DECOMPOSITION OF SOME COMPLEX FUNCTIONS WITH RESPECT TO THE CYCLIC GROUP OF ORDER n

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Abstract

Let f be a function of the complex variable z admitting a Laurent expansion in an annulus I with center in the origin. For an arbitrary positive integer n, Ricci's theorem asserts that the function f can be written as the sum of n functions $f_{[n,k]}$, $k = 0, 1, \cdots, n-1$, defined by

$$f_{[n,k]}(z) \;=\; \frac{1}{n}\sum_{\ell=0}^{n-1}\exp\left(-\frac{2i\pi k\ell}{n}\right)\; f\left(z\exp\left(\frac{2i\pi\ell}{n}\right)\right), \quad z \in I.$$

In this paper, we shall establish certain results to derive some properties and formulas pertaining to $f_{[n,k]}$ from f ones and to express some identities of f as functions of the components $f_{[n,k]}$. As an illustration, we consider the function $f(z) = \exp(z)$. The components of this function are the hyperbolic functions of order n and k-th kind. For those functions, we obtain alternative proofs of known identities and other properties which are believed to be new.

Key words and phrases: Laurent expansion, Ricci's Theorem, hyperbolic functions.

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1. *Introduction*

Let n be an arbitrary positive integer. Denote by $\omega_n = exp(\frac{2i\pi}{n})$ the complex n-root of unity, and by $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$ the set of the first n integers. Let $\Omega(I) = \Omega$ be the space of functions of the complex variable z admitting a Laurent expansion in an annulus I with center in the origin (in particular, holomorphic in a circular neighbourhood I of the origin). Let $\Omega_{[n,k]}(I) = \Omega_{[n,k]}$, $k \in \mathbb{N}_n$, be the vectorial subspace of Ω, defined by the symmetry property:

$$f \;\in\; \Omega_{[n,k]} \iff f(\omega_n z) \;=\; \omega_n^k f(z). \tag{1.1}$$

The subspaces $\Omega_{[2,0]}$ and $\Omega_{[2,1]}$ amount, respectively, to subspaces of even functions and odd functions.
The following decomposition in direct sum holds:

Theorem 1.1 *(cf. [16, p. 43])*

$$\Omega \ = \ \bigoplus_{k=0}^{n-1} \Omega_{[n,k]}. \tag{1.2}$$

It follows that if n is not a prime number, namely $n = pq$, $(p,q) \in \mathbb{N}^2$, and p and $q \geq 2$, we have

$$\Omega \ = \ \bigoplus_{r=0}^{p-1} \Omega_{[p,r]},$$

where

$$\Omega_{[p,r]} \ = \ \bigoplus_{j=0}^{q-1} \Omega_{[n,pj+r]}, \quad r \in \mathbb{N}_p. \tag{1.3}$$

From (1.2), we deduce that for any function f belonging to Ω there exists a unique sequence $\left(f_{[n,k]}\right)_{k\in\mathbb{N}_n}$, $f_{[n,k]} \in \Omega_{[n,k]}$, such that

$$f \ = \ \sum_{k=0}^{n-1} f_{[n,k]}, \tag{1.4}$$

and

$$f_{[n,k]}(z) \ = \ \Pi_{[n,k]}(f)(z) \ = \ \frac{1}{n}\sum_{\ell=0}^{n-1} \omega_n^{-k\ell} f(\omega_n^{\ell} z), \quad k \in \mathbb{N}_n, \tag{1.5}$$

where $\Pi_{[n,k]}$ is the projection operator on $\Omega_{[n,k]}$ along $\Omega_{[n,k]}^{\perp} \ = \ \bigoplus_{\substack{\ell=0 \\ \ell\neq k}}^{n-1} \Omega_{[n,\ell]}$.
The identity (1.4) is called the decomposition of the function f with respect to the cyclic group $\{\omega_n^k,\ k \in \mathbb{N}_n\}$ and the functions $f_{[n,k]}$ defined by (1.5) will be referred to as the components (with respect to the cyclic group of order n) of the function f.
Notice that if $f(z) \ = \ \sum_{m=0}^{\infty} a_m z^m$, then

$$f_{[n,k]}(z) \ = \ \sum_{m=0}^{\infty} a_{nm+k} z^{nm+k}. \tag{1.6}$$

So, for the generalized hypergeometric functions defined by (see, e.g., [12, p. 136, Eq. (1)]):

$${}_pF_q(z) \ = \ {}_pF_q\left(\begin{matrix} a_1, & \ldots & ,a_p, & \\ & & & z \\ b_1, & \ldots & ,b_q, & \end{matrix}\right) \ = \ \sum_{m=0}^{+\infty} \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\cdot\frac{z^m}{m!}, \tag{1.7}$$

where

• $(a)_m$ is the Pochlammer symbol given by (cf. [1, p. 256, Eq. (6.1.22)]):

$$(a)_m = \begin{cases} 1 & \text{if } \ m = 0, \\ a(a+1)\cdots(a+m-1) & \text{if } \ m = 1, 2, 3, \ldots, \end{cases} \tag{1.8}$$

or in terms of Gamma functions:

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, \quad a \neq 0, -1, -2, \ldots;$$

• p and q are positive integers or 0 (interpreting an empty product as 1);

• z is the complex variable;

• the numerator parameters a_i, $i = 1, \ldots, p$, and the denominator parameters b_j, $j = 1, \ldots, q$, take on complex values, provided that $b_j \neq 0, -1, -2, \ldots, j = 1, \ldots, q$.

We have (cf. [14, p. 890, Eq. (5)] or [17, p. 194, Eq. (12)]):

$$\Pi_{[n,k]}\left(z \to {}_pF_q(z) = {}_pF_q\left(\begin{matrix} a_1, & \ldots & , a_p, & \\ & & & z \\ b_1, & \ldots & , b_q, & \end{matrix}\right)\right) \tag{1.9}$$
$$= (z \to \phi(n, k, a_1, \ldots, a_p, b_1, \ldots, b_q, z)),$$

with

$$\phi(n, k, a_1, \ldots, a_p, b_1, \ldots, b_q, z) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \cdot \frac{z^k}{k!}$$
$$\cdot {}_{np}F_{nq+n-1}\left(\begin{matrix} & \Delta(n, a_1+k), & \ldots & , \Delta(n, a_p+k), & \\ & & & & \frac{z^n}{n^{(1-p+q)n}} \\ \Delta^*(n, k+1), & \Delta(n, b_1+k), & \ldots & , \Delta(n, b_q+k), & \end{matrix}\right),$$

where $\Delta(n, \lambda)$ is the set of n parameters:

$$\Delta(n, \lambda) \equiv \left\{\frac{\lambda}{n}, \frac{\lambda+1}{n} \ldots, \frac{\lambda+n-1}{n}\right\}, \ n \in \mathbb{N}^*,$$

and $\Delta^*(n, k+1) = \Delta(n, k+1) \setminus \{\frac{n}{n}\}$.

Thus, the projection operator $\Pi_{[n,k]}$ can be thought of as a process of augmenting the parameters in the ${}_pF_q$ function obtaining instead ${}_{np}F_{nq+n-1}$. Moreover, there are in the literature numereous processes of augmentation of parameters of the ${}_pF_q$. We cite, for instance:

. The Laplace transform and its inverse (cf. [6, Vol.I, p. 219, Eq.(17) and p. 297, Eq.(1)]);
. The Euler transform (cf. [15, p. 104, Eq. (4)]);
. The d'Abdul-Halim-Al-Salem transform (cf. [1, p. 52, Eq. (2.3)]);
. The Jain transform (cf. [11, p. 18, Eq. (2.3)]);
. The Srivastava-Singhal transform (cf. [21, p. 426, Eq. (1.4)]);
. The Srivastava-Joshi transform (cf. [19, p. 19, Eq. (2.3)]);
. The Srivastava-Panda transform (cf. [20, p. 309, Eq. (1.7)]);

which has numereous applications in the special functions theory. So, it is significant to study these projection operators.

In the sequel, we purpose to establish some rules of calculus to facilitate the use of the projection operators $\Pi_{[n,k]}$. More precisely, we will evaluate the projection of the product of two functions and, under certains conditions, the projection of the composite of two functions. We will introduce a class of linear mappings in Ω, called homogeneous, that contains some usual operators. The composite of the projection operators and a homogeneous mapping will be determined. The decomposition of a linear mapping in Ω as a sum of n homogeneous mappings will be stated and proved. Thereafter, some obtained results will be used to derive certain properties and formulas pertaining to $f_{[n,k]}$ from f ones by a mere mechanical application of the projection operators and to express some identities of f using the components $f_{[n,k]}$. Then, as an application, we will consider the function $f(z) = \exp(z)$. The components of this function are the hyperbolic functions of order n and k-th kind (see for, e.g., [7, p. 213, Eq. (8)]). For those functions, we obtain alternative proofs of known identities and other properties which are believed to be new.

In the forthcoming papers, we shall explore some explicit examples in detail especially the Bessel functions; Laguerre polynomials; Boas-Buck polynomials; Brafman polynomials; and Srivastava-Panda polynomials to illustrate particular points in the general theory. Besides, we shall apply some results established in this paper to some questions arising in special functions theory and in harmonic analysis.

2. *Operations in Ω and projection operators*

From the symmetry property (1.1), one can easily state the following results:

Proposition 2.1 *Let $(f,g) \in \Omega_{[n,k]} \times \Omega_{[n,k']}$, then*

1. $$f \cdot g \in \Omega_{[n,\overbrace{k+k'}]},$$

2. $$f \circ g \in \Omega_{[n, \overbrace{\dot{k \cdot k'}}]},$$

where the notation $f \circ g$ *assumes the existence of this function and* $\dot{r}$, $r \in \mathbb{N}$, *denotes the class of* r *modulo* n.

Using these properties and the decomposition (1.4), we deduce the following

Corollary 2.2 *1/ Let* $(f, g) \in \Omega$, *we have*
$$\Pi_{[n,k]}(f \cdot g) = \sum_{r+r' \equiv k(n)} \Pi_{[n,r]}(f) \cdot \Pi_{[n,r']}(g). \tag{2.1}$$

2/ Let $(f, g) \in \Omega \times \Omega_{[n,q]}$, $q \in \mathbb{N}_n^*$.

- *If there exist* $\tilde{q} \in \mathbb{N}_n^*$ *such that* $q\tilde{q} \equiv 1(n)$, *then*

$$\Pi_{[n,k]}(f \circ g) = \Pi_{[n,k\tilde{q}]}(f) \circ g. \tag{2.2}$$

- *If* $n = pq$ *and* $k = qr$, *then*

$$\Pi_{[pq,qr]}(f \circ g) = \Pi_{[p,r]}(f) \circ g. \tag{2.3}$$

To prove (2.3) it is sufficient to remark that the identity (1.3) implies

$$\Pi_{[p,r]} = \sum_{s \equiv r(p)} \Pi_{[pq,s]}.$$

□

Some useful rules are given by the following

Corollary 2.3 *1/ If* $F = \prod_{j=1}^{r} f_j$, $f_j \in \Omega$, *then*

$$F_{[n,k]}(z) = \sum_{i_1+i_2+\cdots+i_r \equiv k(n)} \prod_{j=1}^{r} \Pi_{[n,i_j]}(f_j). \tag{2.4}$$

2/ If $F(z) = f(\alpha z)$, $\alpha \in \mathbb{C}$ *and* $f \in \Omega$, *then*

$$F_{[n,k]}(z) = f_{[n,k]}(\alpha z). \tag{2.5}$$

3/ If $F(z) = f(z^q)$, $f \in \Omega$ *and* $q \in \mathbb{N}^*$, *then*

$$F_{[pq,\ell q]}(z) = f_{[p,\ell]}(z^q), \quad \ell \in \mathbb{N}_p. \tag{2.6}$$

4/ If $F(z) = f\left(\frac{1}{z}\right)$, $f \in \Omega$, *then*

$$F_{[n,k]}(z) = f_{[n,n-k]}\left(\frac{1}{z}\right). \tag{2.7}$$

5/ If the function f satisfies the functional relation:

$$f(z+z') \;=\; \sum_{p=1}^{r} h_p(z)g_p(z'), \quad r \in \mathbb{N}^*, \tag{2.8}$$

where the functions h_p and g_p, $p = 1, 2, \ldots, r$, belong to Ω, then

$$\Pi_{[n,k]}(f)(z+z') \;=\; \sum_{p=1}^{r} \sum_{s+s' \equiv k(n)} \Pi_{[n,s]} h_p(z)\; \Pi_{[n,s']} g_p(z'). \tag{2.9}$$

Proof.
1/ The reiteration of (2.1) leads to (2.4).
2/ Since the function $z \longrightarrow \alpha z$, $\alpha \in \mathbb{C}$, belongs to $\Omega_{[n,1]}$, the identity (2.5) arises from (2.2).
3/ If we put $g(z) = z^q$ in (2.3), we obtain (2.6).
4/ The function $z \longrightarrow \frac{1}{z}$ belongs to $\Omega_{[n,n-1]}$. Since we have $(n-1)(n-1) \equiv 1(n)$, the identity (2.2) yields the formula (2.7).
5/ From (2.8), we learn

$$f\left((z+z')\xi\right) \;=\; \sum_{p=1}^{r} h_p(z\xi)g_p(z'\xi).$$

If we apply the projection operator $\Pi_{[n,k]}$ to the two members of this identity, considered as functions of the variable ξ, and we use (2.5) and (2.1) we obtain

$$\Pi_{[n,k]}(f)\left((z+z')\xi\right) \;=\; \sum_{p=1}^{r} \sum_{s+s' \equiv k(n)} \Pi_{[n,s]} h_p(z\xi)\; \Pi_{[n,s']} g_p(z'\xi),$$

which, for $\xi = 1$, amounts to (2.9). □

3. *Homogeneous mappings*

To make easier the use of the projection operators, we introduce the following notion :

Definition 3.1 Let φ be a linear mapping in Ω, φ is called *homogeneous of degree* ℓ, $\ell \in \mathbb{N}_n$, if and only if $\varphi(\Omega_{[n,k]}) \subseteq \Omega_{[n,\overbrace{k+\ell}^{\cdot}]}$ for all $k \in \mathbb{N}_n$.

As examples of homogeneous mappings we mention :

1. For each $h \in \mathbb{N}_n$, $\Pi_{[n,h]}$ is homogeneous of degree 0.

2. The scaling operator S_α, $\alpha \in \mathbb{C}^*$, defined by

$$(S_\alpha f)(z) \;=\; f(\alpha z)$$

is homogeneous of degree 0.

3. The n-translation operator ${}_n\tau_\xi$, $\xi \in \mathbb{C}$, defined by

$$({}_n\tau_\xi\, f)(z) \;=\; \frac{1}{n}\sum_{k=0}^{n-1} f(z+\omega_n^k\,\xi) \;=\Pi_{[n,0]}\,(\xi \longrightarrow f(z+\xi))$$

is homogeneous of degree 0.

4. The derivative operator $D_z = \frac{d}{dz}$ is homogeneous of degree $(n-1)$.

5. For each $g \in \Omega_{[n,\ell]}$, the mapping $f \longrightarrow gf$ is homogeneous of degree ℓ.

Notice that the composite of two homogeneous mappings of degree respectively ℓ and ℓ' is homogeneous of degree $(\ell+\ell')$.

Other characterizations of the homogeneous mappings are given by the following

Theorem 3.2 *Let φ be a linear mapping in Ω and $\ell \in \mathbb{N}_n$, the following statements are equivalent :*

(i) φ is homogeneous of degree ℓ.

(ii) $\Pi_{[n,\overbrace{k+\ell}^{\cdot}]} \circ \varphi \;=\; \varphi \circ \Pi_{[n,k]}$ *for all $k \in \mathbb{N}_n$.*

(iii) $S_{\omega_n} \circ \varphi \;=\; \omega_n^\ell\, \varphi \circ S_{\omega_n}$.

Proof. $(i) \Longrightarrow (ii)$: Let $k \in \mathbb{N}_n$. For each $f \in \Omega$, we have

$$f = f_{[n,k]} + f^\perp_{[n,k]}, \quad \text{where} \quad \left(f_{[n,k]}, f^\perp_{[n,k]}\right) \in \Omega_{[n,k]} \times \Omega^\perp_{[n,k]}.$$

The property (i) implies

$$\left(\varphi\left(f_{[n,k]}\right), \varphi\left(f^\perp_{[n,k]}\right)\right) \in \Omega_{[n,\overbrace{k+\ell}^{\cdot}]} \times \Omega^\perp_{[n,\overbrace{k+\ell}^{\cdot}]}.$$

So
$$\Pi_{[n,\overbrace{k+\ell}^{\cdot}]} \circ \varphi\,(f) \;=\; \varphi\left(f_{[n,k]}\right) \;=\; \varphi \circ \Pi_{[n,k]}(f)$$

and (ii) holds.

To prove $(ii) \Longrightarrow (iii)$, it is sufficient to observe that

$$S_{\omega_n} \;=\; \sum_{k=0}^{n-1} \omega_n^k\, \Pi_{[n,k]}.$$

$(iii) \Longrightarrow (i)$: Let $f \in \Omega_{[n,k]}$. We have from the relation (iii) and the symmetry property (1.1),

$$S_{\omega_n}(\varphi(f)) \;=\; \omega_n^{\ell}\, \varphi \,\circ\, S_{\omega_n}(f) \;=\; \omega_n^{k+\ell}\, \varphi\,(f).$$

So $\varphi(f) \in \Omega_{[n,\overbrace{k+\ell}^{\cdot}]}$ and (i) holds. □

Now, we state an usefull result:

Corollary 3.3 *Let φ be a homogeneous mapping of degree ℓ and let $r \in \mathbb{N}$ such that $r\ell \;\equiv\; 0(n)$. If f is an eigenvector of φ associated with the eigenvalue λ then the components $f_{[n,k]}$, $k \in \mathbb{N}_n$, of f are eigenvectors of φ^r associated with the eigenvalue λ^r.*

Proof. The mapping φ^r is homogeneous of degree 0 since $r\ell \equiv 0(n)$. So, according to Theorem 3.1, it commutates with $\Pi_{[n,k]}$ for all $k \in \mathbb{N}_n$. Then we have:

$$\varphi^r(f_{[n,k]}) = \varphi^r \circ \Pi_{[n,k]}(f) = \Pi_{[n,k]} \circ \varphi^r(f) = \Pi_{[n,k]}(\lambda^r f) = \lambda^r f_{[n,k]},$$

which finishes the proof. □

Corollary 3.4 *Let f be a function in Ω such that*

$$f(x+y) \;=\; \sum_{p=1}^{r} h_p(x) g_p(y), \quad r \in \mathbb{N}^*, \tag{3.1}$$

where the functions h_p et g_p, $p = 1, 2, \ldots, r$, belong to Ω, then

$${}_n\tau_y \Pi_{[n,0]}(f)(x) \;= \sum_{p=1}^{r} \Pi_{[n,0]}(h_p)(x) \Pi_{[n,0]}(g_p)(y). \tag{3.2}$$

Proof. From (3.1) and the definition of the n-translation operator, we deduce

$${}_n\tau_y(f)(x) \;=\; \sum_{p=1}^{r} (h_p)(x) \Pi_{[n,0]}(g_p)(y).$$

If we apply the projection operator $\Pi_{[n,k]}$ to the two members of the last identity, considered as functions of the variable x, we obtain (3.2). □

Notice that (3.2) may also be deduced from (2.9).

Next, we use the homogeneous mappings to decompose $\mathcal{L}(\Omega)$, the vector space of linear mappings from Ω into Ω. Define the mapping:

$$\begin{array}{rcccl} \beta \;: & \mathcal{L}(\Omega) & \longrightarrow & \mathcal{L}(\Omega) & \\ & \varphi & \longrightarrow & \beta(\varphi) & = \; S_{\omega_n} \circ \varphi \circ S_{\bar{\omega}_n}. \end{array}$$

It follows immediately from Theorem 3.1, that the subspace $(\mathcal{L}(\Omega))_{[n,\ell]}$ of homogeneous mappings of degree ℓ can be defined by the property:

$$\varphi \in (\mathcal{L}(\Omega))_{[n,\ell]} \iff \beta(\varphi) = \omega_n^\ell \varphi. \tag{3.3}$$

The following decomposition in direct sum holds:

Theorem 3.5

$$\mathcal{L}(\Omega) = \bigoplus_{\ell=0}^{n-1} (\mathcal{L}(\Omega))_{[n,\ell]}. \tag{3.4}$$

Proof. Let φ be an arbitrary mapping in $\mathcal{L}(\Omega)$. φ has the decomposition

$$\varphi = \sum_{\ell=0}^{n-1} \varphi_{[n,\ell]}, \tag{3.5}$$

with

$$\varphi_{[n,\ell]} = \frac{1}{n}\sum_{j=0}^{n-1} \omega_n^{-\ell j}\, \beta^j(\varphi),$$

which belongs to $(\mathcal{L}(\Omega))_{[n,\ell]}$ since $\beta\left(\varphi_{[n,\ell]}\right) = \omega_n^\ell \varphi_{[n,\ell]}$. So

$$\mathcal{L}(\Omega) = \sum_{\ell=0}^{n-1} (\mathcal{L}(\Omega))_{[n,\ell]}. \tag{3.6}$$

Now, let φ be in $(\mathcal{L}(\Omega))_{[n,\ell]} \cap (\mathcal{L}(\Omega))_{[n,\ell']}$, $\ell \neq \ell'$. Then $\beta(\varphi) = \omega_n^\ell \varphi = \omega_n^{\ell'} \varphi$. This relation yields $\varphi = 0$ and hence the decomposition (3.6) is direct. □

As an example we consider the differential operator

$$L = \sum_{r=0}^{m} a_r(z)\, D^r, \qquad D \equiv \frac{d}{dz},$$

where the functions a_r, $r = 0, 1, \ldots, m$, belong to Ω. By virtue of (1.4), L takes the form

$$L = \sum_{r=0}^{m}\sum_{k=0}^{n-1} (a_r)_{[n,k]}(z)\, D^r = \sum_{\ell=0}^{n-1} L_\ell,$$

where

$$L_\ell = \sum_{\substack{k-r\equiv\ell(n)\\ 0\le r\le m\\ 0\le k\le n-1}} (a_r)_{[n,k]}(z)\, D^r,$$

which is homogeneous of degree ℓ.

We leave to a future paper the study, using some results established in this section, of explicit examples of differential operators.

4. *Integral representations*

Proposition 4.1 *Let f be a holomorphic function in a domain $\mathcal{D}$ containing the disc $\mid z \mid \leq R$, we have*

$$\Pi_{[n,k]}(f)(z) \;=\; \frac{1}{2i\pi}\int_{|s|=R}\frac{s^{n-1-k}z^k}{s^n - z^n}f(s)ds,\ \mid z \mid < R, \qquad (4.1)$$

or, equivalently,

$$\Pi_{[n,k]}(f)(re^{i\theta}) \;=\; \int_0^{2\pi} P_{n\,,k}(R,r,\phi-\theta)f(Re^{i\phi})d\phi,\ r<R, \qquad (4.2)$$

with

$$P_{n\,,k}(R,r,\phi-\theta)$$

$$= \frac{(R^{2(n-k)}-r^{2(n-k)})R^k r^k e^{-ik(\phi-\theta)}+(R^{2k}-r^{2k})R^{n-k}r^{n-k}e^{i(n-k)(\phi-\theta)}}{2\pi(R^{2n}+r^{2n}-2R^n r^n\cos n(\phi-\theta))}. \qquad (4.3)$$

Proof. We deduce (4.1) by the use of the definition (1.5) and the well-known Cauchy formula since

$$\frac{s^{n-1-k}z^k}{s^n - z^n} \;=\; \frac{1}{n}\sum_{h=0}^{n-1}\frac{\omega_n^{-kh}}{s-\omega_n^h z}. \qquad (4.4)$$

Now, put in (4.1), $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, and $s = Re^{i\phi}$, $0 \leq \phi \leq 2\pi$. By virtue of (4.4), one obtains:

$$\Pi_{[n,k]}(f)(re^{i\theta}) \;=\; \frac{1}{2n\pi}\sum_{h=0}^{n-1}\omega_n^{-hk}\int_0^{2\pi}\frac{s}{s-\omega_n^h z}f(Re^{i\phi})d\phi, \quad r<R,$$

which can be rewritten as

$$\Pi_{[n,k]}(f)(re^{i\theta}) = \frac{1}{2n\pi}\sum_{h=0}^{n-1}\omega_n^{-hk}\int_0^{2\pi}\left(\frac{s}{s-\omega_n^h z}+\frac{\bar{\omega}_n^h\bar{z}}{\bar{s}-\bar{\omega}_n^h\bar{z}}\right)f(Re^{i\phi})d\phi,\ r<R. \qquad (4.5)$$

Indeed, we have

$$\frac{\bar{\omega}_n^h\bar{z}}{\bar{s}-\bar{\omega}_n^h\bar{z}} \;=\; -\frac{s}{s-\frac{s\bar{s}}{\bar{\omega}_n^h\bar{z}}}, \quad z\neq 0, \quad \text{and} \quad \left|\frac{s\bar{s}}{\bar{\omega}_n^h\bar{z}}\right| \;=\; \frac{R^2}{r} \;>\; R.$$

So $\dfrac{s\bar{s}}{\bar{\omega}_n^h \bar{z}}$ is exterior to the circle $s = Re^{i\phi}$, $0 \leq \phi \leq 2\pi$. It follows then, according to the Cauchy-Goursat theorem, that

$$\int_0^{2\pi} \frac{\bar{\omega}_n^h \bar{z}}{\bar{s} - \bar{\omega}_n^h \bar{z}} \, f(Re^{i\phi}) d\phi \;=\; 0.$$

Denote by $P_{n\,,k}(R, r, \phi - \theta)$ the kernel of the integral transform (4.5). That is

$$P_{n\,,k}(R, r, \phi - \theta) = \frac{1}{2n\pi} \sum_{h=0}^{n-1} \omega_n^{-hk} \left(\frac{s}{s - \omega_n^h z} + \frac{\bar{\omega}_n^h \bar{z}}{\bar{s} - \bar{\omega}_n^h \bar{z}} \right) =$$

$$= \frac{1}{2\pi} \left[\frac{s^{n-k} z^k}{s^n - z^n} + \frac{\bar{s}^k \bar{z}^{n-k}}{\bar{s}^n - \bar{z}^n} \right]$$

$$= \frac{\bar{s}^k z^k (\mid s^2 \mid^{n-k} - \mid z^2 \mid^{n-k}) + s^{n-k} \bar{z}^{n-k} (\mid s^2 \mid^k - \mid z^2 \mid^k)}{2\pi \mid s^n - z^n \mid^2},$$

from which one deduces (4.2) and (4.3). □

Two interseting special cases of (4.3), where the kernel $P_{n\,,k}(R, r, \phi - \theta)$ is real, are worthy to note:

$$(i) \qquad P_{2n\,,n}(R, r, \phi - \theta) \;=\; \frac{(R^{2n} - r^{2n)}) R^n r^n \cos n(\phi - \theta)}{\pi (R^{4n} + r^{4n} - 2R^{2n} r^{2n} \cos 2n(\phi - \theta)};$$

$$(ii) \qquad P_{n\,,0}(R, r, \phi - \theta) \;=\; \frac{R^{2n} - r^{2n}}{2\pi (R^{2n} + r^{2n} - 2R^n r^n \cos n(\phi - \theta))},$$

which can be rewritten as:

$$P_{n\,,0}(R, r, \phi - \theta) \;=\; \frac{1}{2\pi} \Re \left(\frac{s^n + z^n}{s^n - z^n} \right).$$

These kernels may be expressed by the classical Poisson's kernel :

$$\mathcal{K}(\rho, \alpha) \;=\; \frac{1 - \rho^2}{2\pi (1 + \rho^2 - 2\rho \cos \alpha)}, \; 0 \leq \rho < 1 \text{ and } 0 \leq \alpha < 2\pi.$$

Indeed, we have

$$P_{n\,,0}(R, r, \phi - \theta\,) = \; \mathcal{K}\Big((\frac{r}{R})^n, n(\phi - \theta) \Big);$$

$$P_{2n\,,n}(R,r,\phi-\theta\ ) = \ 2\frac{(\frac{r}{R})^n}{1+(\frac{r}{R})^{2n}}\cos(n(\phi-\theta))\mathcal{K}\Big((\frac{r}{R})^2 n, 2n(\phi-\theta)\Big).$$

It follows that some properties of these kernels may be derived from those of Poisson's kernel. For example, from the expansion of $\mathcal{K}(\rho,\alpha)$ (cf. [4, p. 271]), we obtain:

$$P_{n\,,0}(R,r,\phi-\theta) = 1+2\sum_{\nu=1}^{+\infty}(\frac{r}{R})^{n\nu}\ \cos n\nu(\phi-\theta) = \sum_{\nu=0}^{+\infty}(\frac{2r}{R})^{n\nu}\,T_{n\nu}(\cos(\phi-\theta)),$$

where $T_p(x)$ is the pth Tchebycheff polynomial.

Let us return now to (4.2). Since for all function f in $\Omega_{[n,k]}$, $\Pi_{[n,k]}(f) = f$, we state:

Corollary 4.2 *Let f be a holomorphic function in a domain $\mathcal{D}$ containing the disc $\mid z \mid\leq R$. If $f\in\Omega_{[n,k]}$, then*

$$f(re^{i\theta}) \ = \ \int_0^{2\pi} P_{n\,,k}(R,r,\phi-\theta)f(Re^{i\phi})d\phi, \quad r<R.$$

For $f(z)=z^p$, $p\in\mathbb{N}$, we have:

$$\int_0^{2\pi} P_{n\,,k}(R,r,\phi-\theta)s^p d\phi \ = \ \delta_{\dot{p}k}\ z^p,$$

where $s=Re^{i\phi}$, $z=re^{i\theta}$, and δ_{ij} is the Kronecker symbol. Thence :

$$\int_0^{2\pi} P_{n\,,0}(R,r,\phi-\theta)s^{pn} d\phi \ = \ z^{pn}, \qquad \forall p\in\mathbb{N},$$

which reduces, for $n=1$ and $p=0$, to the well-known identity

$$\int_0^{2\pi}\mathcal{K}(\rho,\theta)d\theta \ = \ 1.$$

Proposition 4.3 *Let F be a function in Ω and having the integral representation*

$$F(x) \ = \ \int_{\mathcal{I}} K(x,t)dt, \tag{4.6}$$

where the path $\mathcal{I}$ is independent of the variable x. If the integrals

$$\int_{\mathcal{I}} K(\omega_n^{\ell}x,t)dt \quad \textit{exist for all} \quad \ell\in\mathbb{N}_n,$$

then

$$F_{[n,k]}(x) \ = \ \int_{\mathcal{I}} (x\longrightarrow K(x,t))_{[n,k]}(x,t)dt. \tag{4.7}$$

To prove (4.7) it is sufficient to apply the projection operator $\Pi_{[n,k]}$ to the two members of (4.6).

As an example of application of this result, consider the integral representation (cf. [6, Vol. II, p. 200, Eq. (94)]):

$$ {}_pF_{q+1}\begin{pmatrix} (a_p), & & \\ & & x \\ (b_q), & \alpha+\beta, & \end{pmatrix} = \tag{4.8}$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}{}_pF_{q+1}\begin{pmatrix} (a_p), & & \\ & & xt \\ (b_q), & \alpha, & \end{pmatrix} dt,$$

where (a_p) stands for the set of p parameters $a_1, a_2, \ldots, a_p$. By virtue of (1.9), we have

$$ {}_{np}F_{n(q+1)+n-1}\begin{pmatrix} \Delta[n,a_p+k], & & & \\ & & & x \\ \Delta^*(n,k+1), & \Delta[n,b_q+k], & \Delta(n,\alpha+k+\beta), & \end{pmatrix}$$

$$= \frac{1}{B(\alpha+k,\beta)}\int_0^1 t^{\alpha+k-1}(1-t)^{\beta-1}$$

$$\cdot {}_{np}F_{n(q+1)+n-1}\begin{pmatrix} \Delta[n,a_p+k], & & & \\ & & & xt^n \\ \Delta^*(n,k+1), & \Delta[n,b_q+k], & \Delta(n,\alpha+k), & \end{pmatrix} dt, \tag{4.9}$$

where for the sake of brevity $\Delta[n,a_p]$ stands for the set of np parameters:

$$\bigcup_{i=1}^{p} \Delta(n,a_i) = \{\frac{a_i+j}{n},\ i=1,2,\ldots,p,\ j=0,1,\ldots,n-1\}.$$

Notice that the integral representation (4.9) may be justified by the method, other than the use the classical identity (cf. [15, p. 104, Eq. (5)]):

$$ {}_rF_{s+n}\begin{pmatrix} (a_r), & & \\ & & x \\ (b_s), & \Delta(n,\alpha+\beta), & \end{pmatrix}$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,{}_rF_{s+n}\begin{pmatrix} (a_r), & & \\ & & xt^n \\ (b_s), & \Delta(n,\alpha), & \end{pmatrix} dt.$$

5. *A Parseval Formula*

Proposition 5.1 *Let $(f,g) \in \Omega^2$ and $(x,y) \in I^2$, we have:*

$$\sum_{k=0}^{n-1} f_{[n,k]}(x) \cdot \overline{g_{[n,k]}}(y) \ = \ \frac{1}{n}\sum_{p=0}^{n-1} \ f(\omega_n^p\, x) \cdot \bar{g}(\omega_n^p\, y). \tag{5.1}$$

The special case, corresponding to $f = g$ and $x = y$, amounts to the Parseval formula:

$$\sum_{k=0}^{n-1} \mid f_{[n,k]}(x) \mid^2 \ = \ \frac{1}{n}\sum_{p=0}^{n-1} \ \mid f(\omega_n^p\, x) \mid^2 \ = \ \Pi_{[n,0]} \ (\mid f \mid^2)(x). \tag{5.2}$$

Proof. Let $a = (a_p)_{p\in\mathbb{N}_n}$ and $b = (b_p)_{p\in\mathbb{N}_n}$ be two vectors in $\mathbb{C}^n$, and let $A = (A_k)_{k\in\mathbb{N}_n}$ and $B = (B_k)_{k\in\mathbb{N}_n}$ be their respective images by $\mathcal{F}_n$, the discrete Fourier transform of order n . It is well-known that the image by $\mathcal{F}_n$ of the product $a \cdot b \ = \ (a_p \cdot b_p)_{p\in\mathbb{N}_n}$ is the convolution product

$$C \ = \ (C_m)_{m\in\mathbb{N}_n}, \ \ \text{where} \ \ C_m \ = \sum_{k=0}^{n-1} A_k \cdot \ B_{m-k},$$

with the convention: $B_{-r} = B_{n-r}$ if $r \in \{1, 2, \ldots, n-1\}$. So we have

$$\sum_{k=0}^{n-1} A_k \cdot B_{m-k} \ = \ \frac{1}{n}\sum_{p=0}^{n-1} \ a_p \cdot b_p \ \omega_n^{-pm}. \tag{5.3}$$

If we set

$m = 0$; $\quad a_p \ = \ f(\omega_n^p\, x)$; $\quad b_p \ = \ \bar{g}(\omega_n^p\, y)$; $\quad A_k \ = \ f_{[n,k]}(x)$; and $B_k \ = \ g_{[n,k]}(y)$

in (5.3), we obtain (5.1) by virtue of the identity

$$(\bar{f})_{[n,n-k]} \ = \ \overline{f_{[n,k]}} \quad \text{for all } f \in \Omega,$$

which can be easily justified from the definition (1.5). □

A limiting case of (5.2), corresponding to analytic functions f and g, is stated in the following

Corollary 5.2 *Let f and g be two analytic functions in a neighbourhood of the origin. If $f(x) = \sum_{m=0}^{\infty} a_m x^m$ and $g(x) = \sum_{m=0}^{\infty} b_m x^m$, then*

$$\sum_{k=0}^{\infty} a_k \bar{b}_k x^k \bar{y}^k \ = \ \frac{1}{2\pi}\int_0^{2\pi} f(xe^{i\theta})\bar{g}(ye^{i\theta})d\theta. \tag{5.4}$$

Proof. From the definition (1.5), we deduce

$$f_{[n,k]}(z) \;=\; z^k \left[\frac{1}{n}\sum_{h=0}^{n-1}\frac{1}{(ze^{\frac{2i\pi h}{n}})^k} f(ze^{\frac{2i\pi h}{n}})\right].$$

This, in view of the Cauchy formula, we get

$$\lim_{n\to+\infty} f_{[n,k]}(z) \;=\; z^k\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{f(ze^{i\theta})}{(ze^{i\theta})^k}d\theta \;=\; z^k\cdot\frac{f^{(k)}(0)}{k!} \;=\; a_k z^k, \quad (5.5)$$

which in connection with (5.1) leads to (5.4). □

As an example of application of the Corollary 5.2, let us consider the case

$$f(t) \;=\; g(t) \;=\; {}_pF_q\begin{pmatrix}(a_p), & \\ & t\\ (b_q), & \end{pmatrix}$$

where the a_i and the $b_j$are positive real numbers.

From the definition (1.7) and the identity (5.4), we deduce

$$ {}_{2p}F_{2q+1}\begin{pmatrix}(a_p), & (a_p), & & \\ & & & |t|^2\\ (b_q), & (b_q), & 1, & \end{pmatrix} \;=\; \frac{1}{2\pi}\int_0^{2\pi}\left|{}_pF_q\begin{pmatrix}(a_p), & \\ & te^{i\theta}\\ (b_q), & \end{pmatrix}\right|^2 d\theta. \quad (5.6)$$

We list below three special cases of this identity which may be of interest

Case 1 $f(t) \;=\; \exp(t) \;=\; {}_0F_0\begin{pmatrix}-, & \\ & t\\ -, & \end{pmatrix}, \; t\in\mathbb{R}.$

In this case, the integral representation (5.6) amounts to a familiar formula for Bessel function $I_0(2t)$ (cf. [2, p. 376, Eq. (9.6.16)]):

$$\begin{aligned} I_0(2t) \;&= {}_0F_1\begin{pmatrix}-, & \\ & t^2\\ 1, & \end{pmatrix} = \frac{1}{2\pi}\int_0^{2\pi}\left|e^{te^{i\theta}}\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{t\cos\theta}d\theta\\ &= \frac{1}{\pi}\int_0^{\pi} e^{\pm t\cos\theta}d\theta = \frac{1}{\pi}\int_0^{\pi}\cosh(t\cos\theta)d\theta, \quad t\in\mathbb{R}. \end{aligned}$$

Case 2 $f(t) \;=\; (1-t)^{-a} \;=\; {}_1F_0\begin{pmatrix}a, & \\ & t\\ 1, & \end{pmatrix}, \quad a>0 \textit{ and } |t|<1.$

For any $t \in \mathbb{R}$ we have

$$ {}_2F_1\left(\begin{matrix} a, & a, & \\ & & t^2 \\ & 1, & \end{matrix}\right) = \frac{1}{2\pi}\int_0^{2\pi}\left|1 - te^{i\theta}\right|^{-2a} d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(1+t^2-2t\cos\theta)^a}, $$

which, for $a = \frac{1}{2}$, may be expressed by the complete elliptic integral $K(t)$ defined by (cf. [2, p. 590, Eq. (17.3.1)]):

$$ K(t) \;=\; \int_0^{\frac{\pi}{2}} (1 - t\sin^2\theta)^{-\frac{1}{2}}\, d\theta. $$

That is

$$ {}_2F_1\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \\ & & t^2 \\ & 1, & \end{matrix}\right) \;=\; \frac{2}{\pi}\, K(t^2). $$

Case 3 $f(t) \;=\; {}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, & \frac{\alpha+1}{2}, & \\ & & 2\sqrt{t} \\ & \alpha, & \end{matrix}\right),\; \alpha > 0,\; t \geq 0.$

We have

$$ {}_4F_3\left(\begin{matrix} \frac{\alpha}{2}, & \frac{\alpha}{2}, & \frac{\alpha+1}{2}, & \frac{\alpha+1}{2}, & \\ & & & & 4t \\ & \alpha, & \alpha, & 1, & \end{matrix}\right) = \tag{5.7} $$

$$ = \frac{1}{2\pi}\int_0^{2\pi}\left|{}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, & \frac{\alpha+1}{2}, & \\ & & 2\sqrt{t}\exp(i\theta) \\ & \alpha & \end{matrix}\right)\right|^2 d\theta. $$

Recall here a Burchnall identity (cf. [3, p. 101, Eq. (37)]):

$$ {}_4F_3\left(\begin{matrix} a, & b, & \frac{1}{2}c + \frac{1}{2}c', & \frac{1}{2}c + \frac{1}{2}c' - \frac{1}{2}, & \\ & & & & 4t \\ & c, & c', & c + c' - 1, & \end{matrix}\right) \;=\; F_4(a, b, c, c', t, t), \tag{5.8} $$

where

$$ F_4(a, b, c, c', x, y) = \sum_{m,p=0}^{+\infty}\frac{(a)_{m+p}(b)_{m+p}}{(c)_m(c')_m}\cdot\frac{x^m}{m!}\cdot\frac{y^p}{p!} $$

$$ = \sum_{m=0}^{+\infty}\frac{(a)_m(b)_m}{(c)_m}{}_2F_1\left(\begin{matrix} a+m, & b+m, & \\ & & y \\ & c', & \end{matrix}\right)\cdot\frac{x^m}{m!}, $$

with $\sqrt{|x|} + \sqrt{|y|} < 1$, is an Appell function (cf. [5, Vol. I, p. 224, Eq. (9)]). Combining (5.7) and (5.8), we obtain

$$ F_4\left(\frac{\alpha+1}{2}, \frac{\alpha}{2}, 1, \alpha, t, t\right) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\left|{}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, & \frac{\alpha+1}{2}, & \\ & & 2\sqrt{t}e^{i\theta} \\ & \alpha, & \end{matrix}\right)\right|^2 d\theta. $$

6. *A $n \times n$ circulant matrix*

Let us consider the $n \times n$-matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The powers of this matrix can be easily expressed by I_r the $r \times r$ identity matrix, $r \in \mathbb{N}$, as follows:

$$A^n = I_n, \tag{6.1}$$

and

$$A^\ell = \begin{pmatrix} 0 & \vdots & I_\ell \\ \cdots & + & \cdots \\ I_{n-\ell} & \vdots & o \end{pmatrix}, \quad \ell \in \mathbb{N}_n \setminus \{0\}. \tag{6.2}$$

So the eigenvalues of A are ω_n^k, $k \in \mathbb{N}_n$, and hence the eigenvalues of $f(xA)$, $f \in \Omega$, are $f(x\omega_n^k)$, $k \in \mathbb{N}_n$. Therefore

$$\det\ f(xA) = \prod_{k=0}^{n-1} f(\omega_n^k x). \tag{6.3}$$

Now, in view of (6.1) and (6.2), the $n \times n$ matrix $f(xA)$ can be expressed as the $n \times n$ circulant matrix obtained by the circulation of the vector $\left(f_{[n,k]}(x)\right)_{k \in \mathbb{N}_n}$. That is

$$f(xA) = \begin{pmatrix} f_{[n,0]}(x) & f_{[n,n-1]}(x) & \cdots & f_{[n,1]}(x) \\ f_{[n,1]}(x) & f_{[n,0]}(x) & \cdots & f_{[n,2]}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{[n,n-1]}(x) & f_{[n,n-2]}(x) & \cdots & f_{[n,0]}(x) \end{pmatrix}. \tag{6.4}$$

Remark 1 Many authors dealt with the calculus of $\prod_{k=0}^{n-1} f(\omega_n^k x)$ when f is any hypergeometric function. They express this product by another hypergeometric function. we cite, for example,

- $${}_0F_1\left(\begin{matrix} -, & \\ & x \\ \nu, & \end{matrix}\right) \cdot {}_0F_1\left(\begin{matrix} -, & \\ & -x \\ \nu, & \end{matrix}\right) = {}_0F_3\left(\begin{matrix} -, & \\ & -\frac{x^2}{4} \\ \nu, \frac{1}{2}\nu, \frac{1}{2}(\nu+1), & \end{matrix}\right),$$

(cf.[5,Vol.I, p.186]);

- $${}_1F_1\left(\begin{matrix} a, & \\ & x \\ c, & \end{matrix}\right)\cdot {}_1F_1\left(\begin{matrix} a, & \\ & -x \\ c, & \end{matrix}\right) = {}_2F_3\left(\begin{matrix} a, c-a, & \\ & -\frac{x^2}{4} \\ c, \frac{1}{2}c, \frac{1}{2}(c+1), & \end{matrix}\right),$$

(cf. [12, p. 211]);

- $${}_0F_2\left(\begin{matrix} -, & \\ & x \\ a, b, & \end{matrix}\right)\cdot {}_0F_2\left(\begin{matrix} -, & \\ & -x \\ a, b, & \end{matrix}\right)$$
$$= {}_3F_8\left(\begin{matrix} \frac{1}{3}(a+b-1), \frac{1}{3}(a+b), \frac{1}{3}(a+b+1), & \\ & -\frac{27}{64}x^2 \\ a, b, \frac{1}{2}a, \frac{1}{2}(a+1), \frac{1}{2}b, \frac{1}{2}(b+1), \frac{1}{2}(a+b-1), \frac{1}{2}(a+b), & \end{matrix}\right),$$

(cf.[15, p.106]);

- $${}_0F_1\left(\begin{matrix} -, & \\ & x \\ 6c, & \end{matrix}\right)\cdot {}_0F_1\left(\begin{matrix} -, & \\ & \omega_3 x \\ 6c, & \end{matrix}\right)\cdot {}_0F_1\left(\begin{matrix} -, & \\ & \omega_3^2 x \\ 6c, & \end{matrix}\right)$$
$$= {}_2F_7\left(\begin{matrix} 3c-\frac{1}{4}, 3c+\frac{1}{4}, & \\ & (\frac{4}{9})^3x^3 \\ 6c, 2c, 2c+\frac{1}{3}, 2c+\frac{2}{3}, 4c-\frac{1}{3}, 4c, 4c+\frac{1}{3}, & \end{matrix}\right),$$

(cf.[10, p.1513]).
From (6.3), we see that it is possible to express this product as the nth-order circulant determinant.

Remark 2 The formula (6.4) can be used to derive some identities, satisfied by the components of f. For instance:

1. From the elementary identity $f(xA)\cdot g(xA) = (f\cdot g)(xA)$, we derive (2.1).
2. If the function f satisfies the functional relation $f(x)\,f(y) = f(x+y)$, then

$$\Pi_{[n,k]}(f)(x+y) = \sum_{r+r'\equiv k(n)} \Pi_{[n,r]}(f)(x)\cdot\Pi_{[n,r']}(g)(y). \tag{6.5}$$

7. *An illustration: The Generalized Hyperbolic Functions*

In this section, we illustrate the above process and results by treating the function $f(z) = \exp(z)$. Denote by $h_{n,k}$ the components with respect to the cyclic group of order n of the function f. According to (1.5), we have

$$h_{n,k}(z) \;=\; \frac{1}{n}\sum_{\ell=0}^{n-1}\exp\left(-\frac{2i\pi k\ell}{n}\right)\exp\left(z\exp\left(\frac{2i\pi\ell}{n}\right)\right), \qquad (7.1)$$

which will be referred to as the hyperbolic function of order n and k-th kind. The two hyperbolic functions of order $n=2$ are thus $h_{2,0}(z)=\cosh z$ and $h_{2,1}(z)=\sinh z$.
Many authors dealt with the study of these functions. A large bibliography may be consulted in [13]. We gather below some properties pertaining to the functions given by (7.1). Thereafter, we prove them by use of some results established in this paper. Recall first,

P1. $$h_{n,k}(\omega_n^m z) \;=\; \omega_n^{km}\, h_{n,k}(z), \qquad \text{(cf. [7] p.214 Eq.(11));}$$

P2. $$exp(\omega_n^m z) \;=\; \sum_{k=0}^{n-1}\omega_n^{km}\, h_{n,k}(z), \qquad \text{(cf. [7] p.214 Eq.(10));}$$

P3. $$h_{n,k}(z) \;=\; \sum_{m=0}^{\infty}\frac{z^{nm+k}}{(nm+k)!}, \qquad \text{(cf. [7] p.213 Eq.(8));}$$

P4. $$\frac{d^r h_{n,k}}{dz^r} \;=\; h_{n,\overbrace{k-r}^{\cdot}}, \qquad \text{(cf. [7] p.214 Eq.(12));}$$

P5. $$h_{n,k}(z) \;=\; \frac{1}{2i\pi}\int_{\mathcal{C}}\frac{t^{n-k-1}}{t^n-1}\exp(zt)\,dt,$$

where C is a simple closed curve encircling the unit circle once in the positive sense, (cf. [7] p.213 Eq.(9));

P6. $$\int_0^{+\infty} e^{-st}h_{n,k}(t)dt \;=\; \frac{s^{n-k-1}}{s^n-1},\ \Re s>1, \qquad \text{(cf. [7] p.214 Eq.(15));}$$

P7. $$h_{n,k}(x+y) \;=\; \sum_{j=0}^{n-1} h_{n,j}(x)h_{n,k\dot{-}j}(y), \qquad \text{(cf. [7] p.214 Eq.(13));}$$

P8. $$\frac{1}{n}\sum_{r=0}^{n-1} h_{n,0}(x+\omega_n^r y) \;=\; h_{n,0}(x)\,h_{n,0}(y), \qquad \text{(cf. [16] p.48 Eq.(4.6));}$$

P9. $\displaystyle\sum_{k=0}^{n-1} |h_{n,k}(x)|^2 \;=\; \frac{1}{n} \sum_{s=0}^{n-1} exp[2x \cos \frac{2\pi s}{n}],$ (cf. [8] p.293 Eq.(3.5));

Let F(z) the $n \times n$ circulant matrix

$$F(z) \;=\; \begin{pmatrix} h_{n,0}(z) & h_{n,n-1}(z) & \cdots & h_{n,1}(z) \\ h_{n,1}(z) & h_{n,0}(z) & \cdots & h_{n,2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n,n-1}(z) & h_{n,n-1}(z) & \cdots & h_{n,0}(z) \end{pmatrix},$$

we have

P10. $\det\; F(z) \;=\; 1,$ (cf. [7] p.214 Eq.(14));

P11. $F(z) \;\cdot\; F(z') \;=\; F(z+z'),$ (cf. [22] p.689 Eq.(8)).

Proof.

. To prove **P1**; **P2**; and **P3**, it is sufficient to use, respectively, the identities (1.1); (1.4); and (1.6).

. The derivative operator $\dfrac{d^r}{dz^r}$ is homogeneous of degree $\overbrace{n-r}^{\cdot}$. So, according to Theorem 3.1, we have

$$\Pi_{[n,\overbrace{k-r}^{\cdot}]} \;\circ\; \frac{d^r}{dz^r} \;=\; \frac{d^r}{dz^r} \;\circ\; \Pi_{[n,k]}.$$

Then

$$\frac{d^r}{dz^r}\,(h_{n,k}) = \frac{d^r}{dz^r}\circ\Pi_{[n,k]}(f) = \Pi_{[n,\overbrace{k-r}^{\cdot}]} \circ \frac{d^r}{dz^r}(f) = \Pi_{[n,\overbrace{k-r}^{\cdot}]}(f) = h_{[n,\overbrace{k-r}^{\cdot}]},$$

and **P4** holds.

. If we put $f(z) = e^z$ in the integral representation (4.1), we obtain **P5**.

. The integral representation

$$F(z) \;=\; \frac{1}{s-z} \;=\; \int_0^{+\infty} e^{-st}\, e^{zt} dt, \quad \Re s >| z |,$$

satisfies the conditions of the proposition (4.3). So, using (4.7) and (4.4), we obtain

$$\frac{s^{n-k-1} z^k}{s^n - z^n} \;=\; \int_0^{+\infty} e^{-st} h_{n,k}(zt) dt,$$

which, for $z = 1$, comes to **P6**.

. The properties **P7**; **P8**; **P9**; **P10**; and **P11** may be deduced, respectively, from (2.9); (3.2); (5.2); (4.3); and **P7**. □

Next, we establish other properties which are believed to be new for the functions $h_{n,k}$

1. A curve, in $\mathbb{R}^n$, associated with the function $f(z) = \exp(z)$.

Let γ_n be the mapping:

$$\begin{array}{rcccl} \gamma_n : & \mathbb{R} & \longrightarrow & \mathbb{R}^n & \\ & x & \longrightarrow & \gamma_n(x) & = \Bigg(h_{n,k}(x) \Bigg)_{k \in \mathbb{N}_n} . \end{array}$$

Denote the range of γ_n by Γ_n. Γ_2 is the hyperbola of equation: $x_0^2 - x_1^2 = 1$. From **P10**, we deduce that Γ_n lies on the hypersurface of equation:

$$\begin{vmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \ddots & x_2 \\ \vdots & \ddots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{vmatrix} = 1.$$

Now, define the operation $\diamond$ in $\mathbb{R}^n$ as follow:

Let $(X, Y, Z) \in (\mathbb{R}^n)^3$,with

$$X = (x_i)_{i \in \mathbb{N}_n}; \qquad Y = (y_i)_{i \in \mathbb{N}_n}; \quad \text{and} \quad Z = (z_i)_{i \in \mathbb{N}_n}.$$

We say $Z = X \diamond Y$ if and only if

$$z_r = \sum_{\ell + k \equiv r(n)} x_\ell \, y_k.$$

From **P7**, we deduce the relation $\gamma_n(x) \diamond \gamma_n(y) = \gamma_n(x+y)$. This implies that $(\Gamma_n, \diamond)$ and $(\mathbb{R}, +)$ are two isomorphic groups since γ_n is a one-to-one mapping from $\mathbb{R}$ to Γ_n. So Γ_n is a one dimensional manifold equipped with a group structure.

2. A series of generalized hyperbolic functions.

Let us consider the function $z \longrightarrow \exp(\exp(z))$ which may be expressed by

$$\sum_{m=0}^{\infty} \frac{e^{mz}}{m!} = exp\left(h_{n,0}(z)\right) \prod_{k=1}^{n-1} exp\left(h_{n,k}(z)\right). \tag{7.2}$$

The action of the projection operators $\Pi_{[n,k]}$ on both sides of this identity and use of the rule (2.4) give rise to the relation

$$\sum_{m=0}^{\infty} \frac{h_{n,k}(mz)}{m!} = exp\left(h_{n,0}(z)\right) \cdot \sum_{i_1 + 2i_2 + \cdots + (n-1)i_{n-1} \equiv k(n)} \prod_{j=1}^{n-1} h_{n,i_j}\left(h_{n,j}(z)\right). \tag{7.3}$$

For $n=2$, this identity specializes to (cf. [9, p. 42, Eq. (1.471)]):

$$\sum_{m=0}^{\infty} \frac{\cosh(mz)}{m!} = \exp\left(\cosh(z)\right) \cdot \cosh\left(\sinh(z)\right);$$

$$\sum_{m=1}^{\infty} \frac{\sinh(mz)}{m!} = \exp\left(\cosh(z)\right) \cdot \sinh\left(\sinh(z)\right).$$

We conclude by remarking that the above method, used in this section, of derivation of the properties of the functions $h_{n,k}$ would apply mutatis mutandis to the function

$$z \longrightarrow \exp\left(\alpha\, z\right), \qquad \alpha \in \mathbb{C},$$

whose components are

$$\alpha^k\, F^{\alpha^n}_{n,k},\ k = 0, 1, 2, \ldots, n-1,$$

where the functions $F^{\alpha}_{n,k}$ are defined by

$$F^{\alpha}_{n,k}(z) \ = \ \sum_{m=0}^{\infty} \frac{\alpha^m\, z^{nm+k}}{(nm+k)!},$$

and called the α-hyperbolic functions of order n and k-th kind studied by Ungar (cf. [23]). These functions can be expressed by the generalized hypergeometric function ${}_p\Psi_q$ defined by the series (cf.[5, p. 183]):

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p), \\ \\ (\beta_1, B_1), \ldots, (\beta_q, B_q), \end{matrix} \quad z \right] = \sum_{m=0}^{+\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + A_i m)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j m)} \cdot \frac{z^m}{m!}, \quad (7.4)$$

where the parameters

$$\alpha_i\,,\ i = 1, \ldots, p, \quad \text{and} \quad \beta_j\,,\ j = 1, \ldots, q,$$

are complex numbers, and the associated coefficients

$$A_i\,,\ i = 1, \ldots, p \quad \text{and} \quad B_j\,,\ j = 1, \ldots, q,$$

are positive real numbers such that

$$\Lambda := 1 + \sum_{j=1}^{q} B_j \ - \sum_{i=1}^{p} A_i \ > \ 0,$$

provided that no zeros appear in the denominator of (7.4).

In fact, we have (cf. [18, p. 213, Eq. (17)])

$$F_{n,k}^{\alpha}(z) \;=\; z^k \, {}_1\Psi_1 \left[\begin{array}{cc} (1,1), & \\ & \alpha z^n \\ (k+1,n), & \end{array} \right].$$

For $\alpha = 1$, we have the functions (7.1).
For $\alpha = -1$, we have the generalized trigonometric functions of order n and the k-th kind (cf. [7, p. 215, Eq. (18)]). In this case, both the identities (5.2) and (6.3) are reduced, for $n = 2$ and $z \in \mathbb{R}$, to the famous formula

$$\cos^2 z + \sin^2 z \;=\; 1.$$

References

1. Abdul-Halim N., Al-Salem W.A. *Double Euler transformations of certain hypergeometric functions.* Duke Math. J., 30 (1963), 51-62.
2. Abramowitz M., Stegun I.A. *"Handbook of Mathematical Functions".* National Bureau of Standards, Washington, D.C., 1964.
3. Burchnall J.L. *Differential equations associated with hypergeometric functions.* Quart. J. Math. Oxford Ser., 13 (1942), 90-106.
4. Courant R., Hilbert D. *"Methods of mathematical physics".* Vol.II. Partial différential Equations. Interscience Publishers. Wiley & sons. New York, 1961.
5. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. *"Higher transcendal functions".* Vols. I and II. Mc Graw-Hill New York, 1953.
6. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. *Tables of Integrals Transforms.* Vols. I and II. Mc Graw-Hill New York, 1954.
7. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. *Higher transcendal functions.* Vol. III . Mc Graw-Hill New York, 1955.
8. Good I.J. *A simple generalization of analytic function theory.* Exp. Math., 6 (1988), 289-311.
9. Gradshteyn I.S., Ryzhik I.M. *"Table of Integrals, Series and products".* Academic Press, New York, 1980.
10. Henrici P. *A triple product theorem for hypergeometric series.* SIAM. J. Math. Anal., 18 (1987), 1513-1518.
11. Jain R.N. *Some double integral transformations of certain hypergeometric functions.* Math. Japan., 10 (1965), 17-26.
12. Luke Y.L. *"The Special Functions and their Approximations".* Vol.I. Academic Press, New York, 1969.

13. Muldoon M.E., Ungar A.A. *Generalizing the circular and hyperbolic functions.* Math. Mag., 69 (1996), 3-14

14. Osler T.J., *An Identity for Simplifying Certain Generalized Hypergeometric Functions.* Math. Comput., 29 (1975), 888-893.

15. Rainville E.D. *"Special Functions".* Macmillan compagny, New York, 1960.

16. Ricci P.E. *Le funzioni pseudo-iperboliche e pseudo-trigonometriche.* Publ. Istit. Mat. Appl. Fac. Ingr. Univ. Stud. Roma, 12 (1978), 27-49.

17. Srivastava H.M. *A note on certain identities involving generalized hypergeometric series.* Nedel. Akad. Wetensch. Proc. Ser. A 82 = Indag. Math., 41 (1979), 191-201.

18. Srivastava H.M. *A certain family of sub-exponential series.* Int. J. Math. Educ. Sci. Technol., 25 (1994), 211-216.

19. Srivastava H.M., Joshi C.M. *Certain double Whittaker transforms of generalized hypergeometric functions.* Yokohama Math. J., 15 (1967), 17-32.

20. Srivastava H.M., Panda R. *Some operational techniques in the theory of special functions.* Nedel. Akad. Wetensch. Proc. Ser. A 76 = Indag. Math., 35 (1973), 308-319.

21. Srivastava H.M., Singhal J.P. *Double Meijer transformations of certain hypergeometric functions.* Proc. Cambridge Philos. Soc., 64 (1968), 425-430.

22. Ungar A.A. *Generalized hyperbolic functions.* Amer. Math. Monthly, 89 (1982), 688-691.

23. Ungar A.A. *Higher Order alpha hyperbolic functions.* Indian J. Pure Appl. Math., 15 (1984), 301-304.