THE NECESSARY CONDITION OF THE OPTIMALITY FOR AN OPTIMAL CONTROL PROBLEM FOR HELMHOLTZ EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS AND A NONLINEAR FUNCTIONAL

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Abstract

In the paper, the necessary condition of the optimality for an optimal control problem for Helmholtz equation with non-local boundary conditions has been obtained.

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1°Most processes encountered in practice are controlled and consequently it's important to obtain the optimal variant for their realization in the certain sense.

In the paper the optimal control problem for Helmholtz equation with non-local boundary conditions and nonlinear functional is considered. The first basic paper about the problem with nonlocal boundary conditions was published by Bitsadze and Samarski in 1969 [2]. Then this problem was generalized and investigated by Gordeziani, Skubachevskii,Sapagovas, Paneyakh, [4],[5],[19],[18],[14]. Many interesting works are already devoted to these issues [6],[7],[8],[13].

Following the Plotnikov scheme [15],[16], the necessary conditions of the optimality for the optimal control problem for the system of the first order quasi-linear differential equations with the Bitsadze-Samarski boundary conditions and the necessary and sufficient conditions of the optimality for the optimal control problem for Helmholtz equation with the Bitsadze-Samarski boundary conditions with a quadratic functional have been obtained in [11],[3].

2° Statement of the problem. Let $\overline{D}$ be a rectangle $\overline{D}=[0,l_1]\times[0,l_2]$, Γ the boundary of the rectangular domain, $\gamma=\{(l_1,y):0\le y\le l_2\}$ and

$\gamma_i = \{(\xi_i, y) : 0 \leq y \leq l_2\}$, $i = \overline{1,n},$, ξ_i - the fixed points of interval $]0, l_1[$, V some convex subset in $\mathbf{R}$ and U_{ad} the set of control functions $v : D \to V$, $v\,(x, y) \in L_2(D)$, v is a control parameter from V , the set V is the control domain.

Let $L_2(D, \omega)$ be a weighted space with the real functions defined on D with the norm:

$$\|u\|_{L_{2(D,\omega)}} = \left(\int_D \omega\,(x)\,|u\,(x, y)|^2\, dxdy \right)^{1/2},$$

where the weight function $\omega\,(x) = 1 - x$.

The weighted Sobolev space $W_2^k\,(D, \omega)$ is a linear space of the functions $u\,(x)$ given on D, whose derivatives (in a general sense) $D^s u$ of the order $|s| \leq k$ belong to the space $L_2(D, \omega)$. Let us introduce the norm $\|u\|_{W_2^k(D,\omega)} = \left(\sum\limits_{i=0}^{k} |u|^2_{W_2^i(D,\omega)} \right)^{1/2}$, where $|u|^2_{W_2^i(D,\omega)} = \sum\limits_{|s|=i} \|D^s u\|^2_{L_{2(D,\omega)}}$; $W_2^0\,(D, \omega) = L_2\,(D, \omega)$ [10].

Let's define the subspace of the space $W_2^1\,(D, \omega)$, which is obtained by closing the set

$$\overset{*}{C}{}^{\infty}\left(\overline{D}\right) = \left\{ \begin{array}{l} u \in C^{\infty}\left(\overline{D}\right) : supp\, u \cap (\Gamma \backslash \gamma) = \emptyset, \\ \\ u\,(l_1, y) = \sum\limits_{i=1}^{n} \sigma_i u\,(\xi_i, y)\,, \quad 0 < y < l_2 \end{array} \right\}$$

in the norm of the space $W_2^1\,(D, \omega)$ and let's denote it by $\overset{*}{W}{}_2^1\,(D, \omega)$. Let us $W_{2,*}^2\,(D, \omega) = \overset{*}{W}{}_2^1\,(D, \omega) \cap W_2^2\,(D, \omega)$ [1].

Let us consider the non-local boundary problem for Helmholtz equation [18] for each fixed $v(x, y) \in U_{ad}$ in the domain D [2]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - qu = a(x, y)v(x, y) + b(x, y), \qquad (x, y) \in D, \tag{1}$$

$$u(x, y) = 0, \qquad (x, y) \in \Gamma \setminus \gamma,$$

$$u(l_1, y) = \sum_{i=1}^{n} \sigma_i u(\xi_i, y), \qquad 0 \leq y \leq l_{2,} \tag{2}$$

where $a \in L_\infty(D), b \in L_2(D), \quad 0 \leq q, \quad 0 < \sum\limits_{i=1}^{n} \sigma_i < 1, \sigma_i = const > 0.$

The solution of the problem ( ) exists and is unique and belongs to Sobolev space $W^2_{2,*}(D,\omega)$ [1].

Let us consider the following functional:

$$I(v) = \int\int_D F(x,y,u,v)dxdy. \tag{3}$$

Let the real function $F(x,y,u,v)$ be continuous and continuously differentiable with respect to u, v, and belong to the space $L_2(D)$, $(x,y) \in D$.

Let us formulate the following control problem: find the function v_0, whose corresponding solution u_0 of the boundary value problem (1),(2) together with results in minimal value functional (3). Hence, the obtained pair (u_0, v_0) is called optimal [17].

We obtain the necessary conditions of the optimality using the scheme developed in the work [15],[16].

Let $v_0 \in U_{ad}$ be an optimal control, and $v_\varepsilon \in U_{ad}$ an arbitrary permissible control, $v_\varepsilon = v_0 + \varepsilon\delta v$, where $\delta v \in U_{ad}$, $\varepsilon > 0$ for $\varepsilon \le \varepsilon_0$. Let u_0, u_ε be the corresponding solutions of the problem (1)-(2). Let us introduce the notation:

$$\varepsilon\delta v = v_\varepsilon - v_0, \quad \varepsilon\delta u = u_\varepsilon - u_0.$$

Then we obtain the following problem:

$$\left[\frac{\partial^2\delta u}{\partial x^2} + \frac{\partial^2\delta u}{\partial y^2} - q\delta u\right]\varepsilon = a(x,y)\varepsilon\delta v, \qquad (x,y) \in D, \tag{4}$$

$$\varepsilon\delta u(x,y) = 0, \qquad (x,y) \in \Gamma \setminus \gamma,$$

$$\varepsilon\delta u(l_1,y) = \sum_{i=1}^{n} \sigma_i \varepsilon\delta u(\xi_i,y), \qquad 0 \le y \le l_2.$$

Let $\psi \neq 0, \psi \in W^2_2(D \setminus \bigcup\limits_{i=1}^{n} \gamma_i) \cap W^1_2(D)$. Multiplying equation (3) by ψ and integrating over domain D, the following equality is obtained:

$$\int\int_D \psi(x,y)\left[\frac{\partial^2\delta u}{\partial x^2} + \frac{\partial^2\delta u}{\partial y^2} - q\delta u\right]\varepsilon dxdy = \int\int_D a(x,y)\psi(x,y)\varepsilon\delta v dxdy. \tag{5}$$

The increment of the functional (3) with fixed v_0, v_ε is:

$$\delta_\varepsilon I \;=\; I(v_\varepsilon) - I(v_0) = \int\int_D \left[F(x,y,u_\varepsilon,v_\varepsilon) - F(x,y,u_0,v_0)\right]dxdy$$

$$= \iint_D \varepsilon \left[\frac{\partial F}{\partial u}(\eta)\delta u + \frac{\partial F}{\partial v}(\eta)\delta v \right] dxdy, \tag{6}$$

where $\eta = (x, y, u_0 + \theta\varepsilon\delta u, v_0 + \theta\varepsilon\delta v), \quad 0 \leq \theta < 1.$

From the relations (5),(6) we obtain the following expression for the increment:

$$\delta_\varepsilon I = \iint_D \psi(x,y) \left[\frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial^2 \delta u}{\partial y^2} - q\delta u \right] \varepsilon dxdy$$

$$- \iint_D a(x,y)\psi(x,y)\varepsilon\delta v dxdy + \iint_D \varepsilon \left[\frac{\partial F}{\partial u}(\eta)\delta u + \frac{\partial F}{\partial v}(\eta)\delta v \right] dxdy$$

$$= \int_0^{l_2} \Big[\psi(l_1,y)\delta u_x(l_1,y) - \psi(0,y)\delta u_x(0,y) + \big(\psi(\xi_1^-,y) - \psi(\xi_1^+,y)\big)\delta u_x(\xi_1,y)$$

$$+ \big(\psi_x(\xi_1^+,y) - \psi_x(\xi_1^-,y) - \sigma_1\psi_x(l_1,y)\big)\delta u(\xi_1,y) + \int_0^{\xi_1} \frac{\partial^2\psi}{\partial x^2}\delta u(x,y)dx$$

$$+ \big(\psi_x(\xi_2^+,y) - \psi_x(\xi_2^-,y) - \sigma_2\psi_x(l_1,y)\big)\delta u(\xi_2,y)$$

$$+ \big(\psi(\xi_2^-,y) - \psi(\xi_2^+,y)\big)\delta u_x(\xi_2,y) + \int_{\xi_1}^{\xi_2} \frac{\partial^2\psi}{\partial x^2}\delta u(x,y)dx + ...$$

$$+ \big(\psi_x(\xi_n^+,y) - \psi_x(\xi_n^-,y) - \sigma_n\psi_x(l_1,y)\big)\delta u(\xi_n,y)$$

$$+ \big(\psi(\xi_n^-,y) - \psi(\xi_n^+,y)\big)\delta u_x(\xi_n,y)$$

$$+ \int_{\xi_n}^{l_1} \frac{\partial^2\psi}{\partial x^2}\delta u(x,y)dx \Big] \varepsilon dy + \int_0^{l_1} [\psi(x,l_2)\delta u_y(x,l_2) - \psi(x,0)\delta u_y(x,0)$$

$$-\psi_y(x,l_2)\delta u(x,l_2) + \psi_y(x,0)\delta u(x,0) + \int_0^{l_2} \frac{\partial^2\psi}{\partial y^2}\delta u(x,y)dy]\varepsilon dx$$

$$+ \iint_D \left[\frac{\partial F}{\partial u}(\eta) - q\psi(x,y) \right] \varepsilon\delta u dxdy + \iint_D \left[\frac{\partial F}{\partial v}(\eta) - a(x,y)\psi(x,y) \right] \varepsilon\delta v dxdy$$

Let us consider the first variation [15]:

$$\delta I = \lim_{\varepsilon\to 0}\frac{\delta_\varepsilon I}{\varepsilon}$$

$$= \int_0^{l_2} [\psi(l_1,y)\delta u_x(l_1,y) - \psi(0,y)\delta u_x(0,y) + \left(\psi(\xi_1^-,y) - \psi(\xi_1^+,y)\right)\delta u_x(\xi_1,y)$$

$$+ \left(\psi_x(\xi_1^+,y) - \psi_x(\xi_1^-,y) - \sigma_1\psi_x(l_1,y)\right)\delta u(\xi_1,y)$$

$$+ \int_0^{\xi_1} \frac{\partial^2\psi}{\partial x^2}\delta u(x,y)dx + \left(\psi_x(\xi_2^+,y) - \psi_x(\xi_2^-,y) - \sigma_2\psi_x(l_2,y)\right)\delta u(\xi_2,y)$$

$$+ \left(\psi(\xi_2^-,y) - \psi(\xi_2^+,y)\right)\delta u_x(\xi_2,y) + \int_{\xi_1}^{\xi_2} \frac{\partial^2\psi}{\partial x^2}\delta u(x,y)dx + ...$$

$$+ \left(\psi_x(\xi_n^+,y) - \psi_x(\xi_n^-,y) - \sigma_n\psi_x(l_1,y)\right)\delta u(\xi_n,y)$$

$$+ \left(\psi(\xi_n^-,y) - \psi(\xi_n^+,y)\right)\delta u_x(\xi_n,y) + \int_{\xi_n}^{l_1} \frac{\partial^2\psi}{\partial x^2}\delta u(x,y)dx]dy$$

$$+ \int_0^{l_1} [\psi(x,l_2)\delta u_y(x,l_2) - \psi(x,0)\delta u_y(x,0) - \psi_y(x,l_2)\delta u(x,l_2)$$

$$+\psi_y(x,0)\delta u(x,0) + \int_0^{l_2} \frac{\partial^2\psi}{\partial y^2}\delta u(x,y)dy]dx$$

$$+ \int\int_D \left[\frac{\partial F}{\partial u}(x,y,u_0,v_0) - q\psi(x,y)\right]\delta u dx dy$$

$$+ \int\int_D \left[\frac{\partial F}{\partial v}(x,y,u_0,v_0) - a(x,y)\psi(x,y)\right]\delta v dx dy. \qquad (7)$$

It is important to note, that for $\varepsilon \to 0$ we get [15]:

$$\left\| \frac{\partial F}{\partial u}(\eta) - \frac{\partial F}{\partial u}(x, y, u_0, v_0) \right\|_{L_2(D)} \to 0,$$

$$\left\| \frac{\partial F}{\partial v}(\eta) - \frac{\partial F}{\partial v}(x, y, u_0, v_0) \right\|_{L_2(D)} \to 0,$$

since the function F is continuously differentiable with respect to u, v. Then for $\varepsilon \to 0$, taking into account the embedding theorem [20] and proceeding from (7), we conclude that if ψ_0 is the solution of the following problem:

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial y^2} - q\psi_0 = -\frac{\partial F}{\partial u}(x, y, u_0, v_0), \qquad (x, y) \in D \setminus \sum_{i=1}^{n} \gamma_i, \tag{8}$$

$$\psi_0(x, y) = 0, \qquad (x, y) \in \Gamma \setminus \gamma,$$

$$\psi_{0x}(\xi_i^+, y) - \psi_{0x}(\xi_i^-, y) = \sigma_i \psi_{0x}(l_1, y), 0 \le y \le l_2, \qquad i = \overline{1, n},$$

then the first variation δI will take the form:

$$\delta I = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon I}{\varepsilon} = \int\int_D \left[\frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0(x, y) \right] \delta v dx dy.$$

Let us introduce the notation:

$$F(0) = F(x, y, u_0, v_0), \quad \delta v = v\,(x, y) - v_0\,(x, y)\,, \tag{9}$$

where $v\,(x, y)$ is an arbitrary permissible control.

Taking into account (9), in a similar way, if ψ_0 is the solution of the problem (8), then the first variation will take the form:

$$\delta I = \int\int_D \left[\frac{\partial F}{\partial v}(0) - a(x, y)\psi_0(x, y) \right] (v\,(x, y) - v_0\,(x, y)) dx dy, \tag{10}$$

$$\forall v\,(x, y) \in U_{ad}.$$

From the optimality of the pair (u_0, v_0) we have:

$$\int\int_D \left[\frac{\partial F}{\partial v}(0) - a(x, y)\psi_0(x, y) \right] (v\,(x, y) - v_0\,(x, y)) dx dy \ge 0, \tag{11}$$

$$\forall v\,(x,y) \in U_{ad}.$$

Let us introduce the notation: $T = \frac{\partial F}{\partial v}(0) - a(x,y)\psi_0(x,y) : D_0 = \{(x_0, y_0) \in D$ $is\, a\, Lebesgue's\, point\, [9]\, for\, T\, and\, Tv_0\, functions\}$.

Then $mes\, D_0 = mes\, D$.

Let

$$D_\delta = \{(x,y) \in D, |x - x_0| < \delta, |y - y_0| < \delta\},$$

where (x_0, y_0) is an arbitrary fixed point from D_0. Let us consider the admissible control:

$$v_\delta(x,y) = \left\{ \begin{array}{cc} v_0(x,y), & (x,y) \in D \setminus D_\delta, \\ v, & (x,y) \in D_\delta \end{array} \right\},$$

where v is an arbitrary fixed point from V.

Let us show the equivalence of (11) and the following relation:

$$\inf_{v\in V} \left[\left(\frac{\partial F}{\partial v}(0) - a(x,y)\psi_0\,(x,y) \right) v \right.$$
$$\left. - \left(\frac{\partial F}{\partial v}(0) - a(x,y)\psi_0\,(x,y) \right) v_0\,(x,y) \right] \geq 0. \tag{12}$$

From (11) follows:

$$\int\!\!\int_D T(x,y)v_\delta(x,y)dxdy \geq \int\!\!\int_D T(x,y)v_0(x,y)dxdy.$$

Since $D = D_\delta \cup (D \setminus D_\delta)$, then

$$\int\!\!\int_{D_\delta} T(x,y)vdxdy + \int\!\!\int_{D\setminus D_\delta} T(x,y)v_0(x,y)dxdy \geq$$

$$\geq \int\!\!\int_{D_\delta} T(x,y)v_0(x,y)dxdy + \int\!\!\int_{D\setminus D_\delta} T(x,y)v_0(x,y)dxdy.$$

Hence

$$\int\!\!\int_{D_\delta} T(x,y)vdxdy \geq \int\!\!\int_{D_\delta} T(x,y)v_0(x,y)dxdy, \quad mesD_\delta \leq 4\delta^2.$$

Then, we obtain:

$$\lim_{\delta\to 0}\frac{1}{4\delta^2}\iint\limits_{D_\delta} T(x,y)v dxdy \geq \lim_{\delta\to 0}\iint\limits_{D_\delta} T(x,y)v_0(x,y)dxdy.$$

According to Lebesgue's theorem [12]:

$$T(x_0,y_0)v \geq T(x_0,y_0)v_0(x_0,y_0).$$

Since (x_0,y_0) is an arbitrary from D_0 and v from V, then

$$\inf_{v\in V} T(x_0,y_0)v \geq T(x_0,y_0)v_0(x_0,y_0) \quad foranarbitrary \quad (x_0,y_0)\in D_0,$$

i.e., (11) is valid. Since $mes\, D_0 = mes\, D$, then $\inf\limits_{v\in V} T(x,y)v = T(x,y)v_0(x,y)$ almost everywhere on D.

Let us prove the contrary. From (12) it follows:

$T(x,y)v(x,y) \geq T(x,y)v_0(x,y)$ for any $v(\cdot)\in U_{ad}$ almost everywhere on D, as $T(x,y)v(x,y) \geq \inf\limits_{v\in V} T(x,y)v = T(x,y)v_0(x,y)$ for an arbitrary $v(\cdot)\in U_{ad}$. Hence

$$\iint\limits_{D} T(x,y)v(x,y)dxdy \geq \iint\limits_{D} T(x,y)v_0(x,y)dxdy, \quad \forall v\,(x,y)\in U_{ad}.$$

Thus the equivalence of (11),(12) has been proven. Hence, there follows the necessity of condition (12) for the optimality of pair (u_0,v_0).

Theorem. *Let the cost functional I be defined by formula (3) and let ψ_0 be a solution of the adjoint problem (8), then for (u_0,v_0) to be optimal it is necessary that the following relation be true:*

$$\inf_{v\in V}\left[\frac{\partial F}{\partial v}(x,y,u_0,v_0) - a(x,y)\psi_0\,(x,y)\right]v =$$

$$\left[\frac{\partial F}{\partial v}(x,y,u_0,v_0) - a(x,y)\psi_0\,(x,y)\right]v_0\,(x,y)$$

almost everywhere on D.

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