ON REGULARIZED DIFFERENCE SCHEMA FOR ONE QUASILINEAR PARABOLIC EQUATION

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Abstract

In the present work there is constructed regularized two-layer difference schema for one quasilinear parabolic equation with classical initial boundary conditions. Under the certain assumptions prior estimations are obtained and the convergence of solution of constructed regularized schema is investigated.

Key words and phrases: regularization, difference schema, quasilinear, parabolic equation. Weighted Sobolev space, Bending of an orthotropic cusped plate.

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1. *Introduction*

The paper deals with theregularized two-layer difference schema for the following quasilinear equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k\left(x,t,u\right)\frac{\partial u}{\partial x}\right) = f\left(x,t,u,\frac{\partial u}{\partial x}\right), \tag{1.1}$$

$$(x,t) \in D = \{0 < x < 1, 0 < t \leq T\}$$

with classical initial boundary conditions

$$u\left(0,t\right) = \varphi_1\left(x\right), \qquad u\left(1,t\right) = \varphi_2\left(t\right), \qquad 0 < t \leq T \tag{1.2}$$

$$u\left(x,0\right) = u_0\left(x\right), \qquad 0 < x < 1. \tag{1.3}$$

The idea and methods of regularization of the difference schemas belong to A.A Samarskii [2]. The regularized difference schemas for some certain non-linear equations were investigated by the authors in different works [1],[3].

In the present work the regularized difference schema is constructed for the problem (1.1)-(1.3).Under certain assumptions with respect to k

and f a priori estimations are obtained and the convergence of solution of constructed regularized schema is investigated .

Suppose further, that $k\,(x,t,u)$, $f\,(x,t,u,p)$ have derivatives $\frac{\partial k}{\partial u}$, $\frac{\partial f}{\partial p}$, $p\equiv\frac{\partial u}{\partial x}$ continuous with respect to their arguments and bounded for any x, t, u, p from the range of definition. For construction of difference schema let's consider the grid along space and time variable $\overline{\omega_h}=\left\{x_i=ih,\ i=0,1,...N=\frac{1}{h}\right\}$, $\omega_\tau=\left\{t_j=j\tau, j=0,1,...,K=[\frac{T}{\tau}]\right\}$, $\omega_h=\overline{\omega_h}\backslash\{0;1\}$.

Let us denote the points (x_i,t_j), $x_i\in\overline{\omega_h}$, $(x_i\in\omega_h)$, $t_j\in\omega_\tau$, as $\overline{\Omega}_{h,\tau}$ $(\Omega_{h,\tau})$.

Let us denote the values of the grid function, given on $\overline{\Omega}_{h,\tau}$, in the point (x_i,t_j) as $y_i^j=y(x_i,t_j)\equiv y$ and use the notations [2],[4]

$$\widehat{y}=y(x_i,t_{j+1}),\ \ \overset{\vee}{y}=y(x_i,t_{j-1}),\ \ y_t=\frac{\widehat{y}-y}{\tau}\ \ ,\ \ y_{\overline{t}}=\frac{y-\overset{\vee}{y}}{\tau},$$

$$y^{(-1)}=y(x_{i-1},t_j),\qquad\qquad y^{(+1)}=y(x_{i+1},t_j),$$

$$y_{\overline{x}}=\frac{y-y^{(-1)}}{h},\qquad\qquad y_x=\frac{y^{(+1)}-y}{h}$$

Let us construct the difference schema corresponding to differential problems (1.1)-(1.3)

$$y_t=(ay_{\overline{x}})_x+\varphi(t,x,y,\lambda(y)),\qquad (x,t)\in\Omega_{h,\tau},\tag{1.4}$$

$$y(0,t)=\varphi_1(t),\qquad y(1,t)=\varphi_2(t),\qquad t\in\omega_\tau,\tag{1.5}$$

$$y(x,0)=y^0(x)=u_0(x),\qquad x\in\omega_h,\tag{1.6}$$

where $a=a(x,t,y^*)$, $y^*=0.5(y+y^{(-1)})$, $\lambda(y)=y_{\overset{0}{x}}\equiv 0.5(y_x+y_{\overline{x}})$.

Further we shall use the following property of the coefficients of the difference schema (4)-(6)

a) $0<c_1\leq a<\acute{c_1}$, $c_1,c_1'=const,$

b) $a(x,t,u)-k(x,t,u)=0.5hk'(x,t,u)+O(h^2)$,

c) $a_x=k^{'}(x,t,u)+O(h^2)$,

d) $\varphi(x,t,u,\lambda(u))-f(x,t,u,u_x')=O(h^2)$.

If $k(x,t,u)$, $f(x,t,u,u_x^{'})$ and the solution of the problem (1.1)-(1.3) $u(x,t)$ are sufficiently differentiable with respect to x,t and $u(x,t)$ then the conditions $a)-d)$ are sufficient, for (1.4)-(1.6) schema has the second order precision approximation with respect to x and first order to t.

Besides we demand for schema (1.4) to be homogeneous. In [2] there is indicated the way of calculation of the factors with the help of stencil functionals.

As it is known, the schemas of type (1.4) (explicit schemas) apply some restrictions on the step of grid with respect to time, although implicit schemas do not have this defect, but in our case we should operate with non-linear values.

Our purpose is to conduct regularization so that on the high layer we always have a constant operator. The advantages of such schema are obvious.

To do that let us consider the following operator $\sigma\Lambda$ as a regularization:

$$Ry \equiv \sigma\Lambda y \equiv \sigma y_{\overline{x}x}, \tag{1.7}$$

where $\sigma = const$ with indefinite parameter. Then regularized difference schema which is equivalent to schema (1.4), will be written in the following way

$$y_t = \tau R y_t + (a y_{\overline{x}})_x + \varphi(t, x, y, \lambda(y)), \qquad (x, y) \in \Omega_{h,\tau}. \tag{1.8}$$

It is easy to see, that the difference equation (1.8) approximates the equation (1.1) with error $O(\tau + h^2)$.

Joining the boundary conditions (1.5) and the initial condition (1.6) to equation (1.8), we shall receive desirable schema.

2. *The difference problem for the error*

Let $z = y - u$. Substituting $y = z + u$ in (1.8), (1.5), (1.6), and taking into account (1.1)-(1.3), we have the following condition for the error z

$$z_t = \tau R z_t + (a(t, x, y^*) z_{\overline{x}})_x + Q(z) + \Psi \quad in \quad \Omega_{h,\tau}, \tag{2.1}$$

$$z(0, t) = z(1, t) = 0, \qquad t \in \Omega_\tau, \tag{2.2}$$

$$z(x, 0) = 0, \qquad x \in \omega_h, \tag{2.3}$$

where $Q(z) = (g z^*)_x + b z_{\overset{0}{x}} + dz$, $g = \overline{\frac{\partial a}{\partial u}} u_{\overline{x}}$, $b = \overline{\frac{\partial \varphi}{\partial \lambda}}$, $d = \overline{\frac{\partial \varphi}{\partial u}}$, and Ψ is an error of approximation of the solution for the problem (1.1)-(1.3). This feature means, that the derivatives are taken in the some average values of arguments u and λ .

For some restrictions on the k, f and u we have the following increment of approximation $\Psi = O(\tau + h^2)$.

3. *A Priori estimation for the problem (9)-(11)*

Note, that from the a priori estimations for (2.1)-(2.3) the convergence of the method (1.8), (1.5), (1.6) follows only with the following requirements:

$$0 < c_1 \leq a \leq c_1', \quad \{|g|, |b|, |d|\} \leq M = const. \tag{3.1}$$

Let us multiply scalarly (2.1) on $2(\alpha\widehat{z} + \beta z)$, where $\alpha + \beta = 1$, α and β are yet indefinite parameters, we shall have

$$2(\alpha\widehat{z}+\beta z, z_t) = 2\tau(\alpha\widehat{z}+\beta z, Rz_t)+2(\alpha\widehat{z}+\beta z, (az_{\overline{x}})_x)+2(\alpha\widehat{z}+\beta z, Q(z)+\Psi). \tag{3.2}$$

As for our case $z_0 = z_N = 0$, using the well-known inequalities and identities [2], [4] it is easy to get:

$$2(\alpha\widehat{z} + \beta z, z_t) = (\|z^2\|)_t + (\alpha - \beta)\tau \|z_t\|^2, \tag{3.3}$$

$$2\tau(\alpha\widehat{z} + \beta z, Rz_t) = -\tau\sigma(||z_{\overline{x}}||^2)_t - (\alpha - \beta)\tau^2\sigma||z_{t\overline{x}}||^2, \tag{3.4}$$

$$2(\alpha\widehat{z} + \beta z, (az_{\overline{x}})_x) = -0.5\alpha(a, (\widehat{z}_{\overline{x}}+z_{\overline{x}})^2]+0.5\alpha\tau^2(a, z_{\overline{x}t}^2]-2\beta(a, z_{\overline{x}}^2]. \tag{3.5}$$

For illustration let us check up (3.4). Other equations, i.e. (3.3) and (3.5) , can be proved analogously. So it is easy to observe

$$2\tau(\alpha\widehat{z} + \beta z, Rz_t) = 2\tau(\alpha\widehat{z} + \beta z, \sigma z_{t\overline{x}x}) = -2\alpha\tau\sigma(\widehat{z}_{\overline{x}}, z_{t\overline{x}}] - 2\beta\tau\sigma(z_{\overline{x}}, z_{t\overline{x}}] =$$

$$= -\alpha\tau\sigma((\widehat{z}_{\overline{x}} + z_{\overline{x}}) + (\widehat{z}_{\overline{x}} - z_{\overline{x}}), z_{t\overline{x}}] - \beta\tau\sigma((\widehat{z}_{\overline{x}} + z_{\overline{x}}) - (\widehat{z}_{\overline{x}} - z_{\overline{x}}), z_{t\overline{x}}] =$$

$$= -(\alpha + \beta)\tau\sigma(\widehat{z}_{\overline{x}} + z_{\overline{x}}), z_{t\overline{x}}] - (\alpha - \beta)\tau\sigma(\widehat{z}_{\overline{x}} - z_{\overline{x}}), z_{t\overline{x}}] =$$

$$= -\sigma\tau(||z_{\overline{x}}||^2)_t - (\alpha - \beta)\sigma\tau^2||z_{t\overline{x}}||^2.$$

Further, substituting (3.3)-(3.5) in (3.2) we obtain the energetical identity

$$(\|z\|^2)_t + (\alpha - \beta)\tau \|z_t\|^2 =$$

$$= -\sigma\tau(||z_{\overline{x}}||^2)_t - (\alpha - \beta)\sigma\tau^2||z_{t\overline{x}}||^2-0.5\alpha(a, \widehat{z}_{\overline{x}} + z_{\overline{x}})^2]+$$

$$+0.5\alpha\tau^2(a, z^2{}_{\overline{x}t}] - 2\beta(a, z^2{}_{\overline{x}}] + K(z),$$

where

$$K(z) = 2(\alpha\widehat{z} + \beta z, Q(z) + \Psi).$$

Taking into account the definition of the difference derivative, we shall have

$$\|\widehat{z}\|^2 + \sigma\tau||\widehat{z}_{\overline{x}}||^2+(\alpha - \beta)\tau^2 \|z_t\|^2 = \|z\|^2 + \sigma\tau||z_{\overline{x}}||^2- \tag{3.6}$$

$$-\tau^3((\alpha - \beta)\sigma-0.5\alpha a, z_{\overline{x}t}^2] - 0.5\alpha\tau(a, (\widehat{z}_{\overline{x}} + z_x)^2] - 2\beta\tau(a, z_{\overline{x}}^2] + \tau K(z).$$

To obtain the a prior estimations, we have to impose some limitations on *alpha*, β and σ.

Let

$$\alpha > \beta, \quad i.e. \;\; 0.5 < \alpha < 1, \quad 0 < \beta \leq 0.5 \tag{3.7}$$

and

$$\sigma \geq 0.5(1-\delta)^{-1} a_{\max}, \quad i.e. \quad \sigma \geq 0.5(1-\delta)^{-1} c_1', \tag{3.8}$$

where

$$\delta \equiv \frac{\beta}{\alpha} \quad and \quad 0 < \delta < 1.$$

Then from (3.6) it is easy to obtain

$$\|\widehat{z}\|^2 + \sigma\tau||\widehat{z}_{\overline{x}}||^2 \leq \|z\|^2 + \sigma\tau||z_{\overline{x}}||^2 - 2\beta\tau(a, z_{\overline{x}}^2] + \tau K(z). \tag{3.9}$$

Let us pass on now to the estimation of the following expression $\tau K(z)$.

$$K(z) = 2(\alpha\widehat{z} + \beta z, Q(z)) + 2(\alpha\widehat{z} + \beta z, \Psi) = 2\alpha(\widehat{z}, (gz^*)_x + 2\alpha(\widehat{z}, bz_{\overset{\circ}{x}}) +$$

$$+ 2\alpha(\widehat{z}, dz) + 2\alpha(\widehat{z}, \Psi) + 2\beta(z, (gz^*)_x + 2\beta(z, bz_{\overset{\circ}{x}}) + 2\beta(z, dz) + 2\beta(z, \Psi).$$

Let's estimate each addend of right part separately, using the condition (3.1).

So we have

$$2(\widehat{z}, (gz^*)_x) = -2(\widehat{z}_{\overline{x}}, gz^*] = -(\widehat{z}_{\overline{x}}, g(z + z^{(-1)})] = -(\widehat{z}_{\overline{x}}, gz] - (\widehat{z}_{\overline{x}}, gz^{(-1)}],$$

but

$$|(\widehat{z}_{\overline{x}}, gz]| \leq \varepsilon||\widehat{z}_{\overline{x}}||^2 + \frac{1}{4\varepsilon} M \|z\|^2, as \;\; z_0 = z_N = 0, \quad M = \max\{M, M^2\},$$

$$\left|(\widehat{z}_{\overline{x}}, gz^{(-1)}]\right| \leq \varepsilon||\widehat{z}_{\overline{x}}||^2 + \frac{1}{4\varepsilon} M \|z\|^2, as \;\; z_0 = z_N = 0,$$

i.e.

$$2\left|(\widehat{z}, (gz^*)_x)\right| \leq 2\varepsilon||\widehat{z}_{\overline{x}}||^2 + \frac{1}{2\varepsilon} M \|z\|^2. \tag{3.10}$$

Similarly

$$2(\widehat{z}, bz_{\overset{\circ}{x}}) = (\widehat{z}, bz_x) + (\widehat{z}, bz_{\overline{x}}) = -(\widehat{z}_{\overline{x}}, bz] + (b\widehat{z}, z_{\overline{x}}),$$

$$|(\widehat{z}_{\overline{x}}, bz]| \leq \varepsilon||\widehat{z}_{\overline{x}}||^2 + \frac{1}{4\varepsilon} M \|z\|^2,$$

$$|(b\widehat{z}, z_{\overline{x}})| \leq \varepsilon \|z_{\overline{x}}\|^2 + \frac{1}{4\varepsilon} M \|\widehat{z}\|^2 \leq \varepsilon||z_{\overline{x}}||^2 + \frac{1}{4\varepsilon} M \|\widehat{z}\|^2,$$

i.e.

$$2\left|(\widehat{z}, bz_{\overset{0}{x}})\right| \leq \varepsilon||\widehat{z}_{\overline{x}}||^2 + \varepsilon||z_{\overline{x}}||^2 + \frac{1}{4\varepsilon}M(\|\widehat{z}\|^2 + \|z\|^2). \tag{3.11}$$

Further

$$2\left|(\widehat{z}, dz)\right| \leq \varepsilon\|\widehat{z}\|^2 + \frac{1}{\varepsilon}M\|z\|^2, \tag{3.12}$$

$$2\left|(z, (gz^*)_x\right| \leq 2\varepsilon||z_{\overline{x}}||^2 + \frac{1}{2\varepsilon}M\|z\|^2, \tag{3.13}$$

$$2(z, bz_{\overset{0}{x}}) = (z, bz_x) + (z, bz_{\overline{x}}) = -(z_{\overline{x}}, bz] + (z_{\overline{x}}, bz) = 0, \tag{3.14}$$

$$2\left|(z, dz)\right| \leq 2M\|z\|^2, \tag{3.15}$$

$$\left|(\widehat{z}, \Psi)\right| \leq \varepsilon\|\widehat{z}\|^2 + \frac{1}{4\varepsilon}\|\Psi\|, \tag{3.16}$$

$$\left|(z, \Psi)\right| \leq \varepsilon\|z\|^2 + \frac{1}{4\varepsilon}\|\Psi\|. \tag{3.17}$$

Substituting the estimation (3.10)-(3.17) in the expression for $K(z)$, we shall have

$$|K(z)| \leq 2\alpha\varepsilon||\widehat{z}_{\overline{x}}||^2 + \frac{\alpha}{2\varepsilon}M\|z\|^2 + \alpha\varepsilon||\widehat{z}_{\overline{x}}||^2 + \alpha\varepsilon||z_{\overline{x}}||^2 + \frac{\alpha}{4\varepsilon}M\|\widehat{z}\|^2 +$$

$$+\frac{\alpha}{4\varepsilon}M\|z\|^2 + \alpha\varepsilon\|\widehat{z}\|^2 + \frac{\alpha}{\varepsilon}M\|z\|^2 + 2\alpha\varepsilon\|\widehat{z}\|^2 + \frac{2\alpha}{4\varepsilon}\|\Psi\| +$$

$$+2\beta\varepsilon||z_{\overline{x}}||^2 + \frac{\beta}{2\varepsilon}M\|z\|^2 + 2\beta M\|z\|^2 + 2\beta\varepsilon\|z\|^2 + \frac{2\beta}{4\varepsilon}\|\Psi\|^2$$

or

$$|K(z)| \leq 3\alpha\varepsilon||\widehat{z}_{\overline{x}}||^2 + (\alpha + 2\beta)\varepsilon||z_{\overline{x}}||^2 + M_1\|z\|^2 + M_2\|\widehat{z}\|^2 + \frac{1}{2\varepsilon}\|\Psi\|^2, \tag{3.18}$$

where

$$M_1 \equiv 1.75\alpha M\varepsilon^{-1} + \beta(0.5M\varepsilon^{-1} + 2\varepsilon + 2M),$$

$$M_2 \equiv \alpha(0.25\varepsilon^{-1}M + 3\varepsilon).$$

Substituting (3.18) in (3.9) we get:

$$(1 - \tau M_1)\|\widehat{z}\|^2 + \tau(\sigma - 3\alpha\varepsilon)||\widehat{z}_{\overline{x}}||^2 \leq \tag{3.19}$$

$$\leq (1 + \tau M_2)\|z\|^2 + \tau[\sigma + (\alpha + 2\beta)\varepsilon - 2\beta c_1]||z_{\overline{x}}||^2 + \frac{\tau}{2\varepsilon}\|\Psi\|^2,$$

we shall choose such ε , that

$$2\alpha\varepsilon = 2\beta c_1 - (\alpha + 2\beta)\varepsilon.$$

Hence we have

$$\varepsilon = \frac{2\beta c_1}{2(2\alpha+\beta)} = \frac{\beta c_1}{2\alpha+\beta} = \frac{\delta c_1}{2+\delta}, \quad where \quad \delta = \frac{\beta}{\alpha}. \tag{3.20}$$

It is easy to test, that if δ is selected from condition (3.8), then $\delta - 3\alpha\varepsilon > 0$, for any ε selected by condition (3.20).

Thus we have

$$(1-\tau M_1)\,\|\widehat{z}\|^{\,2} + \tau M_3 ||\widehat{z}_{\overline{x}}||^2 \le (1+\tau M_2)\,\|z\|^{\,2} + \tau M_3 ||z_{\overline{x}}||^2 + \tau M_4\,\|\Psi\|^{\,2},$$

here $\quad M_3 \equiv \sigma - 3\alpha\varepsilon, \quad M_4 = 0.5\varepsilon^{-1}.$

Let

$$\|z\|^{\,2} + \tau M_3\,\|z_{\overline{x}}\|^{\,2} = \|z\|^{\,2}_{I},$$

then

$$\|\widehat{z}\|^{\,2}_{I} - \tau M_1\,\|\widehat{z}\|^{\,2} \le \|z\|^{\,2}_{I} + \tau M_2\,\|z\|^{\,2} + \tau M_4\,\|\Psi\|^{\,2}.$$

Strengthening this inequality , we have

$$\|\widehat{z}\|_I^{\;2} - \tau M_1\,\|\widehat{z}\|^{\,2} - \tau^2 M_1 M_3 ||\widehat{z}_{\overline{x}}||^2 \le$$

$$\le \|z\|^{\,2}_{I} + \tau M_2\,\|z\|^{\,2} + \tau^2 M_2 M_3 ||z_{\overline{x}}||^2 + \tau M_4\,\|\Psi\|^{\,2}$$

or

$$\|\widehat{z}\|^2_I - \tau M_1\,\|\widehat{z}\|^2_I \le \|z\|^2_I + \tau M_2\,\|z\|^2_I + \tau M_4\,\|\Psi\|^2,$$

i.e.

$$(1-\tau M_1)\,\|\widehat{z}\|^2_I \le (1+\tau M_2)\,\|z\|^2_I + \tau M_4\,\|\Psi\|^{\,2}.$$

Let

$$\tau \le \tau_0 < \frac{1}{M_1},$$

then

$$\|\widehat{z}\|^2_I \le \frac{1+\tau M_2}{(1-\tau M_1)}\,\|z\|^2_I + \tau\frac{M_4}{(1-\tau M_1)}\,\|\Psi\|^2$$

or

$$\|\widehat{z}\|^2_I \le (1+\tau c_2)\,\|z\|^2_I + \tau c_3\,\|\Psi\|^{\,2}, \tag{3.21}$$

where

$$c_2 = \frac{(M_1+M_2)}{(1-\tau_0 M_1)}, \quad c_3 = \frac{M_4}{(1-\tau_0 M_1)}.$$

From inequality (3.21) there follows

Theorem 3.1. The difference problem (1.8), (1.5), (1.6) with $\tau \le \tau_0$ and condition (3.8) is stable with respect to the initial condition and right part.

Theorem 3.2. In the previous assumptions solution y of the problem (1.8), (1.5), (1.6) converges to the solution u of the problem (1.1)-(1.3) with the rate of $O(\tau + h^2)$.

It is easy to show, that the aboven reasonings are valid for the multidimensional quasilinear equation of parabolic type.

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