EXPONENTIALLY CONVERGENT PARALLEL DISCRETIZATION METHOD FOR THE FIRST ORDER EVOLUTION EQUATION

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Abstract

We propose a new discretization of an initial value problem for differential equations of the first order in a Banach space with a strongly P-positive operator coefficient. Using the strong positiveness we represent the solution as a Dunford-Cauchy integral along a parabola in the right half of the complex plane, then transform it into real integrals over $(-\infty, \infty)$ and finally apply an exponentially convergent Sinc quadrature formula to this integral. The integrand values are the solutions of a finite set of elliptic problems with complex coefficients, which are independent and may be solved in parallel.

Key words and phrases: evolution equations, unbounded operator coefficients, strongly P-positive operators, quadrature rule.

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1. *Introduction*

We consider the initial value problem

$$\dot{u} + Au = 0, \qquad t \in (0, T], \qquad u(0) = u_0, \tag{1.1}$$

where $u : R_+ \to E$ is a vector-valued function, A is a strongly P-positive densely defined closed operator with the spectrum lying in a parabola Γ and with a domain $D(A)$ in a Banach space E. In particular, the equation (1.1) with the Laplace operator $A = -\Delta$ is the well-known heat equation.

Using the improper Dunford- Cauchy integral we will show that one can represent the solution of (1.1) by

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{-zt}(z-A)^{-1}u_0 dz \tag{1.2}$$

Denoting by $\hat{u}(z)$ the solution of the stationary equation

$$(z-A)\hat{u}(z) = u_0 \tag{1.3}$$

we get

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{-zt}\hat{u}(z)dz \tag{1.4}$$

The key steps that allow us to find an approximate solution of (1.1) are :

1. Choose N points $z_1, z_2, ..., z_N$ on the parabola and find the solutions $\hat{u}(z_i)$ of (1.3).
2. Find the approximation $u^N(t)$ for the solution of (1.1) by

$$u^N(t) = \sum_{i=1}^{N} \alpha_i e^{-z_i t}\hat{u}(z_i) \tag{1.5}$$

   with appropriate coefficients α_i.

The next question is how to choose z_i and α_i. This will be discussed in what follows. An analogous idea for equation (1.1) was used in [13]. The operator A was assumed to be selfadjoint and positive definite and the integration in (1.2) was performed along a path $\Gamma = \Gamma_\gamma = \{z = \gamma + \sigma \pm i\sigma : \sigma \geq 0\}$ with $\Im z$ increasing from $-\infty$ to ∞. Using a parametric variable transformation, the contour integral was then transformed into an integral over the finite interval $[0,1]$ with a singular integrand which was treated by the composite trapezoidal rule and the composite Simpson rule with an appropriately chosen parameter. The singularity of higher derivatives of the integrand has implied a rather complex algorithm with the convergence rate bounded by 2 and 4 respectively. Our algorithm presented below possesses an exponential error decay.

2. *Representation of solutions of the first order differential equations with a strongly P-positive operator coefficients*

In order to motivate next definitions we begin with the following examples.

Example 2.1. Let us consider the one-dimensional operator $A : L_1(0,1) \to L_1(0,1)$ with the domain $D(A) = \{u | u \in H_0^2(0,1)\}$ in the Sobolev space $H_0^2(0,1)$ defined by

$$Au = -u'' \quad \forall\, u \in D(A).$$

The eigenvalues $\lambda_k = k^2\pi^2, k = 1,2,...$ of A lie on the real axis inside of the path

$$\Gamma = \{\, z = \eta^2 \pm i\eta, \eta \geq 1, z = 1 \pm i\eta^2, |\eta| \leq 1.$$

The Green function for the problem

$$(zI - Au) \equiv u''(x) + zu(x) = -f(x),\ x \in (0,1); u(0) = u(1) = 0$$

is

$$G(x,\xi) = \frac{1}{\sqrt{z}\sin\sqrt{z}}\,\{\sin\sqrt{z}x\sin\sqrt{z}(1-\xi)x \leq \xi, \sin\sqrt{z}\xi\sin\sqrt{z}(1-x)x \geq \xi,$$

i.e. we have

$$u(x) = (zI - A)^{-1}f = \int_0^1 G(x,\xi)f(\xi)d\xi.$$

Let us estimate the Green function on the parabola $z = \eta^2 \pm i\eta = \sqrt{\eta^4 + \eta^2}(\cos\phi \pm i\sin\phi)$ for $|z|$ large enough, where

$$\cos\phi = \frac{\eta}{\sqrt{\eta^2+1}},\ \sin\phi = \frac{1}{\sqrt{\eta^2+1}}.$$

Actually, we have $\sqrt{z} = \sqrt[4]{\eta^4+\eta^2}(\cos\frac{\phi}{2} \pm i\sin\frac{\phi}{2}) = a \pm ib$ with

$$\cos\frac{\phi}{2} = \frac{\sqrt{\eta^2 + \sqrt{\eta^4+\eta^2}}}{\sqrt{2}\sqrt[4]{\eta^4+\eta^2}},\ \sin\frac{\phi}{2} = \frac{\sqrt{\sqrt{\eta^4+\eta^2} - \eta^2}}{\sqrt{2}\sqrt[4]{\eta^4+\eta^2}},$$

$$a = \frac{\sqrt{\eta^2 + \sqrt{\eta^4+\eta^2}}}{\sqrt{2}},\ b = \frac{\sqrt{\sqrt{\eta^4+\eta^2} - \eta^2}}{\sqrt{2}}.$$

The following estimates hold for $x \leq \xi$ and for η large enough

$$\left|\frac{\sin\sqrt{z}x\sin\sqrt{z}(1-\xi)}{\sqrt{z}\sin\sqrt{z}}\right| =$$

$$= \frac{[\sin^2 ax + \sinh^2 bx]^{\frac{1}{2}}[\sin^2 a(1-\xi) + \sinh^2 b(1-\xi)]^{\frac{1}{2}}}{\sqrt[4]{\eta^4+\eta^2}[\sin^2 a + \sinh^2 b]^{\frac{1}{2}}}$$
$$\leq \frac{c}{\eta}$$

with an absolute constant c. The case $\xi \le x$ can be considered analogously. The last estimate implies that $\|(zI - A)^{-1} f\|_{L_1} \le \frac{M}{1+\sqrt{|z|}} \|f\|_{L_1} \forall f \in L_1(0,1)$, $\forall z \in\in \mathbb{C} \setminus \not\leq_{\lneqq}$ where Ω_Γ is the domain inside the parabola. The same estimates for the Green function imply the analogous estimate in the norm of $L_\infty(0,1)$.

Example 2.2. This example deals with a differential operator which considered in the Hilbert space $L_2(0,1)$ is not symmetric.

Let $D(A) = \{v(x) \in W^2_\infty(0,1)| : v(0 = 0, v'(0) = v'(1))\}$ be the domain of the operator A defined by

$$Au \equiv -u'''(x) - u(x), \qquad \forall u \in D(A)$$

It is easy to find that the spectrum of A consists of the eigenvalues $\lambda_k = (2k\pi)^2 + 1, k = 0, 1, 2, ...$ which are enveloped by the parabola $\Gamma = \{z = \xi + i\eta : \xi = \eta^2\}$. Each eigenvalue corresponds to one eigenfunction and one joint function. We denote by Ω_Γ the domain inside of the parabola. The solution of the problem $(zI - A)u = -f(x), x \in (0,1)$ can be represented by

$$u(x) = \frac{1}{\sqrt{z-1}(1 - \cos\sqrt{z-1})} \left\{ \int_0^x [\sin(\sqrt{z-1}(\xi - x)) - \right.$$
$$- \cos(\sqrt{z-1}(1-x)) \sin(\sqrt{z-1}\xi)] f(\xi) d\xi -$$
$$\left. - \int_x^1 \sin(\sqrt{z-1}x) - \cos(\sqrt{z-1}(1-\xi)) f(\xi) d\xi \right\}.$$

Analogously as in Example 1 one can show that for all $z \in \mathbb{C} : \setminus \not\leq_{\lneqq}$ the estimate

$$\|(Iz - A)^{-1} f\|_\infty = \|u\|_\infty \le \frac{M}{\sqrt{|z-1|}} \|f\|_\infty, \ \forall f(x) \in L_\infty(0,1)$$

holds with a positive constant M. Using the estimate

$$\sqrt{|z-1|} \ge \frac{1+\sqrt{|z|}}{4}, \qquad \forall z \in \mathbb{C} \setminus \not\leq_{\lneqq}$$

we get

$$\|(Iz - A)^{-1} f\|_{L_\infty(0,1) \to L_\infty(0,1)} \le \frac{M_1}{1+\sqrt{|z|}}.$$

Example 2.3. Let Ω be a bounded domain with a Lipschitz boundary in $\mathbf{R}^d, d = 2, 3$ and let

$$\begin{aligned} L(u) \quad &= \quad -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} a_i(x) \frac{\partial u}{\partial x_i} \\ &+ \quad a_0(x)u, x \in \Omega \subset R^d_0, \ d = 2, 3 \end{aligned} \tag{2.1}$$

be a strongly elliptic differential operator, i.e. we have

$$\sum_{i,j=1}^{d} a_{ij} y_i y_j \geq c_1 \sum_{i=1}^{d} y_i^2 \tag{2.2}$$

and $a_{pq} = a_{qp}$. We define the sesquilinear form $a(u,v)$ by

$$\begin{aligned}
a(u,v) &= \int_\Omega L(u)\overline{v}dx \\
&= -\int_\Omega \sum_{p,q} a_{pq} \frac{\partial u}{\partial x_p}\frac{\partial \overline{v}}{\partial x_q}dx + \int_\Omega \sum_p a_p \frac{\partial u}{\partial x_p}\overline{v}dx + \int_\Omega a_0(x) u\overline{v}dx \\
&= -\int_\Omega \sum_{p,q} a_{pq} \frac{\partial u}{\partial x_p}\frac{\partial \overline{v}}{\partial x_q}dx - \frac{1}{2}\sum_p \int_\Omega (\frac{\partial a_p}{\partial x_p} u\overline{v} + a_p u \frac{\partial \overline{v}}{\partial x_p})dx \\
&+ \int_\Omega a_0(x) u\overline{v}dx + \frac{1}{2}\int_\Omega \sum_p a_p \frac{\partial u}{\partial x_p}\overline{v}dx \qquad (2.3) \\
&= -\int_\Omega \sum_{p,q} a_{pq} \frac{\partial u}{\partial x_p}\frac{\partial \overline{v}}{\partial x_q}dx - \int_\Omega (\frac{1}{2}\sum_p \frac{\partial a_p}{\partial x_p} - a_0) u\overline{v}dx \\
&+ \frac{1}{2}\int_\Omega \sum_p a_p (\frac{\partial u}{\partial x_p}\overline{v} - u\frac{\partial \overline{v}}{\partial x_p})dx
\end{aligned}$$

from where

$$a(u,u) = \Re a(u,u) + \Im a(u,u) \tag{2.4}$$

with

$$\begin{aligned}
\Re a(u,u) &= \int_\Omega \sum_{p,q} a_{p,q} \frac{\partial u}{\partial x_p}\frac{\partial \overline{u}}{\partial x_p}dx - \int_\Omega (\frac{1}{2}\sum_p \frac{\partial a_p}{\partial x_p} - a_0)|u|^2 dx, \qquad (2.5) \\
\Im a(u,u) &= \frac{1}{2}\int_\Omega \sum_p a_p (\frac{\partial u}{\partial x_p}\overline{u} - u\frac{\partial \overline{u}}{\partial x_p})dx
\end{aligned}$$

One gets from (2.2)

$$\int_\Omega \sum_{p,q} \frac{\partial u}{\partial x_p}\frac{\partial \overline{u}}{\partial x_q}dx \geq c_1 |u|_1^2, \tag{2.6}$$

where

$$|u|_1^2 = \sum_p \left| \frac{\partial u}{\partial x_p} \right|^2$$

is the semi-norm of the Sobolev space $H^1(\Omega)$. Let $\|\cdot\|_k$be the norm of the Sobolev space $H^k(\Omega),\ k = 0, 1, ..., H^0(\Omega) = L_2(\Omega)$ and let A be the edge of the cube containing the domain Ω, then the Friedrichs inequality implies

$$|u|_1^2 \geq c_F \|u\|_0^2,\ u \in H_0^1(\Omega)$$

with $c_F = 1/(4A^2)$. Denoting

$$c_2 = \min_x |\frac{1}{2}\sum_p \frac{\partial a_p}{\partial x_p} - a|$$

we obtain from (2.5)

$$\Re a(u,u) \quad \geq \quad c_1|u|_1^2 - c_2\|u\|_0^2. \tag{2.7}$$

Note that (2.7) is the well known Gardings inequality. Further we have

$$\begin{aligned}
|\Im m\ a(u,u)| &\leq \max_{x,p}|a_p(x)|\int_\Omega |u|\sum_p |\frac{\partial u}{\partial x_p}|dx \\
&\leq \max_{x,p}|a_p(x)|\|u\|_0\left(\int_\Omega\left(\sum_p|\frac{\partial u}{\partial x_p}|^2dx\right)\right)^{\frac{1}{2}} \\
&\leq \sqrt{d}\max_{x,p}|a_p(x)|\|u\|_0\left(\sum_p\int_\Omega|\frac{\partial u}{\partial x_p}|^2dx\right)^{\frac{1}{2}} \\
&= c_3\|u\|_0|u|_1
\end{aligned} \tag{2.8}$$

with $c_3 = \sqrt{d}\max_{x,p}|a_p(x)|$. If the coefficients $a_{pq}, a_p, \frac{\partial a_p}{\partial x_p}$ are bounded in Ω then it is easy to see that

$$|a(u,v)| \leq c\|u\|_1\|v\|_1, \tag{2.9}$$

i.e. the sesquilinear form $a(u,v)$ is bounded in H^1 and defines a bounded operator $A : H^1 \to H^{-1}$.This operator with the domain $D(A) = \{u \mid u \in H^2(\Omega)\cap \overset{\circ}{H}{}^1(\Omega)\}$ considered in $L_2(\Omega)$ is unbounded. Then for the numerical range $\{a(u,u) = \xi + i\eta\ \forall u \mid \|u\|_{L_2} = 1\}$ of this operator we have

$$\begin{aligned}
\xi &= \Re a(u,u) \geq c_1|u|_1^2 - c_2 \\
|\eta| &= |\Im a(u,u)| \leq c_3|u|_1,
\end{aligned} \tag{2.10}$$

from where

$$\begin{aligned}
\xi &> c_1c_F - c_2,\ |u|_1^2 \leq \frac{\xi + c_2}{c_1}, \\
&|\eta| < c_3\sqrt{\frac{\xi + c_2}{c_1}}.
\end{aligned} \tag{2.11}$$

It follows from the first and the last inequalities that the numerical range (and the spectrum) of A are enveloped by the parabola $\eta^2 = k(\xi - \xi_0)$ with

$$\begin{aligned} k &= \frac{c_3^2}{c_1} = \frac{\sqrt{d}\max_{x,p}|a_p(x)|}{c_1}, \\ \xi_0 &= -c_2 = -\min_x |\frac{1}{2}\sum_p \frac{\partial a_p}{\partial x_p} - a|, \end{aligned} \tag{2.12}$$

i.e. the parabola is completely determined by the coefficients of the differential equation. It was proved in [5] that A possesses only a discrete spectrum. Supposing that the spectrum of A lies in the right half-plane one can easily see that there exists a parabola $\Gamma = \{z = \xi + i\eta |\ \xi = a\eta^2 + b, \}$ with $a, b > 0$ enveloping the spectrum of A. Analogously to [7] one can prove that

$$\|(z - A)^{-1}\| \leq \frac{M}{1 + \sqrt{|z|}},\ \forall z \in \mathbb{C} \setminus \not\leq_{\lneqq} \tag{2.13}$$

holds with a positive constant M.

Note that the inequality $c_1 c_F - c_2 > 0$ is sufficient to guarantee that the spectrum of A lies in the right half-plane. This example gives a motivation for the following generalization. Let $V \subset H \subset V^*$ be a triple of Hilbert spaces and let $a(\cdot,\cdot)$ be a sesquilinear form on V. We denote by c_e the constant from the imbedding inequality $\|u\|_X \leq c_e \|u\|_V$. Assume that $a(\cdot,\cdot)$ is bounded, i.e.

$$|a(u, v)| \leq c\|u\|_V \|v\|_V,\ u, v \in V. \tag{2.14}$$

The boundedness of $a(\cdot,\cdot)$ implies the well-posedness of a bounded operator $A : V \to V^*$ through the identity

$$a(u, v) =_{V^*} < Au, v >_V, u, v \in V. \tag{2.15}$$

One can restrict A to a domain $D(A) \in V$ and consider this operator (perhaps as unbounded) acting in X. The assumptions

$$\Re a(u, u) \geq \delta_0 \|u\|_V^2 - \delta_1 \|u\|_X^2\ \forall u \in X \tag{2.16}$$

and

$$|\Im a(u, u)| \leq \kappa \|u\|_V \|u\|_X \tag{2.17}$$

guarantee that the numerical range $\{a(u, u),\ \forall u\ |\ \|u\|_X = 1\}$ of A (and the spectrum $\Sigma(A)$) lies inside of a parabola determined by the constants $\delta_0, \delta_1, \kappa$. Actually, if $a(u, u) = \xi + i\eta$ then we get

$$\begin{aligned} \xi &= \Re e\ a(u, u) \geq \delta_0 N_V - \delta_1, \\ |\eta| &= |\Im m\ a(u, u)| \leq \kappa\sqrt{N_V}, \end{aligned} \tag{2.18}$$

where $N_V = \|u\|_V^2$. It implies

$$\xi > \delta_0 c_e^{-2} - \delta_1, \; N_V \le \frac{1}{\delta_0}(\xi + \delta_1),$$
$$|\eta| \le \kappa \sqrt{\frac{\xi + \delta_1}{\delta_0}}. \tag{2.19}$$

The first and the last inequalities mean that the parabola $\Gamma_\delta = \{z = \xi + i\eta : \xi = \frac{\delta_0}{\kappa^2}\eta - \delta_1\}$ envelopes the numerical range of A . It is easy to see that under the assumption $\Re\, \Sigma(A) > 0$ there exists another parabola $\Gamma_0 = \{z = (\xi, \eta) : \xi = a\eta^2 + \gamma_0\}$ with $a, \gamma_0 > 0$ containing the spectrum of A. We denote by Ω_{Γ_0} the domain inside of this parabola. Now, we are in the position to give the following general definition.

Definition 2.4. *We say that an operator $A : E \to E$ is strongly P-positive if its spectrum $\Sigma(A)$ lies in the domain Ω_{Γ_0} enveloped by the parabola Γ_0 and on Γ_0 and outside of Γ_0 the estimate*

$$\|(z - A)^{-1}\|_{E \to E} \le \frac{M}{1 + \sqrt{|z|}} \tag{2.20}$$

holds true with a positive constant M (see [7]).

Note, that there is another important class of operators in the mathematical literature, namely, the strongly positive operators which play a significant role in the theory of semigroups, theory of the strongly elliptic operators, theory of finite-element method and other fields. Contrary to strongly P-positive operators these operators possess a spectrum Σ placed in a symmetric (with respect to the positive x-axis) angle 2ϕ, $\phi < \frac{\pi}{2}$ with the vertex at the origin, $\Re\Sigma > 0$ and having a resolvent satisfying

$$\|(z - A)^{-1}\| \le \frac{M}{1 + |z|}$$

on the edges and outside of the angle.

The strongly elliptic partial differential operators with $\Re\Sigma > 0$ are important examples of both the strongly P-positive and strongly positive operators. Clearly, good finite-difference or finite-element approximations of these partial differential operators have to possess the analogous spectral properties.

In this section we show that the solution of (1.1) can be represented by (1.2) with an integration parabola $\Gamma = \{z = (\xi, \eta) : \xi = \tilde{a}\eta^2 + b, \; \tilde{a} > a, b < \gamma_0\}$ containing the spectral parabola $\Gamma_0 = \{z = (\xi, \eta) : \xi = a\eta^2 + \gamma_0\}$ and under assumptions that A is a strongly P-positive operator and $u_0 \in D(A^\epsilon)$ for any $\epsilon > \frac{1}{2}$ (no other information about the operator is necessary). In fact, after parametrization the path Γ and using the strong P-positivity of A we have

$$\begin{aligned}
\|u(t)\| &= \|\frac{1}{2\pi i}\int_\Gamma e^{-zt}(z-A)^{-1}u_0 dz\| \\
&= \|\frac{1}{2\pi i}\int_\infty^0 e^{-(a\eta^2+b+i\eta)t}(a\eta^2+b+i\eta-A)^{-1}(2a\eta+i)d\eta\, u_0 \\
&+ \frac{1}{2\pi i}\int_0^\infty e^{-(\tilde{a}\eta^2+b-i\eta)t}(a\eta^2+b-i\eta-A)^{-1}(2a\eta-i)d\eta\, u_0\| \\
&\le c\int_0^\infty e^{-(a\eta^2+b)t}\frac{\sqrt{4a^2\eta^2+1}}{1+((a\eta^2+b)^2+\eta^2)^{1/4+\epsilon/2}}\|A^\epsilon u_0\|d\eta. \quad (2.21)
\end{aligned}$$

It is easy to see that this integral converges for all $t>0$ if $\epsilon=0$ and for $t=0$ provided that $\epsilon>\frac{1}{2}$. Analogously we get for the derivative of u

$$\|u'(t)\| = \|\frac{1}{2\pi i}\int_\Gamma -ze^{-zt}(z-A)^{-1}u_0 dz\|$$

$$= \| -\frac{1}{2\pi i}\int_\infty^0 (\tilde{a}\eta^2+b+i\eta)e^{-(\tilde{a}\eta^2+b+i\eta)t}(a\eta^2+b+i\eta-A)^{-1}(2a\eta+i)d\eta\, u_0$$

$$+\frac{1}{2\pi i}\int_0^\infty (\tilde{a}\eta^2+b-i\eta)e^{-(\tilde{a}\eta^2+b-i\eta)t}(a\eta^2+b-i\eta-A)^{-1}(2a\eta-i)d\eta\, u_0\|$$

$$\le c\int_0^\infty \sqrt{(\tilde{a}\eta^2+b)^2+\eta^2}e^{-(a\eta^2+b)t}\frac{\sqrt{4a^2\eta^2+1}}{1+((a\eta^2+b)^2+\eta^2)^{1/4+\epsilon/2}}\|A^\epsilon u_0\|d\eta. \quad (2.22)$$

This integral converges for $t>0$ and $\epsilon=0$. The convergence of these integrals implies that $u(0)=u_0$ and moreover

$$\begin{aligned}
u'(t)+Au(t) &= \frac{1}{2\pi i}\int_\Gamma -ze^{-zt}(z-A)^{-1}u_0dz + A(\frac{1}{2\pi i}\int_\Gamma e^{-zt}(z-A)^{-1}u_0dz) \\
&= \frac{1}{2\pi i}\int_\Gamma -ze^{-zt}(z-A)^{-1}u_0dz + \\
&= \frac{1}{2\pi i}\int_\Gamma ze^{-zt}(z-A)^{-1}u_0dz = 0, \quad (2.23)
\end{aligned}$$

i.e. (1.2) is the solution of (1.1). The parametrized integral (1.2) can be represented in another form

$$u(t) = \frac{1}{2\pi i}\int_{-\infty}^\infty \tilde{F}(\eta,t)d\eta, \quad (2.24)$$

where

$$\tilde{F}(\eta,t) = e^{-zt}\hat{u}(z)\frac{dz}{d\eta} = e^{-(\tilde{a}\eta^2+b-i\eta)t}\hat{u}(\tilde{a}\eta^2+b-i\eta)(2a\eta-i), \quad (2.25)$$

and $\hat{u}(z)$ with $z = \tilde{a}\eta^2 + b - i\eta$ is the solution of the stationary equation

$$(z-A)\hat{u}(z) = u_0. \quad (2.26)$$

A unifying numerical algorithm using the representation (2.24) as well as its analysis will be described in the next section where we will consider the case $u_0 \in X$.

3. *Representation of the solution of the first order equations with an initial function from X*

Let us show that the solution of the problem (1.1) and its derivatives can be represented by the formulas

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{-zt}(z-A)^{-1}u_0 dz, \quad (3.1)$$

$$u'(t) = \frac{1}{2\pi i}\int_\Gamma z e^{-zt}(z-A)^{-1}u_0 dz. \quad (3.2)$$

also in the case $x_0 \in X$ provided that A possesses a discrete spectrum consisting of the eigenvalues $\lambda_j = \mu_j + i\nu_j, j = 1,2,...$ with $\Re\lambda_j > \gamma$ which correspond to the eigenfunctions $e_j, j = 1,2,...$ being a basis of X. In this case there exists a biorthogonal system $f_j, j = 1,2,...$ in the dual space X^* such that $< e_k, f_j >= \delta_{k,j}$, where $<,>$ denotes the duality relation. One can represent

$$u_0 = \sum_{j=1}^{\infty}\alpha_j e_j, \ \alpha_j =< u_0, f_j > \quad (3.3)$$

with

$$\|u_0\|^2 = \sum_{j=1}^{\infty} | < u_0, f_j > |^2 \quad (3.4)$$

i.e.

$$u(t) = \frac{e^{-bt}}{2\pi i}\sum_{j=1}^{\infty}\alpha_j e_j I_{1,j}(t), \quad (3.5)$$

where

$$I_{1,j}(t) = \int_\Gamma e^{-(z-b)t}(z-\lambda_j)^{-1}dz.$$

We have to show that the integrals in the last formula converge uniformly with respect to t. We represent $\Gamma = \Gamma_+ + \Gamma_-$ in the parametric form with

$$\Gamma_\pm = \{z = \tilde{a}\eta^2 + b \pm i\eta, \eta \geq 0\}.$$

Then

$$\begin{aligned} I_{1,j}(t) &= \int_{\Gamma_+} e^{-(z-b)t}(z-\lambda_j)^{-1}dz + \int_{\Gamma_-} e^{-(z-b)t}(z-\lambda_j)^{-1}dz \quad (3.6) \\ &= \int_0^\infty e^{-\tilde{a}\eta^2 t}[f_+(\eta,t) + f_-(\eta,t)]d\eta \\ &= \int_0^\infty e^{-\tilde{a}\eta^2 t} F(\eta,t)d\eta, \end{aligned}$$

where

$$\begin{aligned} f_\pm(\eta,t) &= \mp e^{\mp i\eta t}\frac{2\tilde{a}\eta \pm i}{\tilde{a}\eta^2 + b - \mu_j \pm i(\eta \mp \nu_j)}, \\ F(\eta,t) &= [f_+(\eta,t) + f_-(\eta,t)]. \end{aligned} \tag{3.7}$$

It is easy to see that for $t > 0$ one can also write down

$$I_{1,j}(t) = -\int_{-\infty}^\infty e^{-(\tilde{a}\eta^2 + i\eta)t}\frac{2\tilde{a}\eta + i}{\tilde{a}\eta^2 + b - \mu_j + i(\eta - \nu_j)}d\eta. \tag{3.8}$$

For $t = 0$ we have

$$\begin{aligned} I_{1,j}(t) &= -\int_0^\infty \frac{2\tilde{a}\eta + i}{\tilde{a}\eta^2 + b + i\eta - \lambda_j} + \int_0^\infty \frac{2\tilde{a}\eta - i}{\tilde{a}\eta^2 + b - i\eta - \lambda_j}d\eta \\ &= 2i\int_0^\infty \frac{2\tilde{a}\eta^2 - b + \lambda_j}{(\tilde{a}\eta^2 + b + i\eta - \lambda_j)(\tilde{a}\eta^2 + b - i\eta - \lambda_j)}d\eta, \end{aligned} \tag{3.9}$$

which means that the integral (3.6) converges also in this case. We see from (3.6),(3.9) that each of integrals (3.6) converges uniformly with respect to $t \in (0,\infty)$ and there exists a finite one

$$\begin{aligned} &\sup_j |I_{1,j}(t)| \leq \int_0^\infty e^{-\tilde{a}\eta^2 t}\left\{\frac{|2\tilde{a}\eta + i|}{|\tilde{a}\eta^2 + b + i\eta - \lambda_+|}\right. \\ &\left. + \frac{|2\tilde{a}\eta - i|}{|\tilde{a}\eta^2 + b - i\eta - \lambda_-|}\right\} d\eta, \end{aligned}$$

for $t > 0$ where $\lambda_+,\ \lambda_-$ denote the nearest eigenvalues to Γ_+ and Γ_- respectively. It follows from (3.5) that

$$\|u(t)\|^2 = \frac{e^{-bt}}{4\pi^2}\sum_j {\alpha_j}^2 I_{1,j}^2 \leq c^2 e^{-bt}\sum_j {\alpha_j}^2 = c^2 e^{-bt}\|u_0\|^2 \tag{3.10}$$

with $c^2 = \frac{1}{4\pi^2} \sup_j I_{1,j}^2$. In order to show that the function (3.1) satisfies the differential equation we consider the integral (3.2) in the form

$$u'(t) = -\frac{1}{2\pi i} \sum_j \alpha_j e_j I_{1,j}^{(1)}(t) \tag{3.11}$$

with

$$\begin{aligned}
I_{1,j}^{(1)}(t) &= \int_\Gamma z e^{-zt}(z-\lambda_j)^{-1} dz \\
&= \int_{\Gamma_+} z e^{-zt}(z-\lambda_j)^{-1} dz + \int_{\Gamma_-} z e^{-zt}(z-\lambda_j)^{-1} dz \\
&= \int_0^\infty e^{-(a\eta^2+b)t} F_1(\eta,t) d\eta, \\
F_1(\eta,t) &= -(\tilde{a}\eta^2 + b + i\eta) e^{-i\eta t} \frac{2\tilde{a}\eta + i}{a\eta^2 + b - \mu_j + i(\eta - \nu_j)} \\
&+ (\tilde{a}\eta^2 + b - i\eta) e^{i\eta t} \frac{2\tilde{a}\eta - i}{a\eta^2 + b - \mu_j - i(\eta + \nu_j)}.
\end{aligned}$$

Analogously as above one can see that the integral $I_{1,j}^{(1)}(t)$ converges uniformly for all $t > 0$. Therefore, the function (3.1) satisfies the initial condition $u(0) = u_0$ and the differential equation

$$\begin{aligned}
u'(t) + Au(t) &= -\frac{1}{2\pi i} \int_\Gamma z e^{-zt}(z-A)^{-1} u_0 dz \\
&+ \frac{1}{2\pi i} \int_\Gamma z e^{-zt}(z-A)^{-1} u_0 dz = 0.
\end{aligned}$$

Thus, we have proved the following result.

Theorem 3.1. *Let Γ be an integration parabola containing the spectral parabola Γ_0 of the operator A. Then the solution of problem (1.1) can be represented by the integral (1.2).*

4. *Computational algorithm and analysis*

Following [14], we construct a quadrature rule for the integral in (2.24) by using the Sinc approximation on $(-\infty, \infty)$. For $1 \le p \le \infty$, we introduce the family $\mathbf{H}^p(D_d)$ of all vector-valued functions, which are analytic in the infinite strip D_d,

$$D_d = \{z \in \mathbb{C} : -\infty < \Re F < \infty, |\Im F| < \}, \tag{4.1}$$

such that if $D_d(\epsilon)$ is defined for $0 < \epsilon < 1$ by

$$D_d(\epsilon) = \{z \in \mathbb{C} : |\Re F| < \nVdash/\epsilon, |\Im F| < (\nVdash - \epsilon)\} \tag{4.2}$$

then for each $\mathcal{F} \in \mathbf{H}^p(D_d)$ there holds $\|\mathcal{F}\|_{\mathbf{H}^p(D_d)} < \infty$ with

$$\|\mathcal{F}\|_{\mathbf{H}^p(D_d)} = \begin{cases} \lim\limits_{\epsilon\to 0}(\int\limits_{\partial D_d(\epsilon)} \|\mathcal{F}(z)\|^p |dz|)^{1/p} & if \quad 1 \le p < \infty, \\ \lim\limits_{\epsilon\to 0} \sup_{z\in D_d(\epsilon)} \|\mathcal{F}(z)\| & if \quad p = \infty. \end{cases} \tag{4.3}$$

Let

$$S(k,h)(x) = \frac{\sin\left[\pi(x-kh)/h\right]}{\pi(x-kh)/h} \tag{4.4}$$

be the k-th Sinc function with step size h, evaluated in x. Given $\mathcal{F} \in \mathbf{H}^p(D_d), h > 0$ and positive integer N, let us use the notations

$$I(\mathcal{F}) = \int_{\mathbb{R}} \mathcal{F}(x)dx, \qquad T_N(\mathcal{F},h) = h\sum_{k=-N}^{N} \mathcal{F}(kh),$$

$$T(\mathcal{F},h) = h\sum_{k=-\infty}^{\infty} \mathcal{F}(kh),$$

$$C(\mathcal{F},h) = \sum_{k=-\infty}^{\infty} \mathcal{F}(kh)S(k,h),$$

$$\eta_N(\mathcal{F},h) = I(\mathcal{F}) - T_N(\mathcal{F},h), \qquad \eta(\mathcal{F},h) = I(\mathcal{F}) - T(\mathcal{F},h).$$

Applying the quadrature rule T_N with the vector-valued function

$$F(\eta,t) = (2\tilde{a}\eta - i)\varphi(\eta)\hat{u}(\eta) \tag{4.5}$$

where

$$\varphi(\eta) = e^{-t\psi(\eta)}, \quad \psi(\eta) = \tilde{a}\eta^2 + b - i\eta, \tag{4.6}$$

we obtain for integral (2.24)

$$u(t) = T(t)u_0 \equiv exp(-tA)u_0 \approx u_N(t) =$$

$$= T_N(t)u_0 \equiv exp_N(-tA)u_0 = h\sum_{k=-N}^{N} F(kh,t). \tag{4.7}$$

This quadrature rule allows to introduce the following algorithm for the solution of problem (1.1) at a given time value t.

Algorithm 4.1. Parallel solving of problem (1.1).

1. Given a, γ_0, choose $k > 1, \tilde{a} = \frac{a}{k}$, $d = (1 - \frac{1}{\sqrt{k}})\frac{k}{2a}$, N and determine z_p, α_p $(p = -N, \ldots, N)$ by $z_p = \frac{a}{k}(ph)^2 + b - iph$, $\alpha_p = 2ph - i$, where $h = \sqrt[3]{\frac{2\pi dk}{a}}(N+1)^{-2/3}$ and $b = \gamma_0 - \frac{k-1}{4a}$.

2. Solve the equations $(z_p - A)\hat{u}(z) = u_0$, $p = -N, \ldots, N$ (note that it can be done in parallel).

3. Find the approximation u_N for the solution of (1.1) in the form

$$u_N(t) = h \sum_{j=-N}^{N} \alpha_j e^{-z_j t} \hat{u}(z_j). \tag{4.8}$$

Remark 4.2. *The above algorithm possesses two sequential levels of parallelism: first, one can compute all $\hat{u}(z_p)$ at Step 2 in parallel and, second, each operator exponent at different time values $(t_1, t_2, \ldots, t_M)$).*

Adapting the ideas of [14] one can prove the following approximation results for functions from $\mathbf{H}^1(D_d)$.

Lemma 4.3. *For any vector-valued function $f \in \mathbf{H}^1(D_d)$, there holds*

$$\eta(f,h) = \frac{i}{2}\int_{\mathbb{R}} \left\{ \frac{f(\xi - id^-)e^{-\pi(d+i\xi)/h}}{\sin\left[\pi(\xi - id)/h\right]} - \frac{f(\xi + id^-)e^{-\pi(d-i\xi)/h}}{\sin\left[\pi(\xi + d)/h\right]} \right\} d\xi \tag{4.9}$$

which yields the estimate

$$\|\eta(f,h)\| \le \frac{e^{-\pi d/h}}{2\sinh\left(\pi d/h\right)} \|f\|_{\mathbf{H}^1(D_d)}. \tag{4.10}$$

If in addition, f satisfies on $\mathbb{R}$ the condition

$$\|f(x)\| < ce^{-\alpha x^2}, \quad \alpha, c > 0, \tag{4.11}$$

then

$$\|\eta_N(f,h)\| \le c\sqrt{\pi}\left[\frac{exp(-2\pi d/h)}{\sqrt{\alpha}(1 - exp(-2\pi d/h))} + \frac{exp[-\alpha(N+1)^2h^2]}{\alpha h(N+1)}\right]. \tag{4.12}$$

Proof. Let $E(f,h)$ be defined as follows

$$E(f,h)(z) = f(z) - C(f,h)(z).$$

Analogously to [14] (see Theorem 3.1.2) one can get

$$\begin{aligned} E(f,h)(z) &= f(z) - C(f,h)(z) \\ &= \frac{\sin\left(\pi z/h\right)}{2\pi i}\int_{\mathbb{R}} \left\{ \frac{f(\xi - id^-)}{(\xi - z - id)\sin\left[\pi(\xi - id)/h\right]} \right. \\ &\quad \left. - \frac{f(\xi + id^-)}{(\xi - z + id)\sin\left[\pi(\xi + id)/h\right]} \right\} d\xi \end{aligned} \tag{4.13}$$

and upon replacing z by x we have

$$\eta(f,h) = \int_{\mathbb{R}} E(f,h)(x)dx. \tag{4.14}$$

After interchanging the order of integration and using the identities

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\sin(\pi x/h)}{\pm(\xi - x) - id}dx = \frac{i}{2}e^{-\pi(d\pm i\xi)/h}, \tag{4.15}$$

we obtain (4.9). Using the estimate (see [14], p.133) $\sinh(\pi d/h) \leq |\sin[\pi(\xi \pm id)/h]| \leq \cosh(\pi d/h)$, the assumption $f \in \mathbf{H}^1(D_d)$ and the identity (4.9), we obtain the desired bound (4.10). The assumption (4.11) now implies

$$\begin{aligned}\|\eta_N(f,h)\| &\leq \|\eta(f,h)\| + h\sum_{|k|>N}\|f(kh)\| \\ &\leq \frac{exp(-\pi d/h)}{2\sinh(\pi d/h)}\|f\|_{\mathbf{H}^1(D_d)} + c\,h\sum_{|k|>N} exp[-\alpha(kh)^2].\end{aligned} \tag{4.16}$$

For the last sum we use the simple estimate

$$\sum_{|k|>N} e^{-\alpha(kh)^2} = 2\sum_{k=N+1}^{\infty} e^{-\alpha(kh)^2} \leq$$

$$\leq 2\int_{N+1}^{\infty} e^{-\alpha h^2 x^2}dx = \frac{2}{\sqrt{\alpha}h}\int_{\sqrt{\alpha}h(N+1)}^{\infty} e^{-x^2}dx = \tag{4.17}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\alpha}h}erfc(\sqrt{\alpha}h(N+1)) =$$

$$= \frac{\sqrt{\pi}}{\sqrt{\alpha}h}e^{-(N+1)^2\alpha h^2}\psi(\frac{1}{2},\frac{1}{2};(N+1)^2\alpha h^2), \tag{4.18}$$

where $\psi(\frac{1}{2},\frac{1}{2};(N+1)^2\alpha h^2)$ is the Whittecker's function with the asymptotics [3]

$$\psi(\frac{1}{2},\frac{1}{2};x^2) = \sum_{n=0}^{M}(-1)^n\left(\frac{1}{2}\right)_n x^{-(2n+1)} + \mathcal{O}(|x|^{-2M-3}). \tag{4.19}$$

This yields

$$\sum_{|k|>N} e^{-\alpha(kh)^2} \leq \frac{\sqrt{\pi}}{\alpha h^2(N+1)}e^{-\alpha(N+1)^2h^2}. \tag{4.20}$$

It follows from (4.11) that

$$\|f\|_{\mathbf{H}^1(D_d)} \le 2c\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \frac{2c}{\sqrt{\alpha}}\sqrt{\pi} \tag{4.21}$$

which together with (4.16) and (4.20) implies

$$\|\eta_N(f,h)\| \le c\sqrt{\pi}\left[\frac{exp(-\pi d/h)}{\sqrt{\alpha}\sinh(\pi d/h)} + \frac{exp[-\alpha(N+1)^2h^2]}{\alpha h(N+1)}\right],$$

which completes the proof.

Theorem 4.4. *Choose* $k>1,\ \tilde{a}=a/k,\ h=\sqrt[3]{2\pi dk/((N+1)^2a)},\ b=b(k)=\gamma_0-(k-1)/(4a)$ *and the integration parabola* $\Gamma_{b(k)}=\{z=\tilde{a}\eta^2+b(k)-i\eta:\eta\in(-\infty,\infty)\}$, *then there holds*

$$\|u(t)-u_N(t)\| \equiv \|(exp(-tA)-\exp_N(-tA))u_0\| \le$$

$$\le Mc\sqrt{\pi}\left[\frac{2\sqrt{k}exp[-s(N+1)^{2/3}]}{\sqrt{at}(1-exp(-s(N+1)^{2/3}))} + \frac{kexp[-ts(N+1)^{2/3}]}{t(N+1)^{1/3}\sqrt[3]{2\pi dka^2}}\right]\|u_0\|, \tag{4.22}$$

where

$$s=\sqrt[3]{(2\pi d)^2a/k},$$
$$c=M_1e^{t[ad^2/k+d-b]},\quad d=(1-\frac{1}{\sqrt{k}})\frac{k}{2a}, \tag{4.23}$$
$$M_1=\max_{z\in\overline{D}_d}\frac{|2\frac{a}{k}z-i|}{1+\sqrt{|\frac{a}{k}z^2+b-iz|}}$$

and M *is the constant from the inequality of the strong P-positiveness.*

Proof. First, we note that one can choose as integration path any parabola

$$\Gamma_b=\{z=\frac{a}{k}\eta^2+b+i\eta:\eta\in(-\infty,\infty),k>1,b<\gamma_0\}, \tag{4.24}$$

which contains the spectral parabola

$$\Gamma_0=\{z=a\eta^2+\gamma_0+i\eta:\eta\in(-\infty,\infty)\}. \tag{4.25}$$

In order to apply Lemma 4. to the quadrature rule u_N we have to provide that the integrand $F(\eta,t)$ can be analytically extended in a strip D_d around the real axis η. It is easy to see that it is sufficient that there exists $d>0$ such that for $|\nu|<d$ the function (transformed resolvent)

$$R(\eta+i\nu,A)=[\psi(\eta+i\nu)I-A]^{-1},\ \eta\in(-\infty,\infty),\ |\nu|<d \tag{4.26}$$

has a bounded norm $\|R\|_{X\to X}$. Due to the strong P-positivity of A, the latter can be easily verified if the parabola set

$$\Gamma_b(\nu) = \{z = \frac{a}{k}(\eta + i\nu)^2 + b + i(\eta + i\nu) : \ \eta \in (-\infty, \infty), \ |\nu| < d\} \tag{4.27}$$
$$= \{z = \frac{a}{k}\eta^2 + b + \frac{k}{4a} - \frac{a}{k}(\nu + \frac{k}{2a})^2 + i\eta(1 + \frac{2a}{k}\nu); \ \eta \in (-\infty, \infty), \ |\nu| < d\}$$

does not intersect Γ_0. We represent parabolae from the set $\Gamma_b(\nu)$ in the form $\xi = a'\eta^2 + b'$ with

$$a' = a\left(k + 4a\nu + \frac{4a^2}{k}\nu^2\right)^{-1}, b' = b + \frac{k}{4a} - \frac{a}{k}\left(\nu + \frac{k}{2a}\right)^2. \tag{4.28}$$

Now, it is easy to see that if we choose

$$\nu = \left(\frac{1}{\sqrt{k}} - 1\right)\frac{k}{2a} \equiv -d, \ b = b(k) = \gamma_0 - \frac{k-1}{4a} \tag{4.29}$$

then

$$\Gamma_{b(k)}(-d) = \{z = \frac{a}{k}\eta^2 + b + \frac{k-1}{4a} + i\frac{\eta}{\sqrt{k}} : \ \eta \in (-\infty, \infty)\}$$
$$= \{z = \tilde{a}\eta_*^2 + \gamma_0 + i\eta_* : \ \eta_* \equiv \frac{\eta}{\sqrt{k}} \in (-\infty, \infty)\} \equiv \Gamma_0. \tag{4.30}$$

From (4.28), one can see that $a' \to 0, \ b' \to 0$ monotonically with respect to ν as $\nu \to \infty$, i.e. the parabolae from $\Gamma_b(\nu)$ move away from the spectral parabola Γ_0 monotonically. This means that the parabolae set $\Gamma_b(\nu)$ for $b = b(k), \ |\nu| < d$ lies outside of the spectral parabola Γ_0, i.e. we can extend the integrand into the strip (4.1) with d given by (4.29). Note, that the choice $\nu = d = (1 - 1/\sqrt{k})\frac{k}{2a}$ selects from the family $\Gamma_{b(k)}(\nu)$ the particular parabola

$$\Gamma_{b(k)}(d) = \{z = a\eta^2/k + b_+ + i\eta(2 - 1/\sqrt{k}) : \eta \in (-\infty, \infty)\}$$
$$= \{z = a_+\eta_*^2 + b_+ + i\eta_* : \eta_* \equiv \eta(2 - 1/\sqrt{k}) \in (-\infty, \infty)\} \tag{4.31}$$

with

$$a_+ = \frac{a}{k(2 - 1/\sqrt{k})^2}, \ b_+ = b - \frac{3k - 4\sqrt{k} + 1}{4a},$$

which for $|\nu| \le d$ is the most remote from the spectral parabola Γ_0. Due to the strong P-positivity of A there holds for $z = \eta + i\nu \in D_d$

$$\|F(z,t)\| \le M\frac{|(2\frac{a}{k}z - i)|\, exp[-t(\frac{a}{k}z^2 + b - iz)]}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}}\|u_0\| =$$

$$= M\frac{|2\frac{a}{k}z - i| exp\{-t[\frac{a}{k}(\eta^2 - \nu^2) + b + \nu]\}}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}} \in \mathbf{H}^1(D_d), \quad \forall t > 0. \quad (4.32)$$

We also have

$$\|F(\eta, t)\| < ce^{-\alpha\eta^2}, \ \eta \in \mathbb{R} \quad (4.33)$$

with

$$\alpha = t\frac{a}{k}, \quad c = M_1 e^{t[ad^2/k + d - b]}, \quad M_1 = \max_{z \in \overline{D}_d} \frac{|2\frac{a}{k}z - i|}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}}. \quad (4.34)$$

Using Lemma 4. with $\alpha = t\frac{a}{k}$ in (4.12) we get

$$\|\eta_N(F, h)\| \le Mc\sqrt{\pi}\left[\frac{2\sqrt{k} exp(-2\pi d/h)}{\sqrt{at}(1 - exp(-2\pi d/h))} + \frac{k exp[-(N+1)^2 h^2 \frac{a}{k} t]}{ath(N+1)}\right]. \quad (4.35)$$

Equalizing the exponents by setting $-2\pi d/h = -(N+1)^2 h^2 a/k$, we obtain

$$h = \sqrt[3]{\frac{2\pi dk}{a}}(N+1)^{-2/3}. \quad (4.36)$$

Substitution of this value into (4.35) leads to the estimate

$$\|\eta_N(F, h)\| \le Mc\sqrt{\pi}\left[\frac{2\sqrt{k}e^{-s(N+1)^{2/3}}}{\sqrt{at}(1 - e^{-s(N+1)^{2/3}})} + \frac{ke^{-ts(N+1)^{2/3}}}{t(N+1)^{1/3}\sqrt[3]{2\pi dka^2}}\right]\|u_0\|, \quad (4.37)$$

which completes our proof.

Remark 4.5. *Choosing k such that*

$$1 < \sqrt{k} < \frac{1}{2} + \sqrt{\frac{1}{4} + a\gamma_0}$$

we get that $d < \gamma_0/2$ and

$$-\frac{a}{k}\nu^2 + \nu + b > 0 \ \forall z \in [-d, d],$$

i.e. the constant c in (4.34) tends exponentially to 0 when $t \to \infty$.

Remark 4.6. *The theorem 4.4 guarantees the exponential convergence of the algorithm provided that $t > 0$. For $t = 0$ we have to compute the integrals*

$$\begin{aligned} I_{1,j}(0) &= \int_\Gamma (z - \lambda_j)^{-1} dz \\ &= 2i\int_0^\infty \mathcal{F}(\eta, \lambda_j) d\eta, \end{aligned} \quad (4.38)$$

where

$$\mathcal{F}(\eta,\lambda_j) = \frac{2\tilde{a}\eta^2 - b + \lambda_j}{(\tilde{a}\eta^2 + b + i\eta - \lambda_j)(\tilde{a}\eta^2 + b - i\eta - \lambda_j)}. \tag{4.39}$$

The algorithm for this integral reads as follows

$$I_{1,j}^{(N)}(0) = 2ih\sum_{p=-N}^{N} \frac{2\tilde{a}(ph)^2 - b + \lambda_j}{(\tilde{a}(ph)^2 + b + iph - \lambda_j)(\tilde{a}(ph)^2 + b - iph - \lambda_j)} =$$

$$= 2ih\left[\frac{-1}{(b-\lambda_j)} + \sum_{p=1}^{N} \frac{2\tilde{a}(ph)^2 - b + \lambda_j}{(\tilde{a}(ph)^2 + b + iph - \lambda_j)(\tilde{a}(ph)^2 + b - iph - \lambda_j)}\right]. \tag{4.40}$$

The error is given by

$$I_{1,j}(0) - I_{1,j}^{(N)}(0) = \int\limits_0^{(N+1)h} \mathcal{F}(\eta,\lambda_j)d\eta -$$

$$-h\left\{\frac{1}{2}\mathcal{F}(0,\lambda_j) + \sum_{p=1}^{N}\mathcal{F}(ph,\lambda_j) + \frac{1}{2}\mathcal{F}(N+1,\lambda_j)\right\} + h\mathcal{F}(N+1,\lambda_j) +$$

$$+ \int\limits_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta = Tr_N(\lambda_j) + h\mathcal{F}(N+1,\lambda_j) + \int\limits_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta. \tag{4.41}$$

where

$$Tr_N(\lambda_j) = \int\limits_0^{(N+1)h} \mathcal{F}(\eta,\lambda_j)d\eta -$$

$$-h\left\{\frac{1}{2}\mathcal{F}(0,\lambda_j) + \sum_{p=1}^{N}\mathcal{F}(ph,\lambda_j) + \frac{1}{2}\mathcal{F}(N+1,\lambda_j)\right\}$$

stays for the error of the trapezoidal rule. Then we get

$$h\left|\mathcal{F}(N+1,\lambda_j)\right| \le hM\frac{1}{\left|\tilde{a}\left[(N+1)h\right]^2 + b - i(N+1)h - \lambda_-\right|} \le C(N+1)^{-4/3},$$

$$\left|\int\limits_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta\right| \le M \int\limits_{\sqrt[3]{\frac{2\pi dk}{a}}(N+1)^{1/3}}^{\infty} \frac{d\eta}{|\tilde{a}\eta^2 + b - i\eta + \gamma_-|} \le$$

$$\leq M \int\limits_{\sqrt[3]{\frac{2\pi dk}{a}}(N+1)^{1/3}}^{\infty} \frac{d\eta}{\tilde{a}\eta^2} = C(N+1)^{-1/3},$$

$$|Tr_N(\lambda_j)| \leq c \sum_{p=0}^{N} \left| \int\limits_{ph}^{(p+1)h} \mathcal{F}(\eta, \lambda_j) d\eta - h \left[\mathcal{F}(ph, \lambda_j) + \mathcal{F}((p+1)h, \lambda_j) \right] \right| \leq$$

$$\leq c \sum_{p=0}^{N} \left| \int\limits_{ph}^{(p+1)h} \left\{ \mathcal{F}(\eta, \lambda_j) - \frac{1}{2} \left[\mathcal{F}(ph, \lambda_j) + \mathcal{F}((p+1)h, \lambda_j) \right] \right\} d\eta \right| \leq$$

$$\leq c \sum_{p=0}^{N} \left| \int\limits_{ph}^{(p+1)h} \left[\int\limits_{ph}^{\eta} (s-\eta) \frac{d^2 \mathcal{F}(s, \lambda_j)}{ds^2} ds + \int\limits_{(p+1)h}^{\eta} (s-\eta) \frac{d^2 \mathcal{F}(s, \lambda_j)}{ds^2} ds \right] \right| \leq$$

$$\leq Ch^2 = C_1(N+1)^{-4/3}. \tag{4.42}$$

Thus, we have for $t = 0$

$$\left| I_{1,j}(\lambda_j) - I_{1,j}^{(N)}(\lambda_j) \right| \leq C(N+1)^{-1/3}. \tag{4.43}$$

We now turn to discretization in both space and time. Let $\hat{u}_{h_1}(z_j)$ be the solution of the discrete problem

$$(z_j - A_{h_1}) \hat{u}_{h_1}(z_j) = P_{h_1} u_0$$

with a discretization A_{h_1} for A and a projection operator P_{h_1} so that

$$\| \hat{u}_{h_1}(z_j) - \hat{u}(z_j) \| \leq c h_1^k \| u_0 \|$$

holds uniformly in z. The fully discrete approximation for the solution of (1.1) is then defined by

$$u_{N,h_1}(t) = h \sum_{j=-N}^{2N-1} \alpha_j \hat{u}_{h_1}(z_j) e^{-z_j t} \tag{4.44}$$

and , since the errors are additive, this will give a complete error estimate

$$\| u(t) - u_{N,h_1}(t) \| \leq c(\eta_N(F,h) + h_1^k) \| u_0 \|,$$

where $\eta_N(F,h)$ decays for $t > 0$ with the order $\mathcal{O}(e^{-cN^{3/2}})$.

5. *Numerical examples*

Example 5.1. Let us consider the following problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t) = u(1,t) = 0, u(x,0) = \sin \pi x$$

with the exact solution $u(x,t) = e^{-\pi^2 t} \sin \pi x$. The numerical solution was computed in accordance with Algorithm ($a = 1, b = 1, k = 2$) where the step 2 was performed using explicit formulas. The error $\varepsilon_N = \varepsilon_N(x,t) = u(x,t) - u_N(x,t)$ for $x = 0.5$ as a function of N is given by Table 1 and is in a good agreement with the estimate (4.22).

t	ε_{16}	ε_{32}	ε_{64}	ε_{128}
0	6.1458e-01	3.2313e-01	1.7426e-01	1.1123e-01
0.2	4.7224e-02	1.3944e-02	1.8814e-03	1.1108e-04
0.4	8.4978e-04	2.3909e-04	3.9416e-06	3.4460e-08
0.6	5.2588e-04	2.0196e-05	2.1643e-07	1.5619e-10
0.8	1.1102e-04	1.9911e-06	2.6417e-09	6.3806e-14
1.0	1.1094e-05	5.8952e-08	2.0010e-11	3.6594e-16

t	ε_{256}	ε_{512}	ε_{1024}
0	7.8104e-02	5.7705e-02	4.3870e-02
0.2	1.6160e-06	2.0480e-09	3.4646e-14
0.4	4.8676e-11	3.9544e-16	3.2618e-25
0.6	3.6177e-16	5.4619e-23	3.4073e-36
0.8	8.8512e-20	4.9334e-30	2.4506e-47
1.0	2.3290e-24	5.1397e-38	3.0872e-58

Table 0.1: The error of the Algorithm 5.1

The next table presents the error of Algorithm at $x = 0.5,\ t = 0$ and the experimental convergence rate with respect to $N^{-\rho}$ indicating the order $\rho \approx 1/3$.

N	ε_N	ρ_N
2048	0.03391	0.369
4096	0.02648	0.356

Table 0.2: The convergence rate at $t = 0$.

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