ANALOGUE OF SAINT-VENANT'S PRINCIPLE FOR THE ONE SPECIAL TYPE 4-TH ORDER ELLIPTIC EQUATION AND ITS APPLICATIONS

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Abstract

In the present article a priori energetic estimates, analogue of Saint-Venants principle in theory of elasticity, for the solutions of Dirichlet problem for special type 4-th order elliptic equation with variable coefficients are proved. On the basis of this estimates asymptotic behaviour of solutions of the corresponding boundary value problem are studied, under weak assumptions regarding the structure of the boundary in the neighbourhood of irregular boundary points.

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The behaviour of the solution of boundary-value problems for the equation of elasticity theory and for high order elliptic differential equations and systems in the neighbourhood of irregular boundary points and at infinity has been studied in a number of papers with different methods (see for example [1-6] or [15-21]). Also in a number of papers with different methods authors justify the Saint-Venant's principle for different materials and domains (see for example [11-14]). In the present article, on the basis of the methods of [7] and [8] we obtain apriori energetic estimates (Analogue of Saint-Venant's Principle in the plane theory of elasticity [11]) of solutions for one special type 4-th order elliptic equation with variable coefficients.

Later we study the character of solution of this equation in the neighbourhood of irregular boundary points, under weak assumptions regarding the structure of the boundary .

Let the domain Ω be simply connected, bounded and situated in the half-plane $R_2^+ = \{x \equiv (x_1, x_2) : \ x_1 > 0\}$ and the intersection S_t of Ω with the straight line $x_1 = t; 0 < t < T, T = const.$, consist of finitely many intervals whose endpoints belong to $\partial\Omega$;$\partial\Omega$- sufficiently smooth boundary of domain Ω; . In the Ω we consider the boundary-value problem:

$$\left(a_{\alpha\beta}(x)\, u_{,\alpha\beta}(x)\right)_{,\alpha\beta} = f(x), \quad x \in \Omega, \quad f \in C^1(\Omega), \tag{1}$$

$$u \mid_{\partial\Omega} = \varphi_1, \frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = \varphi_{2.} \tag{2}$$

Where the Greek indices α and β on the value 1 and 2 and summation over the repeated indices from 1 to 2 is assumed;

$$v_{,\alpha} \equiv \frac{\partial v}{\partial x_\alpha}; v_{,\alpha\beta} \equiv \frac{\partial^2 v}{\partial x_\alpha \partial x_\beta};$$

and ν - is the direction of external normal to the boundary $\partial\Omega$; $a_{\alpha\beta(x)} \in C^1(\Omega)$ for any $\alpha, \beta = 1, 2$ and

$$0 < a_0 = const. \le a_{\alpha\beta} \le a^0 = const. < \infty.$$

Let ω be an open set in the plane R_2 and γ a part of curves of its boundary. By $H_2(\omega, \gamma)$ we designate the Sobolev space obtained by completing in the norm

$$\|v\|_2 = \left[\int\limits_\omega \left(|v|^2 + v_{,\alpha}v_{,\alpha} + v_{,\alpha\beta}v_{,\alpha\beta}\right) dx\right]^{1/2}$$

the set of functions $v(x)$ which are twice continuously differentiable in $\overline{\omega}$ and equal to zero in a neighbourhood of γ.

We call the function $u(x)$ the generalised solution of (1) in Ω with boundary condition $u = \partial u/\partial\nu = 0$ on the $\gamma \in \partial\Omega$ if $u \in H_2(\Omega, \gamma)$ and satisfies the integral identity

$$\int\limits_\Omega a_{\alpha\beta}(x)\, u_{,\alpha\beta}(x)\, v_{,\alpha\beta}(x)\, dx = \int\limits_\Omega f(x)\, v(x)\, dx \tag{3}$$

for every function $v \in H_2(\Omega, \partial\Omega)$.

It is easy to see that the classical (smoth in $\overline{\Omega}$) solution of (1) in Ω satisfying the boundary conditions $u = \partial u/\partial\nu = 0$ on γ is also the generalized solution, if the boundary of Ω is sufficiently smooth.

Theorem 1 (apriori energetic estimate) *Let a bounded domain Ω be situated in the half-plane R_2^+, and let $S_t \equiv \Omega \cap \{x : x_1 = t\}$ be nonempty for every $t \in (0,T), T = const. < T^*$, where $\Omega_{T^*} \cap \Omega = \Omega f(x) = 0$ in $\Omega_T \equiv \Omega \cap \{x : x_1 < T\}$. Let*

$$\frac{\partial a_{\alpha 1}(x)}{\partial x_1} \geq 0 \quad x \in \Omega.$$

Then, for the generalized solution $u(x)$ of equation (1) in the region Ω_T, with the boundary conditions $u = \partial u/\partial \nu = 0$ on $\partial\Omega_T \cap \partial\Omega$ (if it exists), the following estimates are valid:

$$\int_{\Omega_T} u^2 M(x_1)\,\Phi(x_1,T,\varepsilon)\,dx \leq c_1 \int_{\Omega_T} a_{\alpha\beta}(x)\,(u_{,\alpha\beta})^2\,\Phi(x_1,T,\varepsilon)\,dx \leq \varepsilon^{-1} \int_{\Omega_T} W(u)\,dx; \tag{4}$$

$$\int_{\Omega_T} u_{,\alpha} u_{,\alpha} m(x_1)\,\Phi(x_1,T,\varepsilon)\,dx \leq c_2 \int_{\Omega_T} a_{\alpha\beta}(x)\,(u_{,\alpha\beta})^2\,\Phi(x_1,T,\varepsilon)\,dx, \tag{5}$$

where c_1 and c_2 are some positive constants and are independent of solutions and of coefficients of equations; $\varepsilon = const., 0 < \varepsilon < 1$,

$$W(u) = a_{\alpha\beta}(x) u_{,\alpha\beta}(x)\, u_{,\alpha\beta}(x)$$

and the function $\Phi(x_1,T,\varepsilon)$ is a solution of the following Cauchy problem

$$\Phi_{,x_1x_1}(x_1,T,\varepsilon) - (1-\varepsilon)\,\mu(x_1)\,\Phi(x_1,T,\varepsilon) = 0 \tag{6}$$

for $0 < x_1 < T$, and

$$\Phi(T,T,\varepsilon) = 1, \Phi_{,x_1}(T,T,\varepsilon) = 0. \tag{7}$$

The function $\mu(x_1) \in C^1(\Omega)$ and satisfies the relation from $t \in (0,T]$

$$0 < \mu(t) \leq \kappa(t) \equiv \inf_{v \in \aleph} \left\{ \int_{S_t} a_{\alpha\beta}(v_{,\alpha\beta})^2\,dx_2 \left| \int_{S_t} \left(a_{\alpha 1}(v_{,\alpha})^2 - a_{11} v_{,11} v \right) dx_2 \right|^{-1} \right\}, \tag{8}$$

where $\aleph$ is the set of functions $v(x)$ which are twice continuously differentiable in the neigbourhood of $\overline{S_t}$ and such that $v = v_{,1} = v_{,2} = 0$ on $\overline{S_t} \cap \partial\Omega$;

The functions $M(t)$ and $m(t)$ are continuous in $(0,T]$ and satisfy the relations:

$$0 < M(t) \leq \Lambda(t) \equiv \inf_{v\in\aleph}\left\{\int\limits_{S_t} a_{\alpha\beta}(v_{,\alpha\beta})^2\,dx_2 \left|\int\limits_{S_t} v^2 dx_2\right|^{-1}\right\}; \tag{9}$$

$$0 < m(t) \leq \Lambda_1(t) \equiv \inf_{v\in\aleph}\left\{\int\limits_{S_t} a_{\alpha\beta}(v_{,\alpha\beta})^2\,dx_2 \left|\int\limits_{S_t} v_{,\alpha}v_{,\alpha} dx_2\right|^{-1}\right\}. \tag{10}$$

Proof. Let us construct function $\psi(x_1,\delta)$ so that

$$\psi(x_1,\delta) = \begin{cases} \Phi(x_1,T,\varepsilon) \quad for\ \ 0<\delta\leq x_1\leq T,\ \ \delta = const., \\ \\ (x_1-\delta)\,\Phi_{,x_1}(\delta,T,\varepsilon) + \Phi(\delta,T,\varepsilon) \quad for\ \ 0\leq x_1\leq\delta. \end{cases}$$

Substituting the function $v(x) = u(x)\,(\psi(x_1,\delta)-1) \in H_2(\Omega_T,\partial\Omega_T)$ into the integral identity (3) for Ω_T , we obtain

$$0 = \int\limits_{\Omega_T}\Big\{a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1) + a_{\alpha\beta}u_{,\alpha\beta}u_{,\alpha}\psi_{,\beta} + a_{\alpha\beta}u_{,\alpha\beta}u_{,\beta}\psi_{,\alpha} +$$

$$+a_{\alpha\beta}u_{,\alpha\beta}u\psi_{,\alpha\beta}\}\,dx = \int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1)\,dx + \frac{1}{2}\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha})^2_{,\beta}\,\psi_{,\beta}dx +$$

$$+\frac{1}{2}\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\beta})^2_{,\alpha}\,\psi_{,\alpha}dx + \int\limits_{\Omega_T} a_{\alpha\beta}u_{,\alpha\beta}u\psi_{,\alpha\beta}dx = \int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1)\,dx -$$

$$-\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha})^2\,\psi_{,\beta\beta}dx - \frac{1}{2}\int\limits_{\Omega_T} a_{\alpha\beta,\alpha}(u_{,\beta})^2\,\psi_{,\alpha}dx + \int\limits_{\Omega_T} a_{\alpha\beta}u_{,\alpha\beta}u\psi_{,\alpha\beta}dx. \tag{11}$$

In the derivation of the last equality we have used integration by parts, which can easily be justified if we approximate $u(x)$ by functions of class $C^2\left(\overline{\Omega_T}\right)$ equal to zero in the neighbourhood of $\partial\Omega\cap\partial\Omega_T$, and use the fact that $\psi_{,\alpha} = 0$ for $x_1 = T$. Taking into account that ψ is independent of x_2, and

a) When $\partial a_{\alpha 1}(x)/\partial x_1 = 0$ we find that

$$\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1)\,dx = \int\limits_{\Omega_T\setminus\Omega_\delta}\left(a_{\alpha 1}(u_{,\alpha})^2 - a_{11}u_{,11}u\right)\psi_{,11}dx, \tag{12}$$

b)When $\partial a_{\alpha 1}(x)/\partial x_1 \geq 0$, then in the equality (11) we have

$$0=\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1)\,dx-\int\limits_{\Omega_T} a_{\alpha 1}(u_{,\alpha})^2\psi_{,11}dx- \\ -\int\limits_{\Omega_T} a_{\alpha 1,1}(u_{,\alpha})^2\psi_{,1}dx+\int\limits_{\Omega_T} a_{11}u_{,11}u\psi_{,11}dx \tag{13}$$

and

$$\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1)\,dx=\int\limits_{\Omega_T} a_{\alpha 1,1}(u_{,\alpha})^2\psi_{,1}dx+ \\ +\int\limits_{\Omega_T\setminus\Omega_\delta}(a_{\alpha 1}(u_{,\alpha})^2-a_{11}u_{,11}u)\psi_{,11}dx. \tag{14}$$

By the general theory of ordinary differential equations it is easy to show (see for example [22]) if Φ- is a solution of Cauchy problem (6-7) then $\psi_{,1}\leq 0$. Therefore, since $a_{\alpha 1,1}\geq 0$, the first member of the right side equality (14) is nonpositive, and

$$\int\limits_{\Omega_T} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi-1)\,dx\leq\int\limits_{\Omega_T\setminus\Omega_\delta} P(u)\,\psi_{,11}dx. \tag{15}$$

Let us estimate this integral (12) or (15). We put $P(v)\equiv a_{11}(v_{,1})^2+a_{21}(v_{,2})^2-a_{11}v_{,11}v;W(v)\equiv a_{\alpha\beta}(v_{,\alpha\beta})^2$.

Let u_n be a sequence of functions twice continuously differentiable in $\overline{\Omega_T}$, which are equal to zero in the neigbourhood of the set $\partial\Omega\cap\partial\Omega_T$, converging to $u(x)$ in the norm as $n\to\infty$. It is easy to see that

$$\int\limits_{\Omega_T\setminus\Omega_\delta} P(u)\,\psi_{,11}dx=\int\limits_{\Omega_T\setminus\Omega_\delta} P(u_n)\,\psi_{,11}dx+\varepsilon_n,$$

where $\varepsilon_n\to 0$ as $n\to\infty$. From the definition of the function $\mu(t)$ and the equation for $\Phi(x_1,T,\varepsilon)$, it follows that

$$\left|\int\limits_{\Omega_T\setminus\Omega_\delta} P(u_n)\,\Phi_{,x_1x_1}dx\right|\leq\int\limits_\delta^T\Phi_{,x_1x_1}(x_1,T,\varepsilon)\left|\int\limits_{S_{x_1}} P(u_n)\,dx_2\right|dx_1\leq \\ \leq\int\limits_\delta^T\Phi_{,x_1x_1}(x_1,T,\varepsilon)(\mu(x_1))^{-1}\int\limits_{S_{x_1}} E(u_n)\,dx_2dx_1= \\ =(1-\varepsilon)\int\limits_{\Omega_T\setminus\Omega_\delta} W(u_n)\,\Phi(x_1,T,\varepsilon)\,dx. \tag{16}$$

Letting n in this inequality go to ∞, we obtain

$$\left| \int\limits_{\Omega_T \setminus \Omega_\delta} P(u)\, \Phi_{,x_1 x_1} dx \right| \leq (1-\varepsilon) \int\limits_{\Omega_T \setminus \Omega_\delta} W(u_n)\, \Phi(x_x, T, \varepsilon)\, dx.$$

From this and (15) we conclude that

$$\int\limits_{\Omega_T} a_{\alpha\beta} (u_{,\alpha\beta})^2\, \psi(x_1, \delta)\, dx \leq \int\limits_{\Omega_\delta} a_{\alpha\beta} (u_{,\alpha\beta})^2\, \psi(x_1, \delta)\, dx +$$

$$+ \int\limits_{\Omega_T \setminus \Omega_\delta} W(u)\, \Phi(x_1, T, \varepsilon)\, dx \leq \int\limits_{\Omega_T} W(u)\, dx + (1-\varepsilon) \int\limits_{\Omega_T \setminus \Omega_\delta} W(u)\, \Phi(x_1, T, \varepsilon)\, dx$$

and, consequently,

$$\varepsilon \int\limits_{\Omega_T \setminus \Omega_\delta} W(u)\, \Phi(x_1, T, \varepsilon)\, dx \leq \int\limits_{\Omega_T} W(u)\, dx. \tag{17}$$

Letting $\delta \to 0$, we obtain part of inequality (4)

$$\int\limits_{\Omega_T} W(u)\, \Phi(x_1, T, \varepsilon)\, dx \leq \frac{1}{\varepsilon} \int\limits_{\Omega_T} W(u)\, dx.$$

The remaining inequalities for functions u_n follow immediately from the definitions of M and m. Further, passing to the limit as $n \to \infty$, we obtain the desired inequalities for u. The theorem is proved. □

Remark 1 This result is also true, when $x_1 = 0$ the boundary of domain Ω has an irregular point (for example: cusp or corner).

Remark 2 For function $\Phi(x_1, T, \varepsilon)$ it is permissible to take a continuously differentiable function $\Phi(x_1, T, \varepsilon)$ with a piecewise continuous second derivative, satisfying the initial conditions (7) and the inequalities

$$|\Phi_{,x_1 x_1}(x_1, T, \varepsilon)| \leq (1-\varepsilon)\, \mu(x_1)\, \Phi(x_1, T, \varepsilon), \quad \Phi(x_1, T, \varepsilon) > 0, \tag{18}$$

$$\Phi_{,x_1}(x_1, T, \varepsilon) < 0 \quad \text{for} \quad 0 < x_1 < T.$$

Theorem 2 (Analogues of Saint-Venant's principles). *Under conditions of theorem 1 for any $0 < t_0 < t_1 \leq T$ the following estimates are valid:*

$$\int\limits_{\Omega_{t_0}} a_{\alpha\beta}(x)\, (u_{,\alpha\beta})^2\, dx \leq c\, (\Phi(t_0, t_1))^{-1} \int\limits_{\Omega_{t_1}} a_{\alpha\beta}(x)\, (u_{,\alpha\beta})^2\, dx, \tag{19}$$

where c is some positive constant and function $\Phi(x_1, t_1)$ *satisfies, for* $t_0 \leq x_1 \leq t_1$*, the ordinary differential equation*

$$\Phi_{,x_1 x_1}(x_1, t_1) - \mu(x_1)\Phi(x_1, t_1) = 0 \tag{20}$$

and conditions

$$\Phi(t_1, t_1) = 1, \Phi_{,x_1}(t_1, t_1) = 0. \tag{21}$$

where $\mu(x_1)$ *is any continuous function satisfying (8). (see fig. 1)*

Fig.1

Proof. Let us construct function $\psi(x_1)$ assuming that

$$\psi(x_1) = \begin{cases} \Phi(x_1, T)\, for t_0 \leq x_1 \leq t_1, \\ \\ (x_1 - t_0)\Phi_{,x_1}(t_0, t_1) + \Phi(t_0, t_1)\, for 0 \leq x_1 \leq t_0. \end{cases}$$

Substituting the function $v(x) = u(x)(\psi(x_1) - 1) \in H_2(\Omega_{t_1}, \partial\Omega_{t_1})$ into the integral identity (3) for Ω_T , we obtain

$$0 = \int\limits_{\Omega_{t_1}} \left\{ a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi - 1) + a_{\alpha\beta}u_{,\alpha\beta}u_{,\alpha}\psi_{,\beta} + a_{\alpha\beta}u_{,\alpha\beta}u_{,\beta}\psi_{,\alpha} + \right.$$

$$\left. + a_{\alpha\beta}u_{,\alpha\beta}u\psi_{,\alpha\beta} \right\} dx = \int\limits_{\Omega_{t_1}} a_{\alpha\beta}(u_{,\alpha\beta})^2(\psi - 1)\, dx + \frac{1}{2}\int\limits_{\Omega_{t_1}} a_{\alpha\beta}(u_{,\alpha})^2_{,\beta}\psi_{,\beta} dx +$$

$$+\frac{1}{2}\int\limits_{\Omega_{t_1}} a_{\alpha\beta}\left(u_{,\beta}\right)^2_{,\alpha}\psi_{,\alpha}dx+\int\limits_{\Omega_{t_1}} a_{\alpha\beta}u_{,\alpha\beta}u\psi_{,\alpha\beta}dx=\int\limits_{\Omega_{t_1}} a_{\alpha\beta}\left(u_{,\alpha\beta}\right)^2(\psi-1)\,dx-$$

$$-\int\limits_{\Omega_{t_1}} a_{\alpha 1}\left(u_{,\alpha}\right)^2\psi_{,11}dx-\int\limits_{\Omega_{t_1}} a_{\alpha 1,1}\left(u_{,\alpha}\right)^2\psi_{,1}dx+\int\limits_{\Omega_{t_1}} a_{11}u_{,11}u\psi_{,11}dx.$$

By definition $\psi_{,11}=0$ when $0\le x_1\le t_0$, and therefore

$$\int\limits_{\Omega_{t_1}} a_{\alpha\beta}\left(u_{,\alpha\beta}\right)^2(\psi-1)\,dx\le\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} P(u)\,\psi_{,11}dx. \tag{22}$$

Let u_n be a sequence of functions twice continuously differentiable in Ω_T , which are equal to zero in the neigbourhood of the set $\partial\Omega\cap\partial\Omega_T$, converging to $u(x)$ in the norm as $n\to\infty$. It is easy to see that

$$\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} P(u)\,\psi_{,11}dx=\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} P(u_n)\,\psi_{,11}dx+\varepsilon_n, \tag{23}$$

where $\varepsilon_n\to 0$ as $n\to\infty$. From the definition of the function $\mu(t)$ and the equation for $\Phi(x_1,t_1)$, it follows that

$$\left|\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} P(u_n)\,\Phi_{,x_1x_1}dx\right|\le\int\limits_{t_0}^{t_1}\Phi_{,x_1x_1}(x_1,t_1)\left|\int\limits_{S_{x_1}} P(u_n)\,dx_2\right|dx_1\le \tag{24}$$

$$\le\int\limits_{t_0}^{t_1}\Phi_{,x_1x_1}(x_1,t_1)\left(\mu(x_1)\right)^{-1}\int\limits_{S_{x_1}} W(u_n)\,dx_2dx_1=\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} W(u_n)\,\Phi(x_1,t_1)\,dx.$$

Letting n in this inequality go to ∞, we obtain

$$\left|\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} P(u)\,\Phi_{,x_1x_1}dx\right|\le\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} W(u)\,\Phi(x_1,t_1)\,dx.$$

From the last inequality and inequality (22) we have

$$\int\limits_{\Omega_{t_1}} W(u)\,\psi dx=\int\limits_{\Omega_{t_0}} W(u)\,\psi dx+\int\limits_{\Omega_{t_1}\setminus\Omega_{t_0}} W(u)\,\Phi(x_1,t_1)\,dx\le$$

$$\leq \int_{\Omega_{t_1}} W(u)\, dx + \int_{\Omega_{t_1} \backslash \Omega_{t_0}} W(u)\, \Phi(x_1, t_1)\, dx$$

or

$$\int_{\Omega_{t_0}} a_{\alpha\beta}\, (u_{,\alpha\beta})^2\, \Phi dx \leq \int_{\Omega_{t_1}} a_{\alpha\beta}\, (u_{,\alpha\beta})^2\, dx. \tag{25}$$

It is easy to show that $\Phi_{,x_1} < 0$ and we have estimate (19). The theorem is proved. □

Theorem 3 *Suppose that the assumptions of Theorem1 are fulfilled. Then the function $u(x)$ is continuous in Ω_T, and*

$$|u(x)|^2 \leq \frac{3}{\varepsilon}\, (\Phi(x_1, T, \varepsilon))^{-1}\, M(x)^{-1/4}\, m(x)^{-1/2} \int_{\Omega_T} W(u)\, dx, \quad x \in \Omega_T, \tag{26}$$

where the functions Φ, M and m are defined in the Theorem1. Moreover, it is assumed that $M(x)$ and $m(x)$ are nonincreasing functions continuously differentiable for $0 < x_1 \leq T$.

Proof. Since, by definition, $u(x)$ belongs to $H_2(\Omega, \gamma)$,where $\gamma = \partial\Omega \cap \partial\Omega_{T,}$ there exists a sequence of functions u_n such that $u_n \to u$ as $n \to \infty$ in the norm and $u_n = 0$ in a neighbourhood of γ. We define the functions u_n outside the set Ω_T by assigning the value zero. Let $l > 0$ be a sufficiently large constant such that $\Omega_T \in Q = \{x : 0 < x_1 < T, |x_2| < l\}$. If we set $u = 0$ outside Ω_T, it is easy to see that $u \in H_2(Q, \partial Q \cap \{x : 0 < x_1 < T\})$ and, therefore, according to Sobolev's imbedding theorem , $u(x)$ is continuous in Q . We define functions $\Phi\delta, M_\delta$, and m_δ in such a way that $\Phi_\delta = \Phi, M_\delta = M$, and $m_\delta = m$ for $x_1 > \delta$; These functions are bounded, monotone, and continuously differentiable with respect to $x_1$for $0 < x_1 \leq T$; We estimate the function

$$u_n^2(x)\, \Phi_\delta(x_1, T, \varepsilon)\, M_\delta^{1/4}(x_1)\, m_\delta^{1/2}(x_1) \equiv u_n^2(x)\, \varphi_\delta(x_1)\, .$$

We note that for a certain $\sigma = \sigma(n)$ the function $u_n(x)$ is equal to zero in $\Omega_{\sigma.}$ Hence we may write

$$u_n^2(x)\, \varphi_\delta(x_1) = \int_0^{x_1} \frac{\partial}{\partial x_1}\left(u_n^2 \varphi_\delta\right) dx_1 = \int_0^{x_1} 2u_n u_{n,1} \varphi_\delta dx_1 + \int_0^{x_1} u_n^2 \varphi_{\delta,1} dx_1.$$

Since $\varphi_{\delta,1} \leq 0$,

$$u_n^2(x)\,\varphi_\delta(x_1) \leq \int\limits_0^T (u_{n,1})^2\, m_\delta^{\frac{1}{2}} \Phi_\delta dx_1 + \int\limits_0^T (u_n)^2\, m_\delta^{\frac{1}{2}} M_\delta^{\frac{1}{2}} \Phi_\delta dx_1.$$

It is easy to see that

$$(u_{n,1})^2 = \int\limits_{-l}^{x_2} \frac{\partial}{\partial x_2} (u_{n,1})^2\, dx_2,$$

$$\int\limits_0^T (u_{n,1})^2\, m_\delta^{\frac{1}{2}} \Phi_\delta dx_1 \leq \int\limits_0^T \int\limits_{-l}^{l} \left|2u_{n,12} u_{n,1}\right| m_\delta^{\frac{1}{2}} \Phi_\delta dx_2 dx_1 \leq$$

$$\leq \int\limits_{\Omega_T} (u_{n,12})^2\, \Phi_\delta dx + \int\limits_{\Omega_T} (u_{n,1})^2\, m_\delta \Phi_\delta dx.$$

Analogically we obtain

$$\int\limits_0^T (u_n)^2\, m_\delta^{\frac{1}{2}} M_\delta^{\frac{1}{2}} \Phi_\delta dx_1 + \int\limits_0^T \int\limits_{-l}^{x_2} \frac{\partial}{\partial x_2} \left(u_n^2\right) m_\delta^{\frac{1}{2}} M_\delta^{\frac{1}{2}} \Phi_\delta dx_2 dx_1 \leq$$

$$\leq \int\limits_{\Omega_T} (u_n)^2\, M_\delta \Phi_\delta dx + \int\limits_{\Omega_T} (u_{n,2})^2\, m_\delta \Phi_\delta dx.$$

From these inequalities it follows that

$$u_n^2(x)\,\varphi_\delta(x_1) \leq \int\limits_{\Omega_T} u_n^2 M_\delta \Phi_\delta dx + \int\limits_{\Omega_T} (u_{n,\alpha})^2\, m_\delta \Phi_\delta dx + \int\limits_{\Omega_T} (u_{n,12})^2\, \Phi_\delta dx.$$

Thus, for $x \in \Omega_T$,

$$u_n^2(x)\,\varphi_\delta(x_1) \leq \int\limits_{\Omega_T \setminus \Omega_\delta} u_n^2 M \Phi dx + \int\limits_{\Omega_T \setminus \Omega_\delta} u_{n,\alpha} u_{n,\alpha} m \Phi dx + \int\limits_{\Omega_T \setminus \Omega_\delta} W(u_n)\, \Phi dx +$$

$$+ \int\limits_{\Omega_\delta} u_n^2 M_\delta \Phi_\delta dx + \int\limits_{\Omega_\delta} u_{n,\alpha} u_{n,\alpha} m_\delta \Phi_\delta dx + \int\limits_{\Omega_\delta} W(u_n)\, \Phi_\delta dx.$$

Using the deffinition of $M(x_1)$ and $m(x_1)$, we obtain

$$u_n^2(x)\,\varphi_\delta(x_1) \le 3\int\limits_{\Omega_T\setminus\Omega_\delta} W(u_n)\,\Phi dx + \int\limits_{\Omega_\delta} u_{n,\alpha}u_{n,\alpha}\Phi_\delta m_\delta dx +$$
$$+\int\limits_{\Omega_\delta} u_n^2\Phi_\delta M_\delta dx + \int\limits_{\Omega_\delta} W(u_n)\,\Phi_\delta dx.$$

We pass to the limit as $n \to \infty$. For any fixed $x \in \Omega_T$ and $\delta < x_1$, we find that

$$u^2(x)\,\varphi_\delta(x_1) \le 3\int\limits_{\Omega_T\setminus\Omega_\delta} W(u)\,\Phi dx + \int\limits_{\Omega_\delta} u_{,\alpha}u_{,\alpha}\Phi_\delta m_\delta dx + \tag{27}$$
$$+\int\limits_{\Omega_\delta} u^2\Phi_\delta M_\delta dx + \int\limits_{\Omega_\delta} W(u)\,\Phi_\delta dx.$$

Since, by deffinition, $0 < \Phi_\delta \le \Phi$, $0 < m_\delta \le m$, $0 < M_\delta \le M$,

$$\int\limits_{\Omega_\delta} u_{,\alpha}u_{,\alpha}\Phi_\delta m_\delta dx + \int\limits_{\Omega_\delta} u^2\Phi_\delta M_\delta dx + \int\limits_{\Omega_\delta} W(u)\,\Phi_\delta dx \le$$
$$\le \int\limits_{\Omega_\delta} u_{,\alpha}u_{,\alpha}\Phi m dx + \int\limits_{\Omega_\delta} u^2\Phi M dx + \int\limits_{\Omega_\delta} W(u)\,\Phi dx.$$

By vitrue of (4-5), the right-hand side of this inequality tends to 0 as $\delta \to 0$. Hence, passing to the limit in (27) and using (4) and (5) we obtain

$$u^2(x)\,\varphi(x_1) \le 3\varepsilon^{-1}\int\limits_{\Omega_\delta} W(u)\,dx.$$

The theorem is proved. □

Theorem 4 *If $u(x)$ is a generalized solution of equation* (1) *in Ω_T with boundary conditions $u = \partial u/\partial\nu = 0$ on the $\partial\Omega \cap \partial\Omega_T$ and $f(x) = 0$ in Ω_T, then for every t_0 and $t_1 \in (0,T)$ the following estimate holds:*

$$\max_{\Omega_{t_0}} |u|^2 \le p(t_0)\,|\Phi(t_0,t_1)|^{-1}\int\limits_{\Omega_{t_1}} a_{\alpha\beta}(x)\,(u_{,\alpha\beta})^2\,dx, \tag{28}$$

where $\Phi(t_0,t_1)$ is a solution of problem (20-21).

$$p(t_0) = a_0\left(1 + \sup_{0\le x_1\le t_0}(M(x_1))^{-1} + \sup_{0\le x_1\le t_0}(m(x_1))^{-1}\right), \tag{29}$$

where a_0 is a constant of ellipticity and functions M and m are defined by relations (9) and (10).

Proof. $u \in H_2(\Omega_T, \partial\Omega \cap \partial\Omega_T)$ and there exists some sequence $\{u_i\}$ with two conditions

j) $u_i(x) = 0$, if $x \notin \Omega_{t_1}$,

jj)$u_i(x) \to u(x)$,when $i \to \infty$.

Therefore we write

$$u_i^2(x) = \int_0^{x_1} 2u_i u_{i,1} dx_1 \le \int_0^{t_0} u_i^2 dx_1 + \int_0^{t_0} (u_{i,1})^2 dx_1; \qquad (30)$$

$$(u_{i,1}(x))^2 = \int_{-\infty}^{x_2} \frac{\partial}{\partial x_2} (u_{i,1})^2 dx_2 \le \int_{-\infty}^{\infty} (u_{i,1})^2 dx_2 + \int_{-\infty}^{x_2} (u_{i,12})^2 dx_2. \qquad (31)$$

Hence

$$\int_0^{t_0} (u_{i,1})^2 dx_1 \le \int_{\Omega_{t_0}} (u_{i,1})^2 dx + \int_{\Omega_{t_0}} (u_{i,12})^2 dx,$$

and

$$\int_0^{t_0} u_i^2 dx_1 \le \int_{\Omega_{t_0}} u_i^2 dx + \int_{\Omega_{t_0}} (u_{i,2})^2 dx,$$

These two inequalities and relations (30-31) allow us to write

$$u_i^2(x) \le \int_{\Omega_{t_0}} u_i^2 dx + \int_{\Omega_{t_0}} u_{i,\alpha} u_{i,\alpha} dx + \int_{\Omega_{t_0}} (u_{i,2})^2 dx.$$

Now using relations (8-10), we have

$$u_i^2(x) \le a_0 \sup_{0 \le x_1 \le t_0} (M(x_1))^{-1} \int_{\Omega_{t_0}} u_i^2 M(x_1) dx + \qquad (32)$$

$$a_0 \sup_{0 \le x_1 \le t_0} (m(x_1))^{-1} \int_{\Omega_{t_0}} u_{i,\alpha} u_{i,\alpha} m(x_1) dx + a_0 \int_{\Omega_{t_0}} W(u) dx \le p(t_0) \int_{\Omega_{t_0}} W(u) dx.$$

Hence, passing to the limit in (32) when $i \to \infty$ and using (4) and (5) we obtain inequality (28).

The theorem is proved. $\square$

Lemma 1 *If Ω is a bounded domain in R_2^+ . Then for $\mu(t)$ we may take any continuous function for which*

$$0 < \mu(t) \leq \frac{2\pi^2}{[l(t)]^2} A(t), \quad 0 < t \leq T. \tag{33}$$

where $l(t)$ denotes the length of greatest of the intervals constituting S_t;

$$A(t) \equiv \frac{\min\limits_{S_t, \ \alpha,\beta=1,2} \{a_{\alpha\beta}(x)\}}{\max\limits_{S_t, \ \alpha=1,2} \{a_{\alpha 1}(x)\}}.$$

We have also

$$0 < m(t) \leq \frac{2\pi^2}{[l(t)]^2} A^*(t);$$

and

$$0 < M(t) \leq \frac{(4.73)^4}{[l(t)]^4} A^*(t),$$

where

$$A^*(t) \equiv \min_{S_t, \ \alpha,\beta=1,2} \{a_{\alpha\beta}(x)\}.$$

Proof. From the variational theory, it is known ([9]) that if $v(x_1, x_2)$ is twice continuiously differentiable and $v = v_{,1} = v_{,2} = 0$ on $\overline{S_t} \cap \partial\Omega$, then

$$\int_{S_t} (v_{,1})^2 \, dx_2 \leq (\lambda_1(t))^{-1} \int_{S_t} (v_{,12})^2 \, dx_2$$

$$\int_{S_t} (v_{,2})^2 \, dx_2 \leq (\lambda_2(t))^{-1} \int_{S_t} (v_{,22})^2 \, dx_2, \tag{34}$$

$$\int_{S_t} (v)^2 \, dx_2 \leq (\lambda_3(t))^{-1} \int_{S_t} (v_{,22})^2 \, dx_2,$$

where $\lambda_1(t) = \pi^2 (l(t))^{-2}, \lambda_2(t) = 4\pi^2 (l(t))^{-2}, \lambda_3(t) = (4,73)^4 (l(t))^{-4}$. Besides, we know that for every a, b and $c > 0$ the following enequality holds

$$abc \leq \varepsilon c a^2 + \varepsilon^{-1} c b^2 \quad forevery; \quad \varepsilon > 0.$$

Therefore from (8), we obtain

$$\left|\int\limits_{S_t}\left(a_{\alpha 1}\left(v_{,\alpha}\right)^2-a_{11}v_{,11}v\right)dx_2\right|\leq\int\limits_{S_t}a_{11}\left(v_{,1}\right)^2dx_2+\int\limits_{S_t}a_{12}\left(v_{,2}\right)^2dx_2+$$

$$+\frac{\varepsilon}{2}\int\limits_{S_t}a_{11}v^2dx_2+\frac{1}{2\varepsilon}\int\limits_{S_t}a_{11}\left(v_{,11}\right)^2dx_2\leq$$

$$\leq\max_{S_t}\left\{a_{\alpha 1}\left(x\right)\right\}\int\limits_{S_t}\left(\left(v_{,1}\right)^2+\left(v_{,2}\right)^2+\frac{1}{2\varepsilon}u^2+\frac{\varepsilon}{2}\left(v_{,11}\right)^2\right)dx_2\leq \tag{35}$$

$$\leq\max_{S_t}\left\{a_{\alpha 1}\left(x\right)\right\}\left\{\frac{1}{2}\left(\lambda_1\left(t\right)\right)^{-1}\int\limits_{S_t}\left(v_{,12}\right)^2dx_2+\left(\lambda_2\left(t\right)\right)^{-1}\int\limits_{S_t}\left(v_{,22}\right)^2dx_2+\right.$$

$$\left.+\frac{1}{2\varepsilon}\left(\lambda_3\left(t\right)\right)^{-1}\int\limits_{S_t}\left(v_{,22}\right)^2dx_2+\frac{\varepsilon}{2}\int\limits_{S_t}\left(v_{,11}\right)^2dx_2\right\},$$

where $\varepsilon>0$. We choose $\varepsilon=(4,73)^2\left[l\left(t\right)\right]^{-1}$. Then it is easy to see that we may take $\mu\left(t\right)$ any continuous function for which

$$0<\mu\left(t\right)\leq 2\pi^2\left(l\left(t\right)\right)^{-2}\left(\max_{S_t}\left\{a_{\alpha 1}\left(x\right)\right\}\right)^{-1}\left(\min_{S_t}\left\{a_{\alpha\beta}\left(x\right)\right\}\right),\quad\text{when }\ 0<t\leq T. \tag{36}$$

Let us now estimate $\Lambda\left(t\right)$ .We have

$$\int\limits_{S_t}\left(v\right)^2dx_2\leq\left(\lambda_3\left(t\right)\right)^{-1}\int\limits_{S_t}\left(v_{,22}\right)^2dx_2\leq$$

$$\leq\left(\lambda_3\left(t\right)\right)^{-1}\left(\min_{S_t}\left\{a_{\alpha\beta}\left(x\right)\right\}\right)^{-1}\int\limits_{S_t}E\left(u\right)dx_2,$$

Therefore,

$$\Lambda\left(t\right)\geq\left(4,73\right)^4\left(l\left(t\right)\right)^{-4}\left(\min_{S_t}\left\{a_{\alpha\beta}\left(x\right)\right\}\right)$$

and for $M\left(t\right)$ we may take any continuous function, satisfying

$$0<M\left(t\right)\leq\left(4,73\right)^4\left(l\left(t\right)\right)^{-4}\left(\min_{S_t}\left\{a_{\alpha\beta}\left(x\right)\right\}\right),\quad\text{for }\ 0<t\leq T. \tag{37}$$

We estimate $\Lambda_1(t)$. According to (34),

$$\int_{S_t}\left[(v_{,1})^2+(v_{,2})^2\right]dx_2\leq$$

$$\leq\frac{1}{2}(\lambda_1(t))^{-1}\int_{S_t}2(v_{,12})^2\,dx_2+(\lambda_2(t))^{-1}\int_{S_t}(v_{,22})^2\,dx_2\leq$$

$$\leq\max_{S_t}\left\{\frac{1}{2}(\lambda_1(t))^{-1},(\lambda_2(t))^{-1}\right\}\left(\min_{S_t}\{a_{\alpha\beta}(x)\}\right)^{-1}\int_{S_t}E(u)\,dx_2,$$

Therefore, for $m(t)$ we may take any continuous function such that

$$0<m(t)\leq 2\pi^2(l(t))^{-2}\left(\min_{S_t}\{a_{\alpha\beta}(x)\}\right),\quad 0<t\leq T. \tag{38}$$

EXAMPLES:

A) Let the region Ω lie in a neighbourhood of the origin and inside the angle $\{x : |x_2| < Ax_1/2; A = const.\}$ (see fig. 2) and $a_{\alpha\beta}(x) = const$ for any α and β.Then according to (33) we may put $\mu(t) = 2\pi^2 (At)^{-2}$. It is easy to verify that in this case the problem (6), (7) is solved by the function

$$\Phi(x_1, T, \varepsilon) = \frac{1}{s_1 + s_2}\left[s_2\left(\frac{x_1}{T}\right)^{s_1} + s_1\left(\frac{x_1}{T}\right)^{-s_2}\right],$$

Fig.2

where $s_{1,}s_2 = const. > 0$, and s_1 and $-s_2$ are the roots of the equation $s(s-1) = 2\pi^2(1-\varepsilon)A^{-2}$.

But we may put $M(x_1) = (4.73)^4 (Ax_1)^{-4}$ and $m(x_1) = 2\pi^2 (Ax_1)^{-2}$. Thus, by *Theorem 3*, for such a region Ω we have

$$|u(x)|^2 \leq c_0 |x_1|^{2+s_2} \int\limits_{\Omega_{t_1}} a_{\alpha\beta}(x)(u_{,\alpha\beta})^2 dx$$

where the constant c_0 depends only on A and ε, c_0- is a positive constant and $t_1 > x_1$.

B) Let the region Ω lie in a neighbourhood of the origin and inside the cusp

$$\left\{x : |x_2| \leq \frac{\pi}{2\sqrt{k}} x_1^{k+1} \left(k + (k+1)\, x_1^k\right)^{-1/2}\right\} \quad k = const. > 0.$$

It is easy to see that $l\,(x_1) \leq \pi k^{-1} x_1^{k+1}$ (see fig.3). According to (16) we may put $\mu\,(x_1) = 2kx_1^{-2(k+1)} \left(k + (k+1)\, x_1^k\right)$. It is not difficult to verify that for $T = 1$ and $\varepsilon = 1/2$ the function

$$\Phi\left(x_1, 1, \frac{1}{2}\right) = \frac{1}{2}\left[\exp\left(x_1^{-k} - 1\right) + \exp\left(1 - x_1^{-k}\right)\right]$$

satisfies the initial conditions (7) and the inequalities (14) for $0 < x_1 \leq 1, \varepsilon = \frac{1}{2}$.

Fig.3

According to Lemma 1 we may take

$$M\,(x_1) = c_1 x_1^{-4(k+1)}, \ \text{ and } \ m\,(x_1) = c_2 x_1^{-2(k+1)},$$

where $c_1, c_2 = const. > 0$ Therefore, in the case under consideration

$$|u\,(x)|^2 \leq c \exp\left\{-x_1^{-k}\right\} x_1^{2(k+1)} \int\limits_{\Omega_{t_1}} a_{\alpha\beta}\,(x)\,(u_{,\alpha\beta})^2\, dx,$$

where the constant c depends only on k and $t_1 > x_1$.

C) Let the region Ω lie in a neighborhood of the origin and inside the domain

$$\Omega_T \in \left\{x : |x_2| < \frac{\pi}{2}\mathrm{sin}x_1\right\}$$

(see fig. 4) when $T \leq \frac{\pi}{2}$. Then we may take $\mu(t) = \frac{2}{\mathrm{sin}^2 t}$ and therefore see [22]

$$\Phi(x_1, T) = c_1(T)\mathrm{ctg}x_1 + c_2(T)(1 - x_1\mathrm{ctg}x_1),$$

where $c_1(T) = \frac{1}{\mathrm{sin}^2 T}$ and $c_2(T) = \left[-\mathrm{ctg}T + \frac{T}{\mathrm{sin}^2 T}\right]$.

Fig.4

Thus by estimate (19) we have

$$\int\limits_{\Omega_{t_0}} W(u)dx \leq \mathrm{ctg}t_0\mathrm{sin}^2 t_1 \int\limits_{\Omega_{t_1}} W(u)dx$$

where $c = const$.

Therefore, in the case under consideration

$$|u(x)|^2 \leq c\frac{\mathrm{sin}^3 x_1}{\mathrm{cos}x_1} \int\limits_{\Omega_{x_1}} W(u)dx,$$

here c is independent of x and of solution $u(x)$.

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