1 = \max_{1\leq j\leq n}\sum_{i=1}^{n}|u_{ij}|$, where $\underline{v} = (v_i)_{i=1}^n$, $U = (u_{ij})_{i,j=1}^n$, and also the equalities [4]

$$\sum_{l=1}^{n} l^{2p} = \frac{n(n+1)(2n+1)}{6}\left(\frac{3n^2+3n-1}{5}\right)^{p-1}, \qquad (5.18)$$

$$p = 1, 2.$$

From (5.12), (5.17), (5.18) we have

$$||J||_1 \leq \tau^2\frac{n(n+1)(2n+1)}{24}(\max_{1\leq j\leq n} j|w_{nj,k}^m|)$$

$$\times\left\{\frac{\tau^2[3n(n+1)-1]}{30}\left(\sum_{i=1}^{n} i|w_{ni}^{m-1}|\right)\left[\lambda + \sum_{i=1}^{n} i^2\left(\frac{1}{2}(w_{ni,k}^m)^2 + \frac{5}{9}(w_{ni}^{m-1})^2\right)\right]\right.$$

$$\left.+\sum_{i=1}^{n} i(|w_{ni}^{m-1}| + |f_{ni}^m|)\right\}. \qquad (5.19)$$

Let us require that the condition $||J||_1 \leq q$ be fulfilled for $q,\ 0 < q < 1$, and $\underline{w}_{n,k}^m = (w_{ni,k}^m)_{i=1}^n$ belonging to the domain

$$\left\{\underline{v} = (v_i)_{i=1}^n \in R^n\ : \sum_{i=1}^{n} i|v_i - w_{ni,0}^m| \leq \frac{1}{1-q}\sum_{i=1}^{n} i|w_{ni,0}^m - w_{ni,1}^m|\right\}. \quad (5.20)$$

By virtue of (5.19), it is sufficient for this requirement that

$$\tau^2\left\{\frac{\tau^2[3n(n+1)-1]}{30}\left(\sum_{i=1}^{n} i|w_{ni}^{m-1}|\right)\left[\lambda + \frac{1}{2}\left(\sum_{i=1}^{n} i\left(|w_{ni,0}^m|\right.\right.\right.\right.$$

$$+\frac{1}{1-q}|w_{ni,0}^m - w_{ni,1}^m|\Big)\Big)^2 + \frac{5}{9}\Big(\sum_{i=1}^n i|w_{ni}^{m-1}|\Big)^2\Big] + \sum_{i=1}^n i(|w_{ni}^{m-1}| + |f_{ni}^m|)\Big\}$$

$$-\frac{24q}{n(n+1)(2n+1)}\Big[\sum_{i=1}^n i\Big(|w_{ni,0}^m| + \frac{1}{1-q}|w_{ni,0}^m - w_{ni,1}^m|\Big)\Big]^{-1} \leq 0. \quad (5.21)$$

Using (5.21) and applying the principle of compressing mappings [9], we come to a conclusion that equation (4.1) has, in domain (5.20), a unique solution $\underline{w}_n^m = (w_{ni}^m)_{i=1}^n$, which is a limit as $k \to \infty$ of the sequence $\underline{w}_{n,k}^m$ of process (4.5), while the convergence rate is determined by the inequality

$$\sum_{i=1}^n i|w_{ni,k}^m - w_{ni}^m| \leq \frac{q^k}{1-q}\sum_{i=1}^n i|w_{ni,0}^m - w_{ni,1}^m|. \quad (5.22)$$

Our final result can be easily formulated by using the vector norm

$$||\underline{v}|| = \sum_{i=1}^n i|v_i| \quad (5.23)$$

for $\underline{v} = (v_i)_{i=1}^n$. It is clear that, in the sense of this norm, domain (5.20) is the ball

$$\{\underline{v} = (v_i)_{i=1}^n \in R^n \ : ||\underline{v} - \underline{w}_{n,0}^m|| \leq \frac{1}{1-q}||\underline{w}_{n,0}^m - \underline{w}_{n,1}^m||\}. \quad (5.24)$$

We also need the parameters

$$\alpha = \frac{3n^2+3n-1}{30}||\underline{w}_n^{m-1}||[\lambda + \frac{1}{2}(||\underline{w}_{n,0}^m|| + \frac{1}{1-q}||\underline{w}_{n,0}^m - \underline{w}_{n,1}^m||)^2 + \frac{5}{9}||\underline{w}_n^{m-1}||^2],$$

$$\beta = ||\underline{w}_n^{m-1}|| + ||\underline{f}_n^m||, \quad \gamma = \frac{24q}{n(n+1)(2n+1)}(||\underline{w}_{n,0}^m|| + \frac{1}{1-q}||\underline{w}_{n,0}^m - \underline{w}_{n,1}^m||)^{-1},$$

by means of which condition (5.21) can be rewritten as

$$\alpha\tau^4 + \beta\tau^2 - \gamma \leq 0.$$

We solve this inequality with respect to τ and apply norm (5.23) to (5.22). Thus the following theorem is valid.

Theorem. *Let q be an arbitrary number from the interval $(0,1)$ and the pitch*

$$\tau \leq [\frac{-\beta + (\beta^2 + 4\alpha\gamma)^{1/2}}{2\alpha}]^{1/2}.$$

Then in ball (5.24) there exists a unique solution $\underline{w}_n^m$ *of equation (4.1) to which the sequence* $\underline{w}_{n,k}^m$ *of process (4.5) tends as* $k \to \infty$*, while the method error decreases at the geometrical progression rate*

$$||\underline{w}_{n,k}^m - \underline{w}_n^m|| \le \frac{q^k}{1-q} ||\underline{w}_{n,0}^m - \underline{w}_{n,1}^m||\,,$$

$$k = 0, 1, \dots .$$

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