GENERALIZED LUCAS POLYNOMIALS AND NEWTON SUM RULES FOR THE ZERO'S DISTRIBUTION OF POLYNOMIAL SOLUTIONS OF O.D.E.

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Abstract

We show some useful applications of the Generalized Lucas Polynomials of the first and second kind. In fact, by using some preceding results of E. Buendia, J. S. Dehesa, F. J. Gálvez [6], P.E. Ricci [41], P. Natalini [34], and P. Natalini - P.E. Ricci [37], we give explicit representation formulas for the Newton sum rules of polynomial solutions of ordinary differential equations with polynomial coefficients.

As an example, we compute the Newton sum rules of the associated and co-recursive of the classical Hermite, Laguerre and Jacobi polynomials, starting from the differential equations satisfied by the associated (see S. Belmedhi - A. Ronveaux [3], A. Zarzo - A. Ronveaux - E. Godoy [45]) and co-recursive (see A. Ronveaux - F. Marcellan [42], A. Ronveaux - A. Zarzo - E. Godoy [43]) of all classical orthogonal polynomials.

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1. *Introduction*

The Generalized Lucas Polynomials of second kind are a standard mathematical tool for obtaining representation formulas for powers of square matrices (see e.g. I.V.V. Raghavacharyulu - A.R. Tekumalla [40]). They are connected with the multidimensional generalization of the Chebyshev polynomials of the second kind, studied by M. Bruschi - P.E. Ricci [5]. More general definitions and extensions of this class of multidimensional polynomials can be found in papers by R. Lidl [28], R. Lidl et al. [14], [29], T. Koornwinder [20], [21], R.J. Beerends [2].

Namely, the following proposition is valid:

Proposition 1..1 – *Let $\mathcal{A}$ be an $r \times r$ complex matrix and denote by*

$$P(\lambda) := det(\lambda\mathcal{I} - \mathcal{A}) = \sum_{j=0}^{r}(-1)^j u_j \lambda^{r-j}$$

its characteristic polynomial (by convention $u_0 := 1$), then for the powers of $\mathcal{A}$ with integral exponents the following representation formula holds true:

$$\mathcal{A}^n = F_{1,n-1}(u_1, \ldots, u_r)\mathcal{A}^{r-1} + F_{2,n-1}(u_1, \ldots, u_r)\mathcal{A}^{r-2} + \tag{1.1}$$

$$+ \ldots + F_{r,n-1}(u_1, \ldots, u_r)\mathcal{I}$$

The functions $F_{k,n}(u_1, \ldots, u_r)$ which appear as coefficients in the preceding linear combination are defined by the recurrence relation

$$F_{k,n}(u_1, \ldots, u_r) = u_1 F_{k,n-1}(u_1, \ldots, u_r) - u_2 F_{k,n-2}(u_1, \ldots, u_r) +$$

$$+ \ldots + (-1)^{r-1} u_r F_{k,n-r}(u_1, \ldots, u_r), \tag{1.2}$$

$$(k = 1, \ldots, r;\ n \geq -1)$$

and initial conditions:

$$F_{r-k+1,h-2}(u_1, \ldots, u_r) = \delta_{k,h} \tag{1.3}$$

$$(k, h = 1, \ldots, r)$$

Furthermore, if $\mathcal{A}$ is non-singular ($u_r \neq 0$), then formula (1.1) still holds for negative values of n, provided that we define the $F_{k,n}$ functions for negative values of n as follows:

$$F_{k,n}(u_1, \ldots, u_r) = F_{r-k+1,-n+r-3}(\frac{u_{r-1}}{u_r}, \ldots, \frac{u_1}{u_r}, \frac{1}{u_r}),$$

$$(k = 1, \ldots, r;\ n < -1).$$

Another, but less known application of the above mentioned multidimensional Lucas polynomials is the possibility to obtain representation formulas for the Newton sum rules, starting from the coefficients of the differential equations satisfied by the considered set of polynomials (see E. Buendia, J.S. Dehesa, F.J. Gálvez [6], P.E. Ricci [41], P. Natalini [34], and P. Natalini - P.E. Ricci [37]).

The Generalized Lucas polynomials of the first kind are connected in a similar way with the so called multidimensional Chebyshev polynomials of the first kind, which have been introduced and studied by the same Authors.

The above multidimensional polynomials (i.e. Lucas 1st and 2nd kind) have been used for computing the Newton sum rules, starting from entries of the Jacobi matrix, i.e. from the coefficients of the three-term recurrence relation satisfied by all Orthogonal Polynomials (see B. Germano, P.E. Ricci [15], but they can also be used, to the same aim, starting from the coefficients of the differential equations (when this exists) satisfied by the considered set of polynomials.

In the following, we first recall the method of computing the Newton sum rules described in [6], [37], in the general case of polynomial solutions of non-hypergeometric type differential equations. Then we apply this method to the case of the associated and co-recursive classical orthogonal polynomials (see [10], [11], [19], [25], [26], [27], [44], [17], [18]).

2. *Generalized Lucas polynomials of the second kind*

Consider the bilateral linear homogeneous recurrence relation with $r+1$ terms and constant complex coefficients u_k, $(k = 1, 2, \dots, r)$, and suppose $u_r \neq 0$:

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \dots + (-1)^{r-1} u_r X_{n-r}, \quad (n \in \mathbf{Z}) \tag{2.1}$$

this is just the recurrence relation satisfied by the $F_{k,n}$ functions. Furthermore, the $F_{k,n}$ functions are determined by the initial conditions

$$(1, 0, 0, \dots, 0), \quad (0, 1, 0, \dots, 0), \quad \dots, \quad (0, 0, \dots, 0, 1),$$

in fact:

$$F_{r-k+1,h-2}(u_1, \dots, u_r) = \delta_{k,h}, \quad (k, h = 1, \dots, r)$$

Then they constitute a basis of the vectorial space $\mathcal{V}_r$ of all solutions of the recurrence relation (2.1).

Remark 2..1 *Usually a basis of $\mathcal{V}_r$ is obtained by considering the roots of the characteristic equation*

$$z^r - u_1 z^{r-1} + \ldots + (-1)^r u_r = 0 \tag{2.2}$$

In the simplest case, i.e. if eq. (2.2) *admits r distinct roots $z_1, \ldots, z_r$, then the general solution of the recurrence relation* (2.1) *is expressed by*

$$X_n = C_1 {z_1}^n + C_2 {z_2}^n + \ldots + C_r {z_r}^n, \quad (n \in \mathbf{Z}) \tag{2.3}$$

A little bit complicate formula holds when multiple roots appear.

We can show (I.V.V. Raghavacharyulu - A.R. Tekumalla [40]) the following representation formula for the $F_{k,n}$ functions:

$$F_{k,n-1}(u_1, \ldots, u_r) = \frac{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_r \\ \ldots & \ldots & \ldots & \ldots \\ {z_1}^n & {z_2}^n & \ldots & {z_r}^n \\ \ldots & \ldots & \ldots & \ldots \\ {z_1}^{r-1} & {z_2}^{r-1} & \ldots & {z_r}^{r-1} \end{vmatrix}}{V(z_1, z_2, \ldots, z_r)}$$

$$(k = 1, 2, \ldots, r; \; n \in \mathbf{Z}),$$

where the row $\{{z_1}^n, {z_2}^n, \ldots, {z_r}^n\}$ appears instead of the usual $(r-k+1)$-th one, and $V(z_1, z_2, \ldots, z_r)$, denotes the Vandermonde determinant of the numbers $z_1, z_2, \ldots, z_r$.

Another important property is expressed by the relations

$$\begin{cases} F_{1,n} = u_1 F_{1,n-1} + F_{2,n-1} \\ F_{2,n} = -u_2 F_{1,n-1} + F_{3,n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{r-1,n} = (-1)^{r-2} u_{r-1} F_{1,n-1} + F_{r,n-1} \\ F_{r,n} = (-1)^{r-1} u_r F_{1,n-1} \end{cases}$$

by means of which all the $\{F_{k,n}\}_{n\in\mathbf{Z}}$ can be computed in terms of the bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$.

Then it is useful to introduce the following definition

Definition 2..1 *The bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$, i.e. the solution of* (2.1) *corresponding to the initial conditions*

$$F_{1,-1} = 0, \quad F_{1,0} = 0, \quad F_{1,2} = 0, \ldots, F_{1,r-2} = 1$$

is called the fundamental solution of the recurrence relation (2.1) (see É. Lucas [32]).

In what follows we will use the notation:

$$F_{1,n}(u_1, \ldots, u_r) =: \Phi_n(u_1, \ldots, u_r) = \Phi_n, \quad (n \in \mathbf{Z})$$

For $n \geq -1$, the $\Phi_n(u_1, \ldots, u_r)$ functions are a sequence of polynomials called in literature Generalized Lucas Polynomials of second kind.

Remark 2..2 *Note that if $r = 2$, and $u_2 = 1$, by putting $u_1 = x$, we have:*

$$\Phi_n(u_1, 1) = \Phi_n(x, 1) \equiv U_n\left(\frac{x}{2}\right), \quad (n \in \mathbf{N_0}),$$

where $\{U_n(x)\}_{n\in\mathbf{N}_0}$ are the classical Chebyshev polynomials of second kind.

If $r \geq 3$, and putting $u_r = 1$, we obtain a sequence of polynomials in $r - 1$ variables which appear as an extension of the classical Chebyshev polynomials of second kind:

$$\Phi_n(u_1, \ldots, u_{r-1}, 1) =: U_n(u_1, \ldots, u_{r-1}), \quad (n \in \mathbf{N_0}).$$

Some properties of these polynomials have been obtained in M. Bruschi - P.E. Ricci [5]. Further extensions and more general situations have been considered by several authors: R. Lidl [28], K.B. Dunn - R. Lidl [14], R. Lidl - C. Wells [29], T.H. Koornwinder [20], [21], R.J. Beerends [2].

The above mentioned results can be resumed as follows:

The generalized Lucas polynomials of second kind are defined as the solution of the bilateral linear homogeneous recurrence relation (2.1)

$$\Phi_n = u_1\Phi_{n-1} - u_2\Phi_{n-2} + \ldots + (-1)^r u_r \Phi_{n-r}, \quad (n \in \mathbf{Z})$$

corresponding to the initial conditions

$$\Phi_{-1} = 0, \Phi_0 = 0, \Phi_1 = 0, \ldots, \Phi_{r-2} = 1.$$

It is called the *fundamental solution* of the recurrence relation (2.1) since all solutions of (2.1) can be expressed in terms of this particular solution.

3. *Generalized Lucas polynomials of the first kind*

We recall the definition of Lucas polynomials of the first kind in several variables $u_1, \ldots, u_r$.

Among the solutions of the bilateral recurrence relation (2.1) there is the sum of powers (for any fixed integral exponent) of the roots of characteristic equation (2.2). This particular solution was called by É. Lucas , the *primordial solution*. This solution will be introduced here in the following way.

For any integer r, put:

$$\left\{\begin{array}{l}\Psi_{r-1}(u_1,\ldots,u_r)=u_1=\sum_{k=1}^{r}x_{k,r}\\ \Psi_{r}(u_1,\ldots,u_r)=u_1^2-2u_2=\sum_{k=1}^{r}x_{k,r}^2\\ \Psi_{r+1}(u_1,\ldots,u_r)=u_1^3-3u_1u_2+3u_3=\sum_{k=1}^{r}x_{k,r}^3\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \Psi_{2r-2}(u_1,\ldots,u_r)=u_1\Psi_{2r-3}-u_2\Psi_{2r-4}+\ldots+\\ \qquad\qquad +(-1)^{r-2}u_{r-1}\Psi_{r-1}+(-1)^{r-1}ru_r=\sum_{k=1}^{r}x_{k,r}^r\end{array}\right. \tag{3.1}$$

and, for $M>2r-2$:

$$\begin{array}{rcl}\Psi_M(u_1,\ldots,u_r)&=&u_1\Psi_{M-1}-u_2\Psi_{M-2}+\ldots+\\ &+&(-1)^{r-1}u_r\Psi_{M-r}=\sum_{k=1}^{r}x_{k,r}^{M-r+2},\end{array} \tag{3.2}$$

where the $x_{k,r}$ are the roots of the characteristic equation (2.2) related to the recurrence relation (2.1).

Remark 3..1 *The choice of indexes is justified in [5], in order to find, in case $r=2$, the classical Lucas polynomials of the first kind in two variables* (see Lucas [32]).

In fact, if $r=2$, and $u_2=1$, by putting $u_1=x$, we have:

$$\Psi_n(u_1,1)=\Psi_n(x,1)\equiv 2T_n\left(\frac{x}{2}\right),\quad (n\in\mathbf{N_0}).$$

where $\{T_n(x)\}_{n\in\mathbf{N}_0}$ are the classical Chebyshev polynomials of first kind.

If $r\geq 3$, and putting $u_r=1$, we obtain a sequence of polynomials in $r-1$ variables which appear as an extension of the classical Chebyshev polynomials of second kind:

$$\Psi_n(u_1,\ldots,u_{r-1},1)=:T_n(u_1,\ldots,u_{r-1}),\quad (n\geq r-2),$$

(see M. Bruschi - P.E. Ricci [5]).

Further extensions and more general situations have been considered and studied by the same authors mentioned before: R. Lidl [28], R. Lidl et al. [14], [29], T. Koornwinder [20], [21], R.J. Beerends [2] .

4. *distribution of zeros of OPS satisfying an ODE with polynomial coefficients*

4.1. Dehesa's generalization of the Case method (*Representation formulas for the Newton sum rules*)

Consider the polynomial eigenfunctions $P_N(x)$ of a linear differential operator of order m:

$$\sum_{i=0}^{m} g_i(x) f^{(i)}(x) = 0, \tag{4.1}$$

where the coefficients $g_i(x)$ are polynomials of degree c_i:

$$g_i(x) = \sum_{j=0}^{c_i} a_j^{(i)} x^j.$$

We will assume that $P_N(x) = const. \prod_{l=1}^{N}(x - x_l)$, where all x_l are different, so that zeros of $P_N(x)$ are all simple, and we will write in the sequel:

$$P_N(x) = x^N - u_{N,1}x^{N-1} + u_{N,2}x^{N-2} + \ldots + (-1)^N u_{N,N} \tag{4.2}$$

or

$$P_N(x) = x^N - u_1 x^{N-1} + u_2 x^{N-2} + \ldots + (-1)^N u_N,$$

and introduce the Newton sum rules of the zeros of $P_N(x)$, defined by

$$y_s := \sum_{i=1}^{N} x_i^s. \tag{4.3}$$

If $c_i \leq i$ $(i = 0, 1, \ldots, m)$ the differential operator (4.1) is said to be of hypergeometric type. When $m = 2$, and $c_i \leq i$ $(i = 0, 1, \ldots, m)$, polynomial solutions of (4.1) are very classical since they are connected with classical orthogonal polynomials, and have been deeply studied by A.F. Nikiforov - V.B. Uvarov in [38].

The case when $m = 4$ was first considered by H.L. Krall [23]–[24] in the thirties, but more recently in many papers by A.M. Krall [22], L.L. Littlejohn [30]–[31] and others [3]– [42]–[43]–[45]. New classes of orthogonal polynomials can be found in the same way as the Heine polynomials (see T.S. Chihara [9], and some generalizations of the classical polynomials obtained by adding Dirac measures in the support of the corresponding absolutely continuous Borel measure (see R.A. Nodarse - F. Marcellan [39]).

To any polynomial $P_N(x)$ it is possible to associate a normalized discrete density distribution $\rho_N(x)$ defined by

$$\rho_N(x) = \frac{1}{N}\sum_{l=1}^{N}\delta(x-x_l) \qquad (\delta = \text{Dirac delta})$$

whose moments around the origin are given by

$$\mu'_h = \frac{1}{N}y_h = \frac{1}{N}\sum_{i=1}^{N}x_i^h.$$

Computation of the Newton sum rules y_h has been considered by K.M. Case [8] for the hypergeometric case $c_i \leq i$ ($\forall i = 0,1,\dots,m$), and by E. Buendia - J.S. Dehesa - F.J. Gálvez in [7] in the general case.

A computation of the Case method was given by P.E. Ricci [41] and P. Natalini [34] for the hypergeometric case. We used the generalized Lucas polynomials of the second kind in order to represent the Case sum rules.

In the following, starting from the above mentioned paper [6], we first extend our method to this general case. Then, considering the recursive formula representing the coefficients of $P_N(x)$ in terms of the coefficients of the differential operator (4.1), introduced in [6], (formula 13), we simply use the generalized Lucas polynomials of first kind in order to compute numerically the Newton sum rules.

E. Buendia - J.S. Dehesa - F.J. Gálvez [6], by generalizing the Case paper [8], proved the following recursive relation for the y_h Newton sum rules:

$$\sum_{i=2}^{m} i \sum_{l=-1}^{s+c_i-i-1} a_{i+l+1-s}^{(i)} J_{i+l}^{(i)} = -\sum_{j=0}^{c_1} a_j^{(1)} y_{s+j-1}, \qquad (s \geq 1) \tag{4.4}$$

assuming, by definition:

$$J_r^{(i)} = 0 \qquad \text{for} \quad 0 \leq r \leq i-2,$$

and

$$J_r^{(i)} = \sum_{\neq(l_1,\dots,l_i)}^{(1,\dots,N)} \frac{x_{l_1}^r}{\prod_{k=1}^{i}(x_{l_1}-x_{l_k})}. \tag{4.5}$$

The $J_r^{(i)}$ are so called Case sum rules (see [8]), and in the last formula, $\sum_{\neq(l_1,\dots,l_i)}^{(1,\dots,N)}$ means that the sum runs over all l_s ($s = 1,\dots,N$) provided that $\forall i \neq j$, $l_i \neq l_j$.

The Case sum rules $J_r^{(i)}$ can be expressed in terms of the Newton sum rules y_t with $t \leq r-i+1$ by means of the following representation theorem:

Proposition 4..1 – *For any $N \in \mathbf{N}$ $(N \geq 2)$, $r \in \mathbf{N_0} : \mathbf{N} \cup \{0\}$, $i \in \mathbf{N}$, then*

$$J_r^{(i)} = (i-1)! \sum_{k=0}^{N-i} \frac{N-k}{i} u_k \Phi_{N+r-i-k-1}(u_1, u_2, \ldots, u_N), \tag{4.6}$$

where $\Phi_h(u_1, u_2, \ldots, u_N)$ denote the generalized Lucas polynomials of the second kind in N.

Proof is exactly the same as in the above mentioned papers [41]–[34], since formula (4.6) gives a representation of the function $J_r^{(i)}$, which is a symmetric function of the zeros of $P_N(x)$, in terms of the coefficients of $P_N(x)$. The possibility to obtain such a formula is a consequence of a well known Gauss' theorem, and obviously, this formula is independent of the differential equation satisfied by $P_N(x)$. Note that if the polynomials $P_N(x)$ satisfy an hypergeometric type differential equation (i.e. if $c_i \leq i$ $\forall i = 0, 1, \ldots, m$), then equation (4.4) simplifies into:

$$\sum_{i=2}^{m} i \sum_{j=0}^{i} a_j^{(i)} J_{s+j}^{(i)} = -a_0^{(1)} y_s - a_1^{(1)} y_{s+1}, \qquad (s \geq 0) \tag{4.7}$$

and every $J_{s+j}^{(i)}$ can be computed in terms of the y_t $(t \leq s)$ so that starting from

$$y_0 = \sum_{i=1}^{N} x_i^0 = N,$$

the recurrence relation (4.7) permits the computation of all Newton sum rules.

In the general case, since in the right hand side of (4.4) the more general combination

$$-a_0^{(1)} y_{s-1} - a_1^{(1)} y_s - \cdots - a_{c_1}^{(1)} y_{s+c_1-1}$$

occurs, then for computing all Newton sum rules it is necessary to construct separately the first values

$$y_0 = N,\ y_1,\ \ldots,\ y_{c_1-1}.$$

But this is not sufficient, since a similar indeterminacy problem arises in the left hand side, in which quantities $J_r^{(i)}$ appear, involving y_t with $t \leq r-i+1$, so that $J_{i+l}^{(i)}$ is expressed in terms of the y_t with $t \leq i+l-i+1 = l+1 \leq s+c_i-i-1+1 = s+c_i-i$.

Then, in order that recurrence (4.4) works, it is sufficient to know y_t for $t \leq s+q$ $(s \geq 0)$, where

$$q : \max\{c_i - i; \quad i = 0, 1, 2, \ldots, m\}, \tag{4.8}$$

i.e. to know

$$y_0 = N,\ y_1,\ y_2,\ \ldots,\ y_q. \tag{4.9}$$

In the above mentioned paper [6] the Authors give expressions for the initial conditions (4.9) of the recurrence relation (4.4) in terms of the coefficients of the polynomial $P_N(x)$, by using the Newton-Girard formulas:

$$\begin{cases} u_1 = y_1 \\ u_2 = \frac{1}{2}(u_1 y_1 - y_2) = \frac{1}{2}(y_1^2 - y_2) \\ u_3 = \frac{1}{3}(-u_1 y_2 + u_2 y_1 + y_3) = \frac{1}{6}(y_1^3 - 3y_1 y_2 + 2y_3) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_N = \frac{1}{N}\{[(-1)^N u_1 y_{N-1} + (-1)^{N-1} u_2 y_{N-2} + \ldots + u_{N-1} y_1] + (-1)^{N-1} y_N\} \end{cases}$$

More precisely, initial conditions (4.9) are found by using the above Newton-Girard formulas and the following explicit recurrent expressions for the coefficients of $P_N(x)$ in terms of the coefficients of the differential operator (4.1):

$$u_s = -\frac{\sum_{k=1}^{s}(-1)^k u_{s-k} \sum_{i=0}^{m} \frac{(N-s+k)!}{(N-s+k-i)!} a_{i+q-k}^{(i)}}{\sum_{i=0}^{m} \frac{(N-s)!}{(N-s-i)!} a_{i+q}^{(i)}}, \tag{4.10}$$

where $u_0 := 1$.

The Authors also note that the use of equations (4.4)-(4.10) and the Newton-Girard formulas give also the possibility to compute recursively the Newton sum rules y_t, but due to non linearity of relations involved, they use only these equations (4.4) in order to compute initial conditions (4.9), and subsequently, they use the recurrence relation (4.4).

Concluding this section we can say that *even in this more general case* (with respect to the hypergeometric case considered in [41]–[34]), *the representation formula (4.6), the Newton-Girard formulas, and initial conditions obtained by using (4.10) completely solve the problem of computing by recursion the Newton sum rules*, whereas in [6], the problem is solved only in particular (but relevant) cases.

4.2. Computation of Newton sum rules by using generalized Lucas polynomials of the first kind

According to definition of the generalized Lucas polynomial of the first kind, given in Section 3, $\Psi_h(u_1, u_2, \ldots, u_N)$ gives the sum of the $(h-N+2)$-th powers of the roots of $P_N(x)$, i.e. the Newton sum rule y_{h-N+2}.

Then it is possible to formalize connection between coefficients of differential equation (4.1) and Newton sum rules of zeros of $P_N(x)$, via the Newton-Girard formulas, and avoiding the generalized Case method, by using the following

Proposition 4..2 – *Consider a polynomial $P_N(x)$, given by (4.2), which satisfies the differential equation with polynomial coefficients (4.1). Then, coefficients of $P_N(x)$ are recursively linked to the coefficients of (4.1) by formula (4.10), and for the Newton sum rules the following representation formula holds true:*

$$y_h = \sum_{k=1}^{N} x_k^h = \Psi_{h+N-2}(u_1, u_2, \ldots, u_N).$$

This formula, provided that initial conditions (4.9) are computed, permits recursive computation of moments via (3.2).

5. *distribution of zeros of OPS generated by a three-term recurrence relation*

Consider the polynomial three-term recurrence relation

$$\begin{cases} P_{-1} = 0, \qquad P_0 = 1, \\ P_n(x) = (x - \alpha_n)P_{n-1}(x) - \beta_{n-1}^2 P_{n-2}(x), \quad (n \geq 1) \end{cases} \tag{5.1}$$

For any fixed $n \in \mathbf{N}$ we consider in the sequel the density of the zeros $x_{k,n}$, $(k = 1, \ldots, n)$ of $P_n(x)$ and the related moments, defined by (4.1)–(4.2).

5.1. An explicit representation formula of J.S. Dehesa et al.

J.S. Dehesa [12] proved the following theorem:

Proposition 5..1 [(Dehesa)] *The moments of the density of zeros $\rho_n(x)$ of $P_n(x)$ are given by the following formulas:*

$$\mu_q^{(n)} := \frac{1}{n} \sum_{(q)} F_q(q_1', q_1, q_2', q_2, \ldots, q_j', q_j, q_{j+1}') \times$$

$$\times \sum_{i=1}^{n-t} {\alpha_i}^{q_1'} {\beta_i}^{2q_1} {\alpha_{i+1}}^{q_2'} \cdots {\alpha_{i+j-1}}^{q_j'} {\beta_{i+j-1}}^{2q_j} {\alpha_{i+j}}^{q_{j+1}'}$$

$$(q = 1, 2, \ldots, n)$$

where:

- $j := \left[\frac{q}{2}\right]$ *is the greatest integer contained in* $q/2$;

- $\sum_{(q)}$ *denotes the sum running over all partitions of the integer* q, *say it* $(q'_1, q_1, q'_2, q_2, \ldots, q'_j, q_j, q'_{j+1})$, *subject to the following conditions:*

$$\begin{cases} q'_1 + q'_2 + \cdots + q'_{j+1} + 2(q_1 + q_2 + \cdots + q_j) = q \\ \text{if} \quad q_s = 0,\ 1 < s < j, \quad \text{then} \quad q_k = q'_k = 0,\ \forall k > s; \end{cases}$$

- t *denotes the number of non vanishing* q'_i *involved in the corresponding partition of* q;

and lastly:

- $F_q(q'_1, q_1, q'_2, q_2, \ldots, q'_j, q_j, q'_{j+1}) :=$

$$:= q\frac{(q'_1 + q_1 - 1)!(q_{j-1} + q'_j - 1)!}{q'_1! q_1! (q_{j-1} - 1)! q'_j!} \prod_{i=2}^{j-2} \frac{(q_i + q'_{i+1} + q_{i+1} - 1)!}{(q_i - 1)! q'_{i+1}! q_{i+1}!} \ .$$

The first moments are given by

$$\mu_1^{(n)} = \frac{1}{n} \sum_{k=1}^{n} \alpha_k,$$

$$\mu_2^{(n)} = \frac{1}{n} \{ \sum_{k=1}^{n} \alpha_k^2 + 2 \sum_{h=1}^{n-1} \beta_h^2 \},$$

$$\mu_3^{(n)} = \frac{1}{n} \{ \sum_{k=1}^{n} \alpha_k^3 + 3 \sum_{h=1}^{n-1} \beta_h^2 (\alpha_h + \alpha_{h+1}) \},$$

$$\mu_4^{(n)} = \frac{1}{n} \{ \sum_{k=1}^{n} \alpha_k^4 + 4 \sum_{h=1}^{n-1} \beta_h^2 (\alpha_h^2 + \alpha_h \alpha_{h+1} + \alpha_{h+1}^2 + \frac{1}{2}\beta_h^2) + 4 \sum_{h=1}^{n-2} \beta_h^2 \beta_{h+1}^2 \}.$$

The purpose of the Dehesa's investigation is to obtain information about the asymptotic density of zeros directly from the coefficients of the three-term recurrence relation.

However, the preceding formulas become more and more complicate, since they are highly non-linear with respect to the parameters α's and β's, and furthermore the definition of the coefficients F_q is affected by severe ill conditioning, when q increases.

5.2. Explicit formulas for the characteristic polynomial of the Jacobi matrix

A representation formula for the moments $\mu_q^{(n)}$ is given here in terms of Lucas polynomials of the first kind.

For any integer n, consider the three-diagonal symmetric Jacobi matrix:

$$J_n := \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \dots & \beta_{n-1} & \alpha_n \end{vmatrix} \tag{5.2}$$

so that the characteristic polynomial

$$\begin{aligned} &P_n(x) := (-1)^n det(J_n - x\mathcal{I}) = \\ &= (x^n - u_{1,n}x^{n-1} + u_{2,n}x^{n-2} + \ldots + (-1)^n u_{n,n}) = \\ &= (x^n - u_1 x^{n-1} + u_2 x^{n-2} + \ldots + (-1)^n u_n) \end{aligned} \tag{5.3}$$

satisfies for any $n \geq 1$ the recurrence relation (5.1).

Let, for any s such that $1 \leq s \leq n$, and for any integers $k_1, k_2, \ldots, k_s$ such that $1 \leq k_1 < k_2 < \ldots < k_s \leq n$

$$J_{k_1,\ldots,k_s} := \tag{5.4}$$

$$\begin{vmatrix} \alpha_{k_1} & \beta_{k_1}\delta_{k_1,k_2-1} & 0 & 0 & \dots & 0 & 0 \\ \beta_{k_2-1}\delta_{k_1,k_2-1} & \alpha_{k_2} & \beta_{k_2}\delta_{k_2,k_3-1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \beta_{k_s-1}\delta_{k_{s-1},k_s-1} & \alpha_{k_s} \end{vmatrix}$$

where $\delta_{h,k}$ denotes the Kronecker delta, and $J_{1,2,\ldots,n} = J_n$.

The coefficients $u_s := u_{s,n}\,(1 \leq s \leq n)$ are given by the following formulas:

$$\begin{aligned} &u_1 = trJ_n \\ &u_2 = \textstyle\sum_{k_1<k_2} detJ_{k_1,k_2} \\ &\dots\dots\dots\dots\dots \\ &u_s = \textstyle\sum_{k_1<k_2<\ldots<k_s} detJ_{k_1,k_2,\ldots,k_s} \\ &\dots\dots\dots\dots\dots \\ &u_n = detJ_n \end{aligned} \tag{5.5}$$

It is easy to see that all the preceding determinants can be computed by recurrence relations. We have, in fact:

$$\begin{aligned} det J_{k_1,\dots,k_s} = \alpha_{k_s} det J_{k_1,\dots,k_{s-1}} - \\ -\beta_{k_s-1}\beta_{k_{s-1}}\delta_{k_{s-1}}\delta_{k_s-1} det J_{k_1,\dots,k_{s-2}}, \end{aligned} \tag{5.6}$$

and, in particular:

$$u_n = det J_n = \alpha_n det J_{n-1} - \beta_{n-1}^2 det J_{n-2}. \tag{5.7}$$

By (5.6) and using the induction method, the coefficients $u_1, u_2, \dots, u_n$ can be expressed in terms of $\alpha_1, \alpha_2, \dots, \alpha_n;\ \beta_1, \beta_2, \dots, \beta_{n-1}$.

As a matter of fact we have

$$u_1 = \sum_{k=1}^{n} \alpha_k \tag{5.8}$$

$$u_2 = \sum_{k_1<k_2}^{1,n} \alpha_{k_1}\alpha_{k_2} - \sum_{h=1}^{n-1} \beta_h^2 \tag{5.9}$$

$$u_3 = \sum_{k_1<k_2<k_3}^{1,n} \alpha_{k_1}\alpha_{k_2}\alpha_{k_3} - \sum_{h=1}^{n-1} \beta_h^2 \sum_{k=1}^{n} {}^{(h,h+1)} \alpha_k \tag{5.10}$$

And in general, letting $\sigma := \begin{cases} s/2 & \text{if } s \text{ is even} \\ (s-1)/2 & \text{if } s \text{ is odd} \end{cases}$

$$\begin{aligned} u_s = \\ \sum_{k_1<k_2<\cdots<k_s}^{1,n} \alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_s} - \sum_{h-1}^{n-1}\beta_h^2 \sum_{k_1<\cdots<k_{s-2}}^{1,n} {}^{(h,h+1)} \alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_{s-2}} \\ + \sum_{h_1,h_2}^{1,n-1} \beta_{h_1}^2\beta_{h_2}^2 \sum_{k_1<\cdots<k_{s-4}}^{1,n} {}^{(h_1,h_1+1;h_2,h_2+1)} \alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_{s-4}} \\ \begin{cases} (-1)^\sigma \sum_{h_1<\cdots<h_\sigma h_i-h_j\geq 2}^{1,n-1} \beta_{h_1}^2 \cdots \beta_{h_\sigma}^2 \sum_{k=1}^{n} {}^{(h_1,h_1+1;\dots;h_\sigma,h_\sigma+1)}\alpha_k, & (s \text{ odd}) \\ (-1)^\sigma \sum_{h_1<\cdots<h_\sigma h_i-h_j\geq 2}^{1,n-1} \beta_{h_1}^2 \cdots \beta_{h_\sigma}^2\ , & (s \text{ even}) \end{cases} \end{aligned} \tag{5.11}$$

where the symbol $\sum_{k=1}^{n}{}^{(h,h+1)}$ denotes that the sum runs over all indexes k different from h and $h+1$.

Remark 5..2 *Note that the first term in the sum for obtaining u_s represents the elementary symmetric function of order s relative to the numbers $\alpha_1, \ldots, \alpha_n$ (sum of products of s elements chosen in any way among these numbers); the second term, with negative sign, represents the sum of products of each β_h^2 by the elementary symmetric function of order $(s-2)$ of the numbers $\alpha_1, \ldots, \alpha_{h-1}, \alpha_{h+2}, \ldots, \alpha_n$ (these are the elements that belong to the matrix obtained from J_n erasing the rows and columns containing β_h); the third term, with positive sign, represents the sum of products of each $\beta_{h_1}^2 \beta_{h_2}^2$ by the elementary symmetric function of order $(s-4)$ of the numbers $\alpha_1, \ldots, \alpha_{h_1-1}, \alpha_{h_1+2}, \ldots, \alpha_{h_2-1}, \alpha_{h_2+2}, \ldots, \alpha_n$ (belonging to the matrix obtained from J_n erasing the rows and columns containing β_{h_1} and β_{h_2}); and so on.*

5.3. An explicit representation for the moments in terms of the generalized Lucas polynomials of the first kind

As a consequence of the definition given in Section 3, we can proclaim the following

Proposition 5..3 *For any integers $n \geq 1$ and $q \in \mathbf{N}_0$, the following representation formula for the moments of the density of zeros of orthogonal polynomials holds true:*

$$\mu_q^{(n)} := \frac{1}{n} \sum_{k=1}^{n} x_{k,n}^q = \frac{1}{n} \Psi_{q+n-2}(u_1, \ldots, u_n) \tag{5.12}$$

where the variables $u_1, \ldots, u_n$ are given by the preceding formulas (5.11) $(s = 1, \ldots, n)$.

Remark 5..4 *If all $x_{k,n}$ does not vanish, the representation formula* (5.11) *is still true even if $q \in \mathbf{Z}$, since the Lucas polynomials of the first kind in several variables are well defined.*

Remark 5..5 *Observe that, by* (3.2)*, only the first n moments are linearly independent, since the corresponding problem is an algebraic moment problem. All subsequent moments can be computed by formula* (3.2) .

6. *Computation of Newton sum rules for associated and co-recursive of classical OPS*

Starting from the three-term recurrence relation

$$\begin{cases} \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \alpha_{n-1} P_{n-1}(x) = x P_n(x), & n \geq 0 \\ P_{-1} = 0; \; P_0 = 1, \end{cases} \tag{6.1}$$

α_n, β_n real, $\alpha_n > 0$, $\forall n \in \mathbf{N_0} := \mathbf{N} \cup \{0\}$, associated orthogonal polynomials of order c (c integral number) are the set $\{P_n(x,c)\}_{n \in \mathbf{N}_0}$ defined by the following integral perturbation of indexes of (6.1):

$$\begin{cases} \alpha_{n+c}P_{n+1}(x;c) + \beta_{n+c}P_n(x;c) + \alpha_{n+c-1}P_{n-1}(x;c) = xP_n(x;c), \quad n \geq 0 \\ P_{-1}(x;c) = 0; \; P_0(x;c) = 1. \end{cases} \tag{6.2}$$

Obviously $P_n(x;0) \equiv P_n(x)$ $\forall n \in \mathbf{N_0}$.

The associated orthogonal polynomials have been studied by T. S. Chihara [9]-[10]. They appear in connection with stationary birth and death process (see also M.E. Ismail - J. Letessier - D.R. Masson - G. Valent [19]) i.e. Markov process with non negative integral state variables.

Another class of orthogonal polynomials, the so called co-recursive polynomials, are defined by adding a real perturbation β to the first coefficient β_0 of the recurrence relation (6.1), i.e. considering the set $\{Q_n(x;\beta)\}_{n \in \mathbf{N}_0}$ defined by the following recurrence relation:

$$\begin{cases} \alpha_n Q_{n+1}(x;\beta) + (\beta_n + \beta\delta_{n,0})Q_n(x;\beta) + \alpha_{n-1}Q_{n-1}(x;\beta) = xQ_n(x;\beta), \\ Q_{-1}(x;\beta) = 0; \; Q_0(x;\beta) = 1, \end{cases} \tag{6.3}$$

where $n \geq 0$, and $\delta_{n,0}$ denotes the Kronecker delta.

Obviously $Q_n(x;0) \equiv P_n(x)$ $\forall n \in \mathbf{N_0}$.

The co-recursive orthogonal polynomials have been studied by T.S. Chihara [11], J. Letessier [25]-[26]-[27]. They appear in connection with potential scattering (see H.A. Slim [44]). Co-recursive associated polynomials $Q_n(x;\beta,c)$ have also been introduced.

Recently the distribution of zeros and first Newton sum rules of associated, co-recursive and co-recursive associated polynomials in terms of the entries $\{\alpha_n\}$ and $\{\beta_n\}$ of the Jacobi matrix have been studied by E.K. Ifantis - G.K. Kokologiannaki - P.D. Siafarikas [17], E.K. Ifantis - P.D. Siafarikas [18]. They give explicit expressions for the first Newton sum rules of the associated and co-recursive associated of the classical orthogonal polynomials.

The differential equations satisfied by such polynomials, have been introduced by S. Belmedhi - A. Ronveaux [3], A. Zarzo - A. Ronveaux - E. Godoy [45] for the associated of classical polynomials and by A. Ronveaux - F. Marcellan [42] A. Ronveaux - A. Zarzo - E. Godoy [43] for the co-recursive case.

6.1. Associated orthogonal polynomials

In the above mentioned papers E.K. Ifantis - G.K. Kokologiannaki - P.D. Siafarikas [17], E.K. Ifantis - P.D. Siafarikas [18] proved the following results. For any fixed integral $N \in \mathbf{N}$, denote by $\{e_n\}_{n=0,1,\dots,N-1}$ an orthonormal basis of the euclidean space E^N, introduce the operators (matrices):

$$A = \begin{pmatrix} \alpha_c & & \\ & \ddots & \\ & & \alpha_{c+N-1} \end{pmatrix} \tag{6.4}$$

$$B = \begin{pmatrix} \beta_c & & \\ & \ddots & \\ & & \beta_{c+N-1} \end{pmatrix} \tag{6.5}$$

$$V = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \tag{6.6}$$

$$V^* = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \tag{6.7}$$

and consider the operator $T_0 := AV^* + VA + B$. Then, for associated polynomials the following representation formula for the Newton sum rules of polynomials $\{P_n(x,c)\}_{n \in \mathbf{N}}$ holds true:

Proposition 6..1 – *Let $N \in \mathbf{N}$ be a positive integral number. Denote by $\lambda_n(c)$ $(n = 0, 1, \dots, N-1)$ zeros of the associated polynomial $P_N(x; c)$. Then for any $k \in \mathbf{N}$*

$$\sum_{n=0}^{N-1} \lambda_n^k = \sum_{n=0}^{N-1} (T_0^k e_n, e_n). \tag{6.8}$$

Remark 6..2 – *The second hand side of eq. (6.8) is independent on the choice of the orthonormal basis $\{e_n\}_{n=0,1,\dots,N-1}$ of E^N (see [17]).*

Remark 6..3 – *The matrix associated to the operator T_0 is exactly the Jacobi matrix*

$$T_0 = J_{(c)} = \begin{pmatrix} \alpha_c & \beta_c & 0 & & \\ \beta_c & \alpha_{c+1} & \beta_{c+1} & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{c+N-3} & \alpha_{c+N-2} & \beta_{c+N-2} \\ & & 0 & \beta_{c+N-2} & \alpha_{c+N-1} \end{pmatrix}. \tag{6.9}$$

Note that even if T_0 is a three-diagonal matrix his powers are in general full matrices, so that by the computational point of view the representation formula (6.8) is quite expensive.

In the above mentioned papers [17]-[18] the Authors give explicit expression for the first Newton sum rules in terms of the sequences $\{\alpha_{n+c}\}_{n\in\mathbf{N}}$ and $\{\beta_{n+c}\}_{n\in\mathbf{N}}$. In the particular case of classical orthogonal polynomials the following results are shown:

a) associated Laguerre polynomials $L_N^{(a)}(x;c)$.

$$\begin{aligned}
&\textstyle\sum_{n=0}^{N-1}\lambda_n(c) = N(N+2c+a);\\[1ex]
&\textstyle\sum_{n=0}^{N-1}\lambda_n^2(c) = 2N^3 + N^2(6c+3a-1)\\
&\qquad\qquad +N(6c^2+6ca-2c+a^2-a)-2c^2-2ca;\\[1ex]
&\textstyle\sum_{n=0}^{N-1}\lambda_n^3(c) = 5N^4 + N^3(20c+10a-6)\\
&\qquad\qquad +N^2(30c^2+30ca-18c+6a^2-9a+2)\\
&\qquad\qquad +N(20c^3+30c^2a-18c^2+12ca^2-18ca\\
&\qquad\qquad +4c+a^3-3a^2+2a)\\
&\qquad\qquad -(2c^3+18c^2a+6ca^2);\\[1ex]
&\textstyle\sum_{n=0}^{N-1}\lambda_n^4(c) = 14N^5 + N^4(70c+35a-29)\\
&\qquad\qquad +N^3(140c^2+140ca-116c+30a^2-60a+22)\\
&\qquad\qquad +N^2(140c^3+210c^2a-174c^2+90ca^2-174ca\\
&\qquad\qquad\qquad +66c+10a^3-35a^2+33a-6)\\
&\qquad\qquad -N(70c^4+140c^3a-116c32+90c^2a^2-174c^2a\\
&\qquad\qquad\qquad +66c^2+20ca^3-110ca^2+74ca-12c+a^4\\
&\qquad\qquad\qquad -6a^3+11a^2-6a)\\
&\qquad\qquad -(58c^4+116c^3a+70c^2a^2+12c^2+12ca^3+12ca).
\end{aligned} \tag{6.10}$$

Note that the last formula gives correction of small misprints which appear in [17].

b) associated Hermite polynomials $H_N(x;c)$.

$$\begin{aligned}
&\textstyle\sum_{n=0}^{N-1}\lambda_n(c) = \sum_{n=0}^{N-1}\lambda_n^3(c) = 0;\\[1ex]
&\textstyle\sum_{n=0}^{N-1}\lambda_n^2(c) = \frac{1}{2}(N-1)(N+2c);\\[1ex]
&\textstyle\sum_{n=0}^{N-1}\lambda_n^4(c) = \frac{1}{4}N(N-1)(2N-3)+\frac{c}{2}(3N-5)(N+c).
\end{aligned} \tag{6.11}$$

c) associated Jacobi polynomials $P_N^{(a,b)}(x;c)$.

$$\sum_{n=0}^{N-1} \lambda_n(c) = \frac{N(b^2-a^2)}{(2c+a+b)(2N+2c+a+b)};$$

$$\sum_{n=0}^{N-1} \lambda_n^2(c) = \frac{(N-1)[2c(c+a+b+N)+N(a+b+1)]}{(2c+a+b+1)(2N+2c+a+b-1)}+ \tag{6.12}$$

$$\frac{(b-a)^2[(3N-1)(a+b)^2(2c+a+b)(2N+2c+a+b)]}{2(2c+a+b+1)(2N+2c+a+b-1)(2c+a+b)^2(2N+2c+a+b)^2}-$$

$$\frac{(b-a)^2[(N-1)(2c+a+b)^2(2N+2c+a+b)^2+2N^2(2N-1)(a+b)^2]}{2(2c+a+b+1)(2N+2c+a+b-1)(2c+a+b)^2(2N+2c+a+b)^2}.$$

6.2. Scaled co-recursive orthogonal polynomials

Consider now a positive real parameter γ and co-recursive associated polynomials $\{Q_n(x;\beta,c)\}$. Then a new family $\{Q_n(x;\beta,\gamma,c)\}_{n\in\mathbf{N}}$ called scaled co-recursive polynomials can be introduced as follows:

$$\begin{cases} \alpha_{n+c}Q_{n+1} + (\beta_{n+c} + \beta\delta_{n,0})Q_n + \alpha_{n-1+c}Q_{n-1} = x[1-(\gamma-1)\delta_{n,0}]Q_n, \\ Q_{-1}=0;\ Q_0 = 1. \end{cases}$$

where $n \geq 0$, $\delta_{n,0}$ denotes the Kronecker delta, and $Q_n \equiv Q_n(x;\beta,\gamma,c)$.

Introduce the further operators (matrices)

$$C = \begin{pmatrix} \gamma & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = diag(\gamma,1,\dots,1); \tag{6.13}$$

$$T = C^{-\frac{1}{2}}(T_0 + \beta P_0(x;c))C^{-\frac{1}{2}}; \tag{6.14}$$

and consider the generalized eigenvalues problem

$$[T_0 + \beta P_0(x;c)]x = \lambda C x,$$

by definition, for any fixed $N \in \mathbf{N}$, the eigenvalues of these problems are the zeros of polynomial $Q_n(x;\beta,\gamma,c)$ (β, γ real, $\gamma > 0$, c integral number). Then the following proposition holds true:

Proposition 6..4 – *Let $N \in \mathbf{N}$ be a positive integral number, and denote by $\omega_n(\beta,\gamma,c)$ $(n=0,1,\dots,N-1)$ the zeros of $Q_n(x;\beta,\gamma,c)$. Then for any $k \in \mathbf{N}$*

$$\sum_{n=0}^{N-1} \omega_n^k(\beta,\gamma,c) = \sum_{n=0}^{N-1} (T^k e_n, e_n). \tag{6.15}$$

Obviously the above representation formula (6.12) holds true in particular, for the co-recursive associated polynomials $Q_n(x;\beta,c)$. In this case, it is to assume $\gamma=1$, $T=T_0+\beta P_0(z;c)$ and $\omega_n(\beta,1,c)=\omega_n(\beta,c)$.

Even for the considered general case explicit expressions for the first Newton sum rules in terms of the sequence $\{\alpha_{n+c}\}$, $\{\beta_{n+c}\}$ and parameters β, γ can be found in [17]. In the particular case of co-recursive ($c=0$, $\gamma=1$) classical orthogonal polynomials the following results are shown:

a') co-recursive Laguerre polynomials $L_N^{(a)}(x;\beta)$.

$$\begin{aligned}
&\textstyle\sum_{n=0}^{N-1}\omega_n(\beta)=N(N+a)+\beta;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^2(\beta)=2N^3+N^2(3a-1)+N(a^2-a)+\beta^2+2\beta\beta_0;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^3(\beta)=5N^4+N^3(10a-6)+N^2(6a^2-9a+2)\\
&\qquad +N(a^3-3a^2+2a)+\beta^3+3\beta^2\beta_0+3\beta\beta_0^2+3\alpha_0^2\beta;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^4(\beta)=14N^5+N^4(35a-29)+N^3(30a^2-60a+22)\\
&\qquad +N^2(10a^3-35a^2+33a-6)-N(a^4-6a^3+11a^2-6a)\\
&\qquad +\beta^4+4\beta^3\beta_0+6\beta^2\beta_0^2+4\beta\beta_0^3+8\alpha_0^2\beta_0\beta+4\alpha_0^2\beta^2\\
&\qquad +2\alpha_0^2\beta^2+4\alpha_0^2\beta_1\beta.
\end{aligned}\tag{6.16}$$

b') co-recursive Hermite polynomials $H_N(x;\beta)$.

$$\begin{aligned}
&\textstyle\sum_{n=0}^{N-1}\omega_n(\beta)=\beta;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^2(\beta)=\frac{(N-1)N}{2}+\beta^2+2\beta\beta_0;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^3(\beta)=\beta^3+3\beta^2\beta_0+3\beta\beta_0^2+3\alpha_0^2\beta;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^4(\beta)=\frac{N(N-1)(2N-3)}{4}+\beta^4+4\beta^3\beta_0+6\beta^2\beta_0^2\\
&\qquad +4\beta\beta_0^3+8\alpha_0^2\beta_0\beta+4\alpha_0^2\beta^2+2\alpha_0^2\beta^2+4\alpha_0^2\beta_1\beta.
\end{aligned}\tag{6.17}$$

c') co-recursive Jacobi polynomials $P_N^{(a,b)}(x;\beta)$.

$$\begin{aligned}
&\textstyle\sum_{n=0}^{N-1}\omega_n(\beta)=\frac{N(b^2-a^2)}{(a+b)(2N+a+b)}+\beta;\\
&\textstyle\sum_{n=0}^{N-1}\omega_n^2(\beta)=\beta^2+2\beta\beta_0+\frac{(N-1)N(a+b+1)}{(a+b+1)(2N+a+b-1)}\\
&\qquad +\frac{(b-a)^2[(3N-1)(a+b)^3(2N+a+b)]}{2(a+b+1)(2N+a+b-1)(a+b)^2(2N+a+b)^2}\\
&\qquad -\frac{(b-a)^2[(N-1)(a+b)^2(2N+a+b)^2+2N^2(2N-1)(a+b)^2]}{2(a+b+1)(2N+a+b-1)(a+b)^2(2N+a+b)^2}.
\end{aligned}\tag{6.18}$$

7. *Differential equations of the associated and co-recursive OPS*

Recently S. Belmedhi - A. Ronveaux [3], A. Zarzo - A. Ronveaux - E. Godoy [45] and A. Ronveaux - F. Marcellan [42], A. Ronveaux - A. Zarzo - E. Godoy [43] gave explicit differential equations of the fourth order satisfied by the associated or co-recursive of the classical orthogonal polynomials in terms the coefficients of the original differential equations and parameters c or β. In particular they have found the following differential equations:

$$\sum_{i=0}^{4} G_i(x;c,n)y^{(i)} = 0; \tag{7.1}$$

A) associated Laguerre polynomials $L_N^{(a)}(x;c)$.

$$\begin{aligned} G_4(x;c,n) &= x^2, \\ G_3(x;c,n) &= 5x, \\ G_2(x;c,n) &= -x^2 + 2(a+n+2c)x - a^2 + 4, \\ G_1(x;c,n) &= 3(-x+a+n+2c), \\ G_0(x;c,n) &= n(n+2); \end{aligned}$$

B) associated Hermite polynomials $H_N(x;c)$.

$$\begin{aligned} G_4(x;c,n) &= 1, \\ G_3(x;c,n) &= 0, \\ G_2(x;c,n) &= 4(-x^2+n+2c), \\ G_1(x;c,n) &= -12x, \\ G_0(x;c,n) &= 4n(n+2); \end{aligned}$$

C) associated Jacobi polynomials $P_N^{(a,b)}(x;c)$.

$$\begin{aligned} G_4(x;c,n) &= (x^2-1)^2, \\ G_3(x;c,n) &= 10x(x^2-1), \\ G_2(x;c,n) &= [24 - 2n(a+b+n+2c+1) - (a+b+2c)^2]x^2 - 2(a^2-b^2)x + \\ &\quad + 4c^2 + 4(a+b+n)c + 2n(a+b+n+1) - (a-b)^2 - 8, \\ G_1(x;c,n) &= 3[4 - 2n(a+b+n+2c+1) - (a+b+2c)^2]x - 3(a^2-b^2), \\ G_0(x;c,n) &= n(n+2)[(a+b+n+2c)^2 - 1]; \end{aligned}$$

$$\sum_{i=0}^{4} G_i(x;\beta,n)y^{(i)} = 0; \tag{7.2}$$

A') co-recursive Laguerre polynomials $L_N^{(a)}(x;\beta)$.

$$\begin{aligned}
G_4(x;\beta,n) &= x^2[4n\beta^2+2(1+a+4an-x-4nx)\beta+\\
&\quad +(2a+2a^2+4a^2n+x-4ax-8anx+2x^2+4nx^2)],\\
G_3(x;\beta,n) &= 2x[10n\beta^2+(5+5a+20an-4x-16nx)\beta+\\
&\quad +(5a+5a^2+10a^2n+2x-8ax-16anx+3x^2+6nx^2)],\\
G_2(x;\beta,n) &= 4n(4-a^2+2x+2ax+2nx-x^2)\beta^2+2(4+4a-a^2-a^3+\\
&\quad +16an-4a^3n+x+4ax+3a^2x-nx+8anx+12a^2nx+8an^2x-\\
&\quad -3x^2-3ax^2-8nx^2-12anx^2-8n^2x^2+x^3+4nx^3)\beta+(8a+\\
&\quad +8a^2-2a^3-2a^4+16a^2n-4a^4n+2x-ax+5a^2x+8a^3x-\\
&\quad -2anx+8a^2nx+16a^3nx+8a^2n^2x+2x^2-4ax^2-12a^2x^2+\\
&\quad +4nx^2-16anx^2-24a^2nx^2-16an^2x^2+x^3+8ax^3+8nx^3+\\
&\quad +16anx^3+8n^2x^3-2x^4-4nx^4),\\
G_1(x;\beta,n) &= 2[6n(1+a+n-x)\beta^2+\\
&\quad +(4+6a+2a^2+8n+12an+10a^2n+12an^2-4x-4ax+\\
&\quad -3nx-20anx-8n^2x+2x^2+10nx^2)\beta+\\
&\quad +(4a+6a^2+2a^3+8an+6a^2n+4a^3n+6a^2n^2+x-5ax+\\
&\quad -6a^2x+2nx-8anx-12a^2nx-8an^2x-x^2+6ax^2+2nx^2+\\
&\quad +12anx^2+2n^2x^2-2x^3-4nx^3)],\\
G_0(x;\beta,n) &= n(1+n)[4(n-1)\beta^2+2(6-a+4an+x-4nx)\beta+\\
&\quad +(12a+2a^2+4a^2n+3x-4ax-8anx+2x^2+4nx^2)];
\end{aligned}$$

B') co-recursive Hermite polynomials $H_N(x;\beta)$.

$$\begin{aligned}
G_4(x;\beta,n) &= 8n\beta^2-4x(1+4n)\beta+[3+4(1+2n)x^2],\\
G_3(x;\beta,n) &= 4[(1+4n)\beta-2(1+2n)x],\\
G_2(x;\beta,n) &= 2\{16n(1+n-x^2)\beta^2+8x[-1-4n-4n^2+(1+4n)x^2]\beta+\\
&\quad +9(1+2n)+2(8n^2+8n-1)x^2-8(1+2n)x^4\},\\
G_1(x;\beta,n) &= 8\{-12nx\beta^2+[2+9n+4n^2+4(1+5n)x^2]\beta+\\
&\quad -x[7+4n+4n^2+4(1+2n)x^2]\},\\
G_0(x;\beta,n) &= 4n(1+n)[8(n-1)\beta^2+4x(1-4n)\beta+4(1+2n)x^2+15];
\end{aligned}$$

C') co-recursive Chebyshev polynomials $P_N^{(a,b)}(x;c)$, $(a=b=1/2)$.

$$G_4(x;\beta,n)=4(1-x^2)^2\{T_{4,2}^{(0)}\beta^2+T_{4,1}^{(0)}\beta+[T_{4,0}^{(0)}+T_{4,0}^{(1)}(1-x^2)]\},$$

where

$T_{4,2}^{(0)}=32n(2+n)$, $T_{4,1}^{(0)}=-8x(3+8n+4n^2)$,
$T_{4,0}^{(0)}=4x^2(3+4n+2n^2)$,
$T_{4,0}^{(1)}=6$,

$$\begin{aligned}
G_3(x;\beta,n) &= 4(1-x^2)\{T_{3,2}^{(0)}\beta^2+\\
&\quad +8[T_{3,1}^{(0)}+T_{3,1}^{(1)}(1-x^2)]\beta+4[T_{3,0}^{(0)}+T_{3,0}^{(1)}(1-x^2)]\},
\end{aligned}$$

where
$T_{3,2}^{(0)} = -320nx(2+n)$, $T_{3,1}^{(0)} = 10x^2(3+8n+4n^2)$,
$T_{3,1}^{(1)} = 3+8n+4n^2$,
$T_{3,0}^{(0)} = -10x^3(3+4n+2n^2)$, $T_{3,0}^{(1)} = -2x(9+4n+2n^2)$,

$$\begin{aligned} G_2(x;\beta,n) &= 4\{T_{2,2}^{(0)}\beta^2 + 4[T_{2,1}^{(0)} + T_{2,1}^{(1)}(1-x^2)]\beta + \\ &\quad +2[T_{2,0}^{(0)} + T_{2,0}^{(1)}(1-x^2) + T_{2,0}^{(2)}(1-x^2)^2]\}, \end{aligned}$$

where
$T_{2,2}^{(0)} = 32n(2+n)[-6+4n+2n^2+(21-4n-2n^2)x^2]$,
$T_{2,1}^{(0)} = -30x^3(3+8n+4n^2)$,
$T_{2,1}^{(1)} = -4nx(6+19n+16n^2+4n^3)$,
$T_{2,0}^{(0)} = 30x^4(3+4n+2n^2)$,
$T_{2,0}^{(1)} = x^2(45+72n+68n^2+32n^3+8n^4)$,
$T_{2,0}^{(2)} = 9(-1+4n+2n^2)$,

$$\begin{aligned} G_1(x;\beta,n) &= 2\{T_{1,2}^{(0)}\beta^2 + \\ &\quad +8[T_{1,1}^{(0)} + T_{1,1}^{(1)}(1-x^2)]\beta + 4[T_{1,0}^{(0)} + T_{1,0}^{(1)}(1-x^2)]\}, \end{aligned}$$

where
$T_{1,2}^{(0)} = 192nx(2+n)[1-2n(2+n)]$,
$T_{1,1}^{(0)} = 4nx^2(18+57n+48n^2+12n^3)$
$T_{1,1}^{(1)} = 2n(2+n)(3+8n+4n^2)$,
$T_{1,0}^{(0)} = -4nx^3(24+36n+24n^2+6n^3)$,
$T_{1,0}^{(1)} = -4nx(18+17n+8n^2+2n^3)$,

$$G_0(x;\beta,n) = 4n(n+1)^2(n+2)\{T_{0,2}^{(0)}\beta^2 + 8T_{0,1}^{(0)}\beta + 2[T_{0,0}^{(0)} + T_{0,0}^{(1)}(1-x^2)]\},$$

where
$T_{0,2}^{(0)} = 32(n-1)(3+n)$,
$T_{0,1}^{(0)} = -x(3+8n+4n^2)$,
$T_{0,0}^{(0)} = 2x^2(9+4n+2n^2)$,
$T_{0,0}^{(1)} = 15$.

8. *Numerical results*

By using the above mentioned results, starting from representation formulas of the Newton sum rules in terms of the coefficients of differential equation (4.1), we have computed numerically the first moments of the above considered associated or co-recursive of the classical Laguerre, Hermite and Jacobi polynomials. Results are shown in the following tables.

A) associated Laguerre polynomials $L_N^{(a)}(x;c)$ $(a=0)$.

$c=1$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	11	14	17	20
μ_2	$210.\bar{7}$	$351.8\bar{3}$	$528.8\bar{6}$	$741.\bar{8}$
μ_3	$4'909.\bar{6}$	10'829	20'240.2	$33'953.\bar{3}$
μ_4	$125'703.\bar{2}$	$368'090.1\bar{6}$	$857'868.\bar{3}$	$1'724'152.\bar{1}$
μ_5	$3'394'659.\bar{8}$	$13'246'195.\bar{6}$	$38'582'514.\bar{3}$	$93'047'744.\bar{4}$
μ_6	$94724728.\bar{1}$	$494101308.8\bar{3}$	$1802225707.2\bar{6}$	$5222465532.\bar{5}$
μ_7	$2699863878.\bar{5}$	$18877156981.\bar{6}$	86373808509.5	$301110298397.\bar{7}$
μ_8	$78056028739.\bar{6}$	733279293589.5	4215395056098	17698232925166

$c=2$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	13	16	19	22
μ_2	$280.\bar{1}$	$439.\bar{3}$	$634.4\bar{6}$	$865.\bar{5}$
μ_3	$7'258.\bar{3}$	14'656	25'904.6	$41'814.\bar{6}$
μ_4	$206'396.\bar{5}$	$539'462.\bar{6}$	$1'170'741.9\bar{3}$	$2'240'693.\bar{7}$
μ_5	$6'193'790.\bar{7}$	$21'031'829.\bar{3}$	$56'165'513.\bar{6}$	$127'644'400.\bar{8}$
μ_6	$192'243'592.\bar{1}$	$850'487'741.\bar{3}$	$2'799'821'750.4\bar{6}$	$7'565'026'011.\bar{5}$
μ_7	$6100791841.\bar{4}$	$35247863893.\bar{3}$	143264113113.67	$460722344446.\bar{2}$
μ_8	$196558827950.\bar{3}$	1486142896432	7467951710005.4	28612069246011

$c=3$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	15	18	21	24
μ_2	361	538.5	751.8	1'001
μ_3	10'335	19'395	32'663.4	50'952
μ_4	323'673	769'135.5	1'568'640.6	2'874'009
μ_5	10'692'975	32'302'503	79'969'113	172'352'184
μ_6	365'527'497	1'407'730'279.5	4'237'580'539.8	10'756'024'665
μ_7	12784159935	62906325381	230580032697	689968936296
μ_8	454251745017	2861172143509.5	12786018430325	45144281321145

B) associated Hermite polynomials $H_N(x;c)$.

$\mu_{2i+1} = 0 \ \forall c, N, i$

$c=1$	$N=9$	$N=12$	$N=15$	$N=18$
μ_2	$4.\overline{8}$	$6.41\overline{6}$	$7.9\overline{3}$	$9.\overline{4}$
μ_4	$42.\overline{2}$	$74.451\overline{6}$	$115.8\overline{3}$	$166.\overline{1}$
μ_6	$430.\overline{2}$	1'037.6041$\overline{6}$	2'045.38$\overline{3}$	3'554.86$\overline{1}$
μ_8	4'710.7$\overline{2}$	15'691.38541	39'492.058$\overline{3}$	83'534.6$\overline{1}$

$c=2$	$N=9$	$N=12$	$N=15$	$N=18$
μ_2	$5.\overline{7}$	$7.\overline{3}$	$8.8\overline{6}$	$10.3\overline{8}$
μ_4	$56.\overline{8}$	$93.91\overline{6}$	$139.8\overline{3}$	$194.69\overline{4}$
μ_6	$655.\overline{7}$	1'430.8$\overline{3}$	2'650.81$\overline{6}$	4'417.263$\overline{8}$
μ_8	8'119.3$\overline{8}$	23'668.7291$\overline{6}$	54'924.058$\overline{3}$	110'017.5069$\overline{4}$

$c=3$	$N=9$	$N=12$	$N=15$	$N=18$
μ_2	$6.\overline{6}$	8.25	9.8	$11.\overline{3}$
μ_4	74	115.875	166.5	226
μ_6	955	1'922.0625	3'377.85	5'424.5
μ_8	13'218.5	34'562.71875	74'806.725	142'796.5

C) associated Chebyshev 2nd kind polynomials $P_N^{(a,b)}(x;c)$, (a=b=1/2).
$\mu_{2i+1}=0 \ \ \forall c, N, i$

$c=1,2,3$	$N=9$	$N=12$	$N=15$	$N=18$
μ_2	$0.\overline{4}$	$0.458\overline{3}$	$0.4\overline{6}$	$0.47\overline{2}$
μ_4	$0.30\overline{5}$	$0.32291\overline{6}$	$0.\overline{3}$	$0.3402\overline{7}$
μ_6	$0.236\overline{1}$	$0.255208\overline{3}$	$0.2\overline{6}$	$0.27430\overline{5}$
μ_8	$0.192708\overline{3}$	0.212890625	0.225	$0.23307291\overline{6}$

We note that in the last case results are independent of value of c.
A') co-recursive Laguerre polynomials $L_N^{(a)}(x;\beta)$ $(a=0)$

$\beta=1$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$9.\overline{1}$	$12.08\overline{3}$	$15.0\overline{6}$	$18.0\overline{5}$
μ_2	$153.\overline{3}$	276.25	435.2	$630.1\overline{6}$
μ_3	3'178.$\overline{1}$	7'800.8$\overline{3}$	15'555.$\overline{6}$	27'252.$\overline{5}$
μ_4	72'445.$\overline{3}$	243'291.25	615'737.6	1'307'558.1$\overline{6}$
μ_5	1'739'549.$\overline{1}$	8'026'287.08$\overline{3}$	25'847'247.0$\overline{6}$	66'645'298.0$\overline{5}$
μ_6	$43098888.\overline{3}$	274233395.5	1126254441.8	$3531375631.\overline{6}$
μ_7	$1089332225.\overline{1}$	$958938621008\overline{3}$	50326073311.9	192150871420
μ_8	$27897927765.\overline{3}$	34070829362125	2288953302694	10655028348510

$\beta=2$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$9.\overline{2}$	$12.1\overline{6}$	$15.1\overline{3}$	$18.\overline{1}$
μ_2	$153.\overline{8}$	$276.\overline{6}$	$435.5\overline{3}$	$630.\overline{3}$
μ_3	$3'180.\overline{5}$	$7'802.\overline{6}$	$15'557.1\overline{3}$	$27'253.\overline{7}$
μ_4	$72'456.\overline{1}$	$243'299.\overline{3}$	$615'744.0\overline{6}$	$1'307'563.\overline{5}$
μ_5	$1'739'599.\overline{2}$	$8'026'324.\overline{6}$	$25'847'277.1\overline{3}$	$66{\cdot}645'323.\overline{1}$
μ_6	$43'099'143.\overline{2}$	$274'233'586.\overline{6}$	$1'126'254'594.7\overline{3}$	$3'531'375'759.\overline{1}$
μ_7	$1'089'333'689$	$9'589'387'308$	$50'326'074'190.2$	$192'150'872'152$
μ_8	$27897937475.\overline{6}$	340708300904	2288953308520.2	10655028353365

$\beta=3$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$9.\overline{3}$	12.25	15.2	$18.1\overline{6}$
μ_2	$154.\overline{6}$	277.25	436	$630.8\overline{3}$
μ_3	$3'185$	$7'806$	$15'559.8$	$27'256$
μ_4	$72'480$	$243'317.25$	$615'758.4$	$1'307'575.5$
μ_5	$1'739'726$	$8'026'419.75$	$25'847'353.2$	$66{\cdot}645'386.5$
μ_6	$43'099'827$	$274'234'099.5$	$1'126'255'005$	$3'531'376'101$
μ_7	1089337536	9589390193.25	50326076498.4	192150874075.5
μ_8	27897960720	340708318337.3	2288953322467.2	10655028364987

B') co-recursive Hermite polynomials $H_N(x;\beta)$

$\beta=1$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$0.\overline{1}$	$0.08\overline{3}$	$0.0\overline{6}$	$0.0\overline{5}$
μ_2	$4.\overline{1}$	$5.58\overline{3}$	$7.0\overline{6}$	$8.\overline{5}$
μ_3	$0.2\overline{7}$	$0.208\overline{3}$	$0.1\overline{6}$	$0.13\overline{8}$
μ_4	$30.\overline{3}$	58	94.7	$140.41\overline{6}$
μ_5	$0.80\overline{5}$	$0.6041\overline{6}$	$0.48\overline{3}$	$0.402\overline{7}$
μ_6	$268.02\overline{7}$	$730.8958\overline{3}$	$1'549.3\overline{6}$	$2'824.63\overline{8}$
μ_7	$2.736\overline{1}$	2.05208	$1.641\overline{6}$	$1.3680\overline{5}$
μ_8	$2'552.0\overline{5}$	$10'020.2291\overline{6}$	$27'735.258\overline{3}$	$62'409.8402\overline{7}$

$\beta=2$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$0.\overline{2}$	$0.1\overline{6}$	$0.1\overline{3}$	$0.\overline{1}$
μ_2	$4.\overline{4}$	$5.8\overline{3}$	$7.2\overline{6}$	$8.7\overline{2}$
μ_3	$1.\overline{2}$	$0.91\overline{6}$	$0.7\overline{3}$	$0.6\overline{1}$
μ_4	$32.\overline{6}$	59.75	96.1	$141.58\overline{3}$
μ_5	$6.6\overline{1}$	$4.958\overline{3}$	$3.9\overline{6}$	$3.30\overline{5}$
μ_6	$281.\overline{7}$	$741.208\overline{3}$	$1'557.761\overline{6}$	$2'831.513\overline{8}$
μ_7	$35.80\overline{5}$	$26.8541\overline{6}$	$21.48\overline{3}$	$17.902\overline{7}$
μ_8	$2'629.3\overline{8}$	$10'078.2291\overline{6}$	$27'781.658\overline{3}$	$62'448.5069\overline{4}$

$\beta=3$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$0.\overline{3}$	0.25	0.2	$0.1\overline{6}$
μ_2	5	6.25	7.6	9
μ_3	3.5	2.625	2.1	1.75
μ_4	41	66	101.1	145.75
μ_5	35.75	26.8125	21.45	17.875
μ_6	80.25	815.0625	1'616.7	2'880.75
μ_7	362.865	272.15625	217.725	181.4375
μ_8	3'700.5	10'881.5625	28'424.325	62'984.0625

C') co-recursive Chebyshev polynomials of 2nd kind $P_N^{(a,b)}(x;c)$ $(a=b=1/2)$

$\beta=1$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$0.\overline{1}$	$0.08\overline{3}$	$0.0\overline{6}$	$0.0\overline{5}$
μ_2	$0.\overline{5}$	$0.541\overline{6}$	$0.5\overline{3}$	$0.52\overline{7}$
μ_3	$0.19\overline{4}$	$0.1458\overline{3}$	$0.11\overline{6}$	$0.097\overline{2}$
μ_4	$0.52\overline{7}$	0.48958	$0.4\overline{6}$	$0.4513\overline{8}$
μ_5	$0.319\overline{4}$	0.23958	$0.191\overline{6}$	$0.1597\overline{2}$
μ_6	$0.6180\overline{5}$	$0.541\overline{6}$	$0.4958\overline{3}$	$0.4652\overline{7}$
μ_7	0.51215	0.38411	0.30729	0.25607
μ_8	0.8177	0.68164	0.6	0.54557

$\beta=2$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$0.\overline{2}$	$0.1\overline{6}$	$0.1\overline{3}$	$0.\overline{1}$
μ_2	$0.\overline{8}$	$0.791\overline{6}$	$0.7\overline{3}$	$0.69\overline{4}$
μ_3	$1.0\overline{5}$	$0.791\overline{6}$	$0.6\overline{3}$	$0.52\overline{7}$
μ_4	$2.52\overline{7}$	1.98958	$1.\overline{6}$	$1.4513\overline{8}$
μ_5	$4.80\overline{5}$	$3.6041\overline{6}$	$2.88\overline{3}$	$2.402\overline{7}$
μ_6	$10.430\overline{5}$	7.90104	$6.38\overline{3}$	5.37152
μ_7	21.73263	16.29947	13.03958	10.86631
μ_8	46.35937	34.83789	27.925	23.3164

$\beta=3$	$N=9$	$N=12$	$N=15$	$N=18$
μ_1	$0.\overline{3}$	0.25	0.2	$0.1\overline{6}$
μ_2	$1.\overline{4}$	$1.208\overline{3}$	$1.0\overline{6}$	$0.97\overline{2}$
μ_3	3.25	2.4375	1.95	1.625
μ_4	$10.30\overline{5}$	7.82291	$6.\overline{3}$	$5.3402\overline{7}$
μ_5	$30.958\overline{3}$	23.21875	18.575	$15.4791\overline{6}$
μ_6	$95.6736\overline{1}$	$71.8\overline{3}4$	$57.5291\overline{6}$	$47.9930\overline{5}$
μ_7	294.36979	220.77734	176.62187	147.18489
μ_8	907.81770	680.93164	544.8	454.04557

9. *Conclusion*

The *central moments* of the distribution of zeros of the above considered polynomials P_n are defined by

$$\mu_{c,k} := \frac{1}{n}\sum_{i=0}^{n}\left[x_{i,n} - \mu_1^{(n)}\right]^k, \ (k \in \mathbf{N})$$

E.g., by using for shortness the notation $\mu_k := \mu_k^{(n)}$, the first central moments are

$$\begin{aligned}
&\mu_{c,1} := \mu_1\\
&\mu_{c,2} := \mu_2 - \mu_1^2\\
&\mu_{c,3} := \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3\\
&\mu_{c,4} := \mu_4 - 4\mu_1\mu_3 + 6\mu_2\mu_1^2 - 3\mu_1^4
\end{aligned}$$

By using the results of Sections **4** and **5**, the above mentioned central moments, and the most important statistical parameters, such as:

$$\begin{aligned}
&\text{Mean:}\quad M := \mu_{c,1}\\
&\text{Variance:}\quad \sigma^2 := \mu_{c,2}\\
&\text{Fischer coefficient:}\quad \gamma_1 := \tfrac{\mu_{c,3}}{\mu_{c,2}^{3/2}}\\
&\text{Pearson index:}\quad \gamma_2 := \tfrac{\mu_{c,4}}{\mu_{c,2}^2} - 3
\end{aligned}$$

have been computed for all classical and many semi-classical Orthogonal Polynomial Sets (see e.g. [34],[15],[37). The same results have been obtained for associated and co-recursive of all classical polynomials, and for the so called *Relativistic orthogonal polynomials* (see [1],[33],[35],[16]).

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