MATHEMATICAL MODEL OF THE ATMOSPHERE POLLUTION WITH NON-CLASSIC BOUNDARY CONDITIONS

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(Received: September15, 1999)

Abstract

A regional mathematical model of transporting and dispersion of the atmosphere admisture under non-local boundary conditions is discussed in this article. The new three-dimensional mathematical model with non-local boundary conditions is given. In case of two-dimensional model the existence and uniqueness of the regular solution of the problem is proved.

We investigate a regional zonally averaged mathematical model of the Georgian transport corridor pollution. In the mathematical model the influence of orography is taken into account. The mathematical model is based on the solution of primitive equations under non-local boundary conditions.

Key words and phrases: Atmosphere pollution, non-local boundary condition, Georgian transport corridor, orography.

AMS subject classification: Primary 35Q80; secondary 86A30

As is known the European Union (EU) is one of the main ideologists and sponsors of the transport corridor Europe-Caucasus-Asia (TRACECA). EU considers TRACECA as a mechanism of the inter-state and inter-regional collaboration and the guarantee of peace and stability. EU considers Georgia as a partner in the development of the transport networks between the Black Sea and Central Asia because of its geopolitical position. According to experience of European transit countries besides great political and economical benefits the transit of strategic materials causes great losses to the ecological situation in these countries.

Environment protection is one of the most urgent issues of today. The highest speed of production development has caused the environment pollution and ecological disbalance. Thoughtless increase of industry and energy sector leads to such irreversible processes that threaten the life existence on the earth.

So the diagnosis, analysis of adverse substances and prognosis of their space-time distribution is one of the main problems of modern science. And numerical experiment, mathematical and computer simulation is an efficient method for analysis, diagnosis and prognosis of the factors causing ecological balance changes.

We consider the problem of adverse substances transfer through the atmosphere using a new mathematical model.

As is known, a substance transfer through the atmosphere can be described by the following equation [1,3]:

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}+w\frac{\partial S}{\partial z}=K_x\frac{\partial^2 S}{\partial x^2}+K_y\frac{\partial^2 S}{\partial y^2}+\frac{\partial}{\partial z}K_z\frac{\partial S}{\partial z}-\alpha S+F, \quad (1)$$

where S is concentration, u, v, w are the axial components of wind velocity along axis Ox, Oy and Oz; and K_x, K_y and K_z are the coefficients of turbulent diffusion; α is the coefficient that determines the velocity of substance concentration changes during the process of substance decomposition and transformation; $F(x,y,z,t)$ are internal sources.

Let the axis Ox be directed along the earth parallel, the axis Oy be directed along the meridian, and the axis Oz be directed along the earth radius vertically upward. Let $S(x,y,z,t)$ be intensity of aerosol substance that migrates through the atmosphere along with air streams at the velocity of $\vec{v}\ (x,y,z,t)\simeq u\,\vec{i}+v\,\vec{j}+w\,\vec{k}$ and we are looking for its distribution in the cylindrical area G with surface Γ. Let us denote its lateral surface by $\sum$, let the bottom be denoted by $\sum_0$ (when $z=0$) and the top surface by $\sum_H(z=H, H=const)$.

Suppose $\sum_1$ is a surface inside G, $\sum\cap\sum_1=0$ and the distance $\rho(\sum_{1,}\sum)$ between $\sum_1$ and $\sum$ surfaces is strictly positive. Let $\sum$ and $\sum_1$ be diffeomorphic, and $I(x,y,z)$ be diffeomorphism. Denote the coefficient of concentration decrease per unit distance by $q^0(x,y,z,t)$. Naturally, q^0 depends on the physical characteristics of a source, its location, etc. The simplest way to determine q^0 are experimental or empiric methods. It is obvious that $q^0\leq 1$. Naturally, in a normal ecological situation $q^0\prec 1$ if $q^0=[1-\varepsilon(x,y,z,t)]$, where $0\prec\varepsilon\prec 1$.

Let us denote by $q'(xx',yy',zz',t)$ the concentration decrease at point (x,y,z) of the surface $\sum$ at the moment compared with its value in a diffeomorphic point $(x',y',z')\in\sum_1$ at the same moment t. It is obvious that

$$q(x,y,z,t)=[1-\varepsilon(x,y,z,t)]^{p(xx')}\,. \quad (2)$$

Now, let us set forth the following initial boundary value problem: find the solution of the equation (1) that satisfies the following initial condition

$$S(x,y,z,0)=S_0(x,y,z),\quad (x,y,z)\in G, \quad (3)$$

boundary conditions

$$K_z \frac{\partial S}{\partial z} = \varphi S \ \ on \ \sum_0, \ \ 0 \le t \le T, \tag{4}$$

$$\frac{\partial S}{\partial z} = 0 \ \ on \ \sum_H, \ \ 0 \le t \le T \tag{5}$$

and non-local boundary condition

$$\begin{aligned} &S(x, y, z, t) = q(x, y, z, t)S(z', y', z', t) + \Psi(x, y, z, t), \\ &(x, y, z, t) = I(x', y', z'), \ \ 0 \le t \le T, \end{aligned} \tag{6}$$

where Ψ is given concentration, φ is constant, that characterized interection pollution with earth surface.

So the non-local problem (1), (2)-(6) is being solved. Under some general assumptions it can be shown that there exists the unique regular solution of this problem. Problem (1), (2)-(6) can be solved by Decomposition Method. All aforesaid can be discussed in detail for a specific case.

Let $G = [0, L] \times [0, H]$, $Q = G \times \left[0, \bar{T}\right]$, $\bar{G} = G \cup \Gamma$, let Γ be the boundary of G, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$,

$$\Gamma_1 = \{(x, z), \ x \in [0, L], \ z = 0\}, \ \Gamma_2 = \{(x, z), \ z \in [0, H]\},$$

$$\Gamma_3 = \{(x, z), \ x \in [0, L], \ z = H\}, \ \Gamma_4 = \{(x, z), \ x = L, \ z \in [0, H]\},$$

$$\Gamma_0 = \{(x, z), \ x = x_{0,} \ z \in [0, H]\}.$$

Let the axis Ox have the direction of an average velocity of atmospheric stream, Oz be directed vertically upward and Oy be perpendicular to the plane Oxz.

Consider a two-dimensional case of (1) in area G:

$$\frac{\partial S}{\partial t} = K_x \frac{\partial^2 S}{\partial x^2} + \frac{\partial}{\partial z} K_z \frac{\partial S}{\partial z} - u \frac{\partial S}{\partial x} - w \frac{\partial S}{\partial z} - \alpha S + F(x, z, t), \tag{7}$$

with initial and boundary conditions

$$S(x, z, 0) = S_0(x, z), \quad (x, z) \in \bar{G}, \tag{8}$$

$$\frac{\partial S}{\partial z} = \phi_1, \quad (x, z) \in \Gamma_3, \quad 0 \le t \le T, \tag{9}$$

$$K_z \frac{\partial S}{\partial z} = \alpha S + \phi_2, \quad (x, z) \in \Gamma_1, \quad 0 \le t \le T, \tag{10}$$

$$S(x,z,t)=\phi_3(x,z), \quad (x,z)\in\Gamma_2, \quad 0\le t\le T, \tag{11}$$

$$S(L,z,t)=qS(x_{0,}z,t)+\phi_4, \quad z\in(0,H), \quad 0\le t\le T, \tag{12}$$

where functions $F(x,z,t), S_0(x,z), \phi_1, \phi_2, \phi_3, \phi_4$ are given.

So, our aim is to investigate the problem (7)-(12) and find its numerical solution.

For simplicity of further demonstration let us imply that K_x, K_z, u, w , are constants. Assume that instead of (9) and (10) we have

$$S(x,z,t)=\phi_1, \;\; (x,z)\in\Gamma_2, \;\; 0\le t\le T, \tag{13}$$

$$S(x,z,t)=\phi_2, \;\; (x,z)\in\Gamma_3, \;\; 0\le t\le T. \tag{14}$$

Consider the following iteration process

$$\begin{aligned}
&\frac{\partial S^{(k+1)}}{\partial t}=K_x\frac{\partial^2 S^{(k+1)}}{\partial x^2}+K_z\frac{\partial^2 S^{(k+1)}}{\partial z^2}-u\frac{\partial S^{(k+1)}}{\partial x}-w\frac{\partial S^{(k+1)}}{\partial z}\\
&-\alpha S^{(k+1)}+F(x,z,t),\\
&S^{(k+1)}(x,z,0)=S_0(x,z), \quad (x,z)\in\bar{G},\\
&S^{(k+1)}(x,z,t)=\phi_1, \quad (x,z)\in\Gamma_3, \quad 0\le t\le T,\\
&S^{(k+1)}(x,z,t)=\phi_2, \quad (x,z)\in\Gamma_1, \quad 0\le t\le T,\\
&S^{(k+1)}(x,z,t)=\phi_3, \quad (x,z)\in\Gamma_2, \quad 0\le t\le T,\\
&S^{(k+1)}(L,z,t)=qS^{(k)}(x_0,z,t)+\phi_4, \quad z\in(0,H), \quad 0\le t\le T,
\end{aligned} \tag{15}$$

$(k=0,1,...)$, $S^0(x_0,z,t)$, ϕ_1, ϕ_2 are given sufficiently smooth functions, having continuous borders with the functions ϕ_1 and ϕ_2.

In [4,5] it is proved that there exists the only solution of (7), (8), (11)-(14) and $S^{(k)}(x,z,t)\to S(x,z,t)$ at geometric progression speed in metrics $C^{(0)}(\bar{G})$. The uniqueness of the solution of the problem (7), (8), (11)-(14) is proved on the basis of an analog of the first Harnack theorem [4]. On the basis of the same analog (15) iteration process convergence is proved. Using methods presented in [4-6] the existence and uniqueness of the regular solution of the problem (7), (12) can be shown.

In order to solve this problem let us consider the following iteration process:

$$\frac{\partial S^{(k+1)}}{\partial t} = K_x \frac{\partial^2 S^{(k+1)}}{\partial x^2} + K_z \frac{\partial^2 S^{(k+1)}}{\partial z^2} - u\frac{\partial S^{(k+1)}}{\partial x} - w\frac{\partial S^{(k+1)}}{\partial z}$$

$$-\alpha S^{(k+1)} + F(x,y,z,t),$$

$$(x,z) \in \bar{G}, \quad 0 \le t \le T,$$

$$S^{(k+1)}(x,z,0) = S_0(x,z), \quad (x,z) \in \bar{G},$$

$$\frac{\partial S^{(k+1)}}{\partial z} = \phi_1, \quad (x,z) \in \Gamma_3, \quad 0 \le t \le T, \tag{16}$$

$$K_z \frac{\partial S^{(k+1)}}{\partial z} = \alpha S^{(k+1)} + \phi_2, \quad (x,z) \in \Gamma_3, \quad 0 \le t \le T,$$

$$S^{(k+1)}(x,z,t) = \phi_3, \quad (x,z) \in \Gamma_2, \quad 0 \le t \le T,$$

$$S^{(k+1)}(L,z,t) = qS^{(k)}(x_0,z,t) + \phi_4, \quad z \in (0,H), \quad 0 \le t \le T,$$

$k = 1, 2, ...$

Convergence of (16) iteration process has not been studied. It is interesting to prove practical applicability of the process (16) by Calculation Experiment Method, because on the basis of this method solution of non-classical problem (7)-(12) can be turned into classical mixed boundary problem at each time step.

For numerical solution of the problem (7)-(12) let us apply Decomposition Method, on the basis of which we build up averaged additive models. Further, let's continue with building up of decomposition difference schemes of parallel calculation.

Devide interval $[0,T]$ by grid $\omega_r = \left\{t, t = t_j = j\tau, j = 0, 1, ..., K = \left[\frac{T}{\tau}\right]\right\}$. Let the value of the function in a certain time interval $\Delta_{j+1} = (t_j, t_{j+1})$ be denoted as follows: $S^{(j)}(x,z,t) = S^{(j)}$. Assume, that:

$$L_{1=} K_x \frac{\partial^2}{\partial x^2} - u\frac{\partial}{\partial x} - \alpha^1 E,$$

$$L_{2=} K_z \frac{\partial^2}{\partial z^2} - w\frac{\partial}{\partial z} - \alpha_2 E, \tag{17}$$

$$L = L_1 + L_2, F(x,z,t) = F_1(x,z,t) + F_2(x,z,t).$$

Let us formally devide the problem (7) (12) into two subproblems with operators L_1 and L_2 in time interval $[t_j, t_{j+1}]$.

Consider the following additively averaged model for the problem (7)-(12):

$$\frac{1}{2}\frac{\partial S_1^{(j+1)}}{\partial t} L_1 S_1^{(j+1)} + F_1, \quad 0 \le x \le L, \quad z \in [0,H], \quad t \in \Delta_{j+1}, \tag{18}$$

$$S_1^{(j+1)}(x,z,t_j) = P_1^{(j+1)}(x,z,t_j), \quad (x,z) \in \bar{G}, \tag{19}$$

$$S_1^{(j+1)}(0,z,t_j) = \phi_3(z,t), \quad 0 \le z \le H, \quad t \in \Delta_{j+1}, \tag{20}$$

$$S_1^{(j+1)}(L,z,t_j) = qS_1^{(j+1)}(x_0,z,t_j) + \phi_4(z,t), 0 \le z \le H, t \in \Delta_{j+1}, \tag{21}$$

$$\frac{1}{2}\frac{\partial S_2^{(j+1)}}{\partial t} = L_2 S_2^{(j+1)} + F_2, \quad 0 \le z \le H, \quad x \in [0,L], \quad t \in \Delta_{j+1}, \tag{22}$$

$$S_2^{(j+1)}(x,z,t_j) = P^{(j+1)}(x,z,t_j), \quad (x,z) \in \bar{G}, \tag{23}$$

$$\frac{\partial S_2^{(j+1)}}{\partial z} \mid_{z=0} = \phi_1(x,t), \quad 0 \le x \le L, \quad t \in \Delta_{j+1}, \tag{24}$$

$$K_x \frac{\partial S_2^{(j+1)}}{\partial z} \mid_{z=H} = \alpha S_2^{(j+1)} \mid_{z=H} + \phi_2(x,t), \ 0 \le x \le L, \ t \in \Delta_{j+1}, \tag{25}$$

$$\begin{gathered} P^{(j+1)}(x,z,t_j) = 0,5[S_1^{(j)}(x,z,t_j) + S_2^{(j)}(x,z,t_j)], \\ (j = 0,1,2,...,K \equiv \left[\tfrac{T}{\tau}\right]), \end{gathered} \tag{26}$$

$$S_1^{(0)}(x,z,0) = S_2^{(0)}(x,z,0) = S_0(x,z), \quad (x,z) \in \bar{G} \ . \tag{27}$$

The function $P(x,z,t)$ is determined in the following way:

$$P(x,z,t) = 0,5[S_1^{(j+1)}(x,z,t) + S_2^{(j+1)}(x,z,t)], \tag{28}$$

where ($t \in \Delta_{j+1,}j = 0,1,2,...,K-1$), will be called the solution of the split problem.

It can be shown (see [7]) that if initial data are smooth enough, additively averaged model (18)-(28) approximates the problem (7)-(12) and

$$\max_{\overline{Q}} |S(x,z,t) - P(x,z,t)| = 0(\tau).$$

So, in order to solve two-dimensional problem (7)-(12) we have to solve two one-dimensional problem. The first one is the transfer of substance along axis Ox and the second one is the transfer of substance along axis Oz. In the first case we have non-local boundary conditions, and in the second case we have classical boundary conditions. We can solve these problems simultaneously in the interval $\Delta_{(j+1)}$. Therefore, if we replace each problem with its difference analog, we will get decomposition difference scheme or locally one-dimensional parallel calculation scheme.

The next step is the construction of a parallel calculation scheme and its investigation.

Let us introduce the grid $\omega_h \left\{(x,z) \mid x = x_i, z = z_k, x_i = ih_1, z_k = kh_3, i = \overline{0, N_1}, k = \overline{0, N_3}, h_1 = \frac{L}{N_1},\ h_3 = \frac{H}{N_3}\right\}$. Denote the value of the function $W(x,z,t)$ in (x_i, z_k) at moment t_j by $W^j_{i,k}$. We will use the following known notation:

$$\begin{aligned}
&\frac{W^j_{i+1,k} - W^j_{i,k}}{h_1} = W^j_{i,kx}, \quad \frac{W^j_{i,,k} - W^j_{i-1,k}}{h_1} = W^j_{i,k\bar{x}},\\
&\frac{W^j_{i,k+1} - W^j_{i,k}}{h_3} = W^j_{i,kz}, \quad \frac{W^j_{i,,k} - W^j_{i,k-1}}{h_3} = W^j_{i,k\bar{z}},\\
&\frac{W^{j+1}_{i,k} - W^j_{i,k}}{\tau} = W^j_{i,kt}, \quad \frac{W^j_{i,k} - W^{j-1}_{i,k}}{\tau} = W^j_{i,k\bar{t}},\\
&\frac{W_{i+1,k} - W_{i-1,k}}{2h_1} = W^j_{i,k\overset{0}{x}}, \quad \frac{W_{i,k+1} - W_{i,k-1}}{2h_3} = W^j_{i,k\overset{0}{z}}.
\end{aligned} \tag{29}$$

On the basis of the additive model (18)-(27) the following parallel locally one-dimensional calculation scheme can be constructed:

$$\begin{aligned}
&\frac{C_1^{j+1} - P^j}{\tau} = K_x C^{j+1}_{1x\bar{x}} - uC^{j+1}_{1\overset{0}{x}} - \alpha_1 C_1^{j+1} + F_1^*, \quad (x,z,t) \in W_{h,\times} W_\tau,\\
&P^j = \frac{1}{2}(C_1^j + C_2^j),\\
&C_1(0, z_k, t_{j+1}) = \phi_3(z_k, t_{j+1}),\\
&C_1(L, z_k, t_{j+1}) = qC_1(x_0, z_k, t_{j+1}) + \phi_4(z_k, t_{j+1}),\\
&(k = \overline{0, N_3}, \quad j = \overline{0, K})
\end{aligned} \tag{30}$$

$$\frac{C_2^{j+1}-P^j}{\tau}=K_zC^{j+1}_{2\bar{z}z}-wC^{j+1}_{2\overset{0}{z}}-\alpha_2C_2^{j+1}+F_2^*,\quad (x,z,t)\in W_{h,\times}W_\tau,$$

$$P^j=\frac{1}{2}(C_1^j+C_2^j),$$

$$\frac{C_{2,i,1}^{j+1}-C_{2,i,0}^{j+1}}{h_3}=\phi_1(x_i,t_i),\quad (i=\overline{0.N_1},\quad j=\overline{0.K}),\tag{31}$$

$$K_z\frac{C_{2,i,N_3}^{j+1}-C_{2,i,N_3-1}^{j+1}}{h_3}=\alpha C_{2,N_3}^{j+1}+\phi_2(x_i,t_i),\quad (i=\overline{0.N_1},\quad j=\overline{0.K}),\tag{32}$$

where P^j is considered to be the solution of difference scheme $F_\beta^* = F_\beta(x_i,z_k,i_{J+1/2})$ $(\beta=1,2)$. We also assume, that coordinate x_0 coincides with a node of the grid along axis Ox and $x_0=(N_1-m)h_1$; m is the natural number,that changes incompliance with the step of the grid.

In order to solve the system (30) the classical method of sweep is applied. For solving (29) classical formulas should be constructed. Not dwelling on deduction of the latter, let us adduce formulas for general case.

Resolution of difference scheme (29) can be reduced to resolution of the following system.

$$\begin{aligned}&\alpha_1R_{i-1}-C_iR_i+b_iR_{i+1}=b_i\quad (i=\overline{1,N_1-1}),\\ &R_0=+\mu_1,\qquad R_{N_1}=\eta_2R_{N_1-1}+\eta R_{N_1-m}+\mu_2.\end{aligned}\tag{33}$$

The system (31) can be solved on the basis of the following formulas (see [6]):

$$\begin{aligned}&R_1=\alpha_{i+1}R_{i+1}+\beta_{i+1},\\ &\alpha_{i+1}=\frac{b_i}{C_i-a_i\alpha_i},\quad (i=\overline{0,N_1}-1),\\ &\beta_{i+1}=\frac{\alpha_i\beta_i+b_i}{C_i-a_i\alpha_i},\quad (i=\overline{0,N_1}-1),\\ &R_{N_1}=\frac{\eta_2\beta_{N_1}+\eta_1\overline{\beta}_{N_1-m+1}+\mu_z}{1-(\eta_2\alpha N_1+\eta_1\overline{\alpha}_{N_1-m+1})},\\ &\bar{\alpha}_{N_1-m+1}=\alpha_{N_1}\times\cdots\times\alpha_{N_1-m+1},\end{aligned}\tag{34}$$

$$\bar{\beta}_{N_1-m+1} = \alpha_{N_1-m+1} \times \cdots \times \alpha_{N_1-1\beta_N} + \cdots \div \beta_{N-1}.$$

Here α_i, b_i, c_i are expressed trough coefficients of the difference scheme.

Analysis of synoptic process in the Caucasian region shows that Georgian territory is dominated by western and eastern meteorological processes. This is stipulated by the geographical location of Georgia and features of its relief.

The north Georgian territory is bordered by Great Caucasus Range directed along parallel. It is a natural obstacle for meteorological processes coming from the north. Air masses intruded from the north are slowed down by Great Caucasus Range, they pass round it and enter Georgian territory from the west and the east. To the south of Georgia, Thrialeti and Meskheti ranges are stretched along the parallel, which, together with Great Caucasus Range, create natural "canyon", that also is Georgia's transportation corridor (GTC). In this corridor air masses are mainly carried from one place to another zonally. These movements are hindered by Likhi Range directed along meridian (that divides Georgian territory into western and eastern parts).

The above said allows us to make simplifications while investigating the problem of mathematical simulation of adverse substances transference and consider two-dimensional numerical model for air masses zonal transference taking into account characteristics of the relief (Likhi Range).[1-3].

Now let us consider the following specific problem: migration and diffusion of adverse substances ejected from a point source of power m_1 located at altitude h in the region of nonhomogeneous orography. Assume that meteorological situation stipulates transference of substance in the atmosphere in the direction of average wind velocity (zonal transference or winds of prevailing direction), that is $v = 0$. Under these conditions substance transference in the atmosphere is described by equation (7).

Let us remove limitation that we have imposed on u and w components of wind velocity and coefficient K_x being on constant. In (7) determine u and w by means of a complete system of Hydrothermodynamics. In case of homogeneous orography the set problem is described by the following system of equations [8-10].

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + w\frac{\partial S}{\partial z} = K_x\frac{\partial^2 S}{\partial x^2} + \frac{\partial}{\partial z}K_z\frac{\partial S}{\partial z} - \alpha S, \tag{35}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K_x\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial z}K_z\frac{\partial u}{\partial z}, \tag{36}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho w)}{\partial z}=0, \tag{37}$$

$$\frac{\partial p}{\partial z}=-\rho g, \tag{38}$$

$$\frac{\partial \Theta}{\partial t}+u\frac{\partial \Theta}{\partial x}+w\frac{\partial \Theta}{\partial z}=K_x\frac{\partial^2 \Theta}{\partial x^2}+\frac{\partial}{\partial z}K_z\frac{\partial \Theta}{\partial z}, \tag{39}$$

$$P=\rho RT, \tag{40}$$

where P is atmospheric pressure, ρ is air density, R is specific constant of dry air, Θ is potential temperature of air, T is temperature, g is acceleration of gravity.

We consider the problem of transference and diffusion of adverse substances in an atmosphere for the Caucasian region, where one of the determining factors for circulation process particularities is orography. In order to reflect correctly the impact of complex relief in the mathematical model, let us rewrite (34)-(39) system in a coordinate system. $x'=x, z=z-r(x), t=t$. (where function $r(x)$ describes nonhomogenity of the earth surface). If we rewrite the obtained system of equations in isobaric coordinate system (x,ς,t) where connection between ς, and z is determined from (37), then we'll receive [3,10]:

$$\frac{\partial S}{\partial t}+\frac{\partial uS}{\partial x}+\frac{\partial wS}{\partial \varsigma}=AS-\alpha S, \tag{41}$$

$$\frac{\partial u}{\partial t}+\frac{\partial(u^2)}{\partial x}+\frac{\partial(uw)}{\partial \varsigma}=\frac{\partial(\Phi+gr)}{\partial x}+Au, \tag{42}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{43}$$

$$\frac{\partial \Phi}{\partial \varsigma}=\frac{R\Theta P^{1/x}}{P^{x-1/x}}, \tag{44}$$

$$\frac{\partial \Theta}{\partial t}+\frac{\partial(u\Theta)}{\partial x}+\frac{\partial(w\Theta)}{\partial \varsigma}=A\Theta, \tag{45}$$

where $\varsigma=\frac{P}{P_0}$ is a vertical coordinate, P_0 is a value of pressure on 1000 Hpa of surface, Φ is relative geopotential, $A=K_x\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial z}K_z\frac{\partial}{\partial z}$.

While making prognosis of atmosphere pollution it is very important to have an information about adverse substance concentration behavior in the near-earth and border layer of the atmosphere.

In the near-earth layer vertical thermal streams and momentum are maintained according to altitude, and vertical component $K_z(z)$ of turbulent stream variation coefficient increases in proportion of altitude [1,8].

$$K_z(z) = v + K_1 \frac{z}{z_1},$$

where v is coefficient of molecular diffusion; $K_1 = K_z(z_1)$; According to M. Iudin and M. Shvets model above near-earth layer there exists some external scale of turbulence, that helps to restrain vortex movements and above near-earth layer $K_z = const$ and $K_z = v + K_1 \frac{h_1}{z_1}$, where h_1 is the height of near-earth layer [1].

As a result of theoretical studies I. Kibel has received analogous formula for $K_z(z)$ [1].

$$K_z(z) = \begin{cases} K_1 (\frac{z}{h_1})^{1-\varepsilon_2} & when \quad z \leq h_1, \\ \\ K_1 & when \quad z > h_1, \end{cases}$$

where $0 \leq \varepsilon \leq 1$ and by selecting its values it is possible to select thermal stratification of the environment. We will use the last formula in our mathematical model.

(40)-(44) can be integrated in the area G (point source is placed on Γ_1 (at point $x = 0, z = h$)), that satisfies the following initial condition:

$$S\mid_{t=0} = S_0(x, \varsigma), \quad u\mid_{t=0} = u^0(x, \varsigma), \quad \Theta\mid_{t=0} = \Theta^0(x, \varsigma). \tag{46}$$

Because of not having aerological observation results, as a rule, distribution of adverse substances concentration $\varsigma^0(x, \varsigma)$ in the area G is unknown. To avoid this inconvenience while making calculations under the numerical model, we act in the following way. As substance concentration change in atmosphere is of quasi-stationary character over a time period, it is possible to consider a stationary problem. Taking into account the fact, that we gave some information about vertical distribution of wind velocity and from vertical velocity $w = w_{a-}w_b$ (w_a is convectional movement velocity, w_b is velocity of substance precipitation by gravity) we take into consideration only velocity of precipitation by gravity, then we can rewrite (34) in the following form

$$u(z)\frac{\partial S_0}{\partial x} + w\frac{\partial S_0}{\partial z} = \frac{\partial}{\partial z}\left(K_z(z)\frac{\partial S_0}{\partial z}\right) - \alpha S = 0. \tag{47}$$

(46) can be integrated under the following boundary conditions:

$$S_0(x, \infty) = 0, \tag{48}$$

$$\frac{\partial S_0}{\partial z}\mid_{x=0}=0, \tag{49}$$

$$S_0(x,z)\mid_{x=x_0}=S_0^*(x_0,z). \tag{50}$$

In (49) function $S_0^*(x_0,z)$, in its turn, is the solution of the following simple stationary problem:

$$u\frac{\partial S_0^*}{\partial x}=\frac{\partial}{\partial z}K_z\frac{\partial S_0^*}{\partial z}, \tag{51}$$

that can be integrated with the following boundary conditions:

$$S_0^*\mid_{x\to\infty}\to 0, \qquad K_z\frac{\partial S_0^*}{\partial z}\mid_{x\to 0}\to 0, \tag{52}$$

$$S_0^*\mid_{x\to 0}=\frac{m_1}{u}\delta(z-h_1), \tag{53}$$

where m_1 is power of adverse substances cast out from a source located at altitude h

$$\delta(z-h)=\begin{cases}0, & z\neq h,\\ \infty, & z=h.\end{cases}$$

If we assume that $u=const, K_z=const$ in (50), then we will receive the known solution of (49) with the boundary conditions (50) and (51) [1,3]:

$$S_0(x,z)=\frac{m}{\sqrt{\pi K_z u x}}\left[e^{-\frac{u(h-z)^2}{uK_z x}}+e^{-\frac{u(h+z)^2}{uK_z x}}\right]. \tag{54}$$

Stationary problem (46) with boundary conditions (47)-(49) can be integrated by numerical methods using formula (32) that is similar to a classical one.

Generally, problem (40)-(44) is integrated in G with the boundary conditions:

$$\frac{\partial S}{\partial \varsigma}=0, \quad w=0, \quad when \quad \varsigma=1, \tag{55}$$

$$\frac{\partial S}{\partial \varsigma}=0, \quad \frac{\partial u}{\partial \varsigma}=0, \quad when \quad \varsigma=0.5, \tag{56}$$

$$S=S^{0(t,\varsigma)}, \quad u=u^0(t,\varsigma), \quad when \quad x=0, \tag{57}$$

$$\frac{\partial S}{\partial x}=0, \quad \frac{\partial u}{\partial x}=0, \quad \frac{\partial \Theta}{\partial x}=0, \quad when \quad x=L. \tag{58}$$

Boundary conditions (56) and (57) correspond to the case, when substance ejection source is placed on the boundary Γ_1 of area G. If in the area G we have additional m_2 powered source placed at the point $x = x_0, z = h_2$, then in (57) boundary condition $\frac{\partial S}{\partial x} = 0$ is unreal. Indeed, space distribution of substance $S(x, h_2)$, received by solving non-stationary problem (46) with boundary conditions (47)-(52) is represented on Fig.1.

Fig.1.

As seen from Fig.1, $\frac{\partial S}{\partial x} \neq 0$ in the interval $[a, b]$. If in the area G the source of power m_2 has such a location that boundary Γ_3 gets into interval $[a, b]$, then it is necessary to use non-local boundary condition (12). We come to non-local boundary conditions in certain particular problems, when there is, one source at the internal point $x = x_0, z = h_1$ of area G.

We also need non-local boundary conditions, when in the vicinity of boundary Γ_2 or Γ_3 substance concentration variability gradients (by thermal sources or orography) are considerable. Generally, distribution (dissemination) of adverse substances in space at a given moment depends on many meteorological factors (turbulence, wind velocity, clouds, thermal layers restraining aerosol flow etc.).

From experimental observation results, it is know that concentration of any aerosol (among them-radioactive substances) decreases as altitude increases, but this variation is not generally monotonic, it does not conform to any law of nature [1,8]. Aerosol is accumulated in the atmosphere, if there exist isothermal layers or if temperature inversion and, generally, temperature gradient reduction takes place (then in these and below layers the aerosol is intensely accumulated). While simulating these types of problems (problems of mesometeorology), apparently, it would be advisable to apply non-local boundary condition on Γ_3 boundary of G area.

Now let us consider the particular case of (34) integration, when unknown functions are determined from the following equations of movement and continuity:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial \Phi}{\partial x}+K_x\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial z}K_z\frac{\partial u}{\partial z}, \tag{59}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{60}$$

where $\Phi = R\,\bar{T}_{aver}\ln P,\quad \bar{T}_{aver}$ $(z)$is anaveraged along axis Ox, variable according altitude, known for each experiment function.

In (58) we assume, that for every simulated meteorological situation $\frac{\partial\Phi}{\partial x}$ is a known function. (34), (58), (59) describe transference of substance into the area G, having homogeneous orography in case of established meteorological processes. Our aim is to study the features characterizing the problem of adverse substance migration in atmosphere with non-local boundary conditions. (34), (58), (59) are integrated in the area G with the following initial and boundary conditions:

$$S\mid_{t=0}=S_0,\quad u\mid_{t=0}=u_0, \tag{61}$$

$$\frac{\partial S_0}{\partial z}=0,\quad w=0,\quad when\quad z=0, \tag{62}$$

$$S=0,\quad \frac{\partial u}{\partial z}=0,\quad when\quad z=H, \tag{63}$$

$$S=S_0(0,z),\quad u=u_0(z),\quad when\quad x=0, \tag{64}$$

$$\frac{\partial u}{\partial x}=0,\quad S(L,t)=qS(x_0,z,t),\quad when\quad z=L, \tag{65}$$

where, q is determined from (2).

Problem (34), (58), (59) with the initial and boundary conditions (60)-(64) is solved on g_2 grid $x_i = i\times\triangle x,\ z_k = k\times\triangle z,\ t_j = j\times\triangle t,\ i = 1,...,N,\ k = 1,...,M,\ j = 0,1,...$ by numerical methods using Lax-Vendrof finite-difference scheme [8,10].

By numerical experiments quantities and physical constants had the following value $\triangle x = 500m, \triangle z = 50m, \triangle t = 60, K_z = 10, N = 400; M = 100$.

According to the proposed model numerical calculations have been carried out. Those calculations were differ from each other by values of parameters and constants included in the numerical model.

In (60) the value of S_0 is determined from (46) by boundary conditions (47)-(49). In (49) $S^0(x_0, z)$ is given by the following formula:

$$S_0^*(x_0, z) = \frac{m_2}{u} S(z - h_2), \tag{66}$$

where m_2 is the power of the source located at altitude h_2 $(x = x_0, z = h_2) \in G$.

According to the height initial distribution of component u_0 of wind velocity is given depending on the character of model problem (in different experiments u_0 have different model profiles according to height). In order to give horizontal character to the distribution of u_0 we use the values of well-known for meteorologists distribution of the vertical component of the velocity and continuity equation (59). From synoptic materials it is clear that in troposphere w is distributed by height as follows: on the earth surface $w = 0$ (in case of relief $\overline{w} = w$), at higher altitudes its value increases, at medium level of troposphere (3-5 kilometers) it achieves maximum value $W_{\max}$ and at tropopause it is in the vicinity of zero again [1,8]. We have [1]:

$$W_1^0(z) = W_{\min} + W_{\max}(1 + 1/n)^n z(1 - z)^n, \tag{67}$$

where n is the parameter, characterizing the tendency of w fluctuation by height.

Space distribution of u_0 is given by the following formula:

$$u_0(x, z) = u_1^0(z) \cdot u_2^0(x), \tag{68}$$

where value of $u_2^0(x)$ is obtained by means of (58) and (59), and $u_1^0(z)$ is defined from the tested problem.

In order to determine parameters q and ϕ_4 in (64), the following problem is solved at every step t of numerical integration :

$$u^{(j)} \frac{\partial S_1^{(j)}}{\partial x} = \frac{\partial}{\partial z} K_z \frac{\partial S_1^{(j)}}{\partial x} \tag{69}$$

with boundary conditions:

$$S_1^{(j)} \mid_{z=H} = 0, \qquad \frac{\partial S_1^{(j)}}{\partial z} \mid_{x=0} = 0, \qquad S_1^{(j)} \mid_{z=x0} = \frac{m_2}{u} S(z - h_2), \tag{70}$$

where $S_1^{(j)}$-is concentration of substance ejected from the m_2 powered source located in $(x = x_0, z = h_2)$ point, $u^{(j)}$ is determined at each time step from (58), (59).

Values of $S_1(x, z)$ at each j iteration obtained after resolving (68) enables us to determine values of q and ϕ_4.

Numerical experiments were carried out. They deferred from each other by power values of the first and the second sources and location of the second source. Here we adduce results of the experiments. In each case m_1 powered source is located at point $x = 0, z = h_1$, and m_2 powered one at point $x = 9L/10, z = h_2$. Their power values were $m_1 = 2m_2$ in the first case; in the second case $-m_1 = m_2$; in the third case $m_1 = 0.5m_2$.

Let us compare numerical calculation results with the solutions received under classical boundary conditions (the rest of parameters are same). In particular, (34), (58), (59), where solved under boundary conditions (60)-(64), but in (64) we had $\frac{\partial S_1}{\partial x} = 0$ when $x = L$.

In order to compare numerical calculation processes during these experiments, we were calculating integral characteristics of prognosis fields at each time step; particularly, tendencies of meteoelements variation in the area G in average time

$$|\bar{\phi}|= \frac{1}{N_x \triangle t} \sum_{n=1}^{N} | \phi_{n,j+\triangle t} - \phi_{i,j} |, \tag{71}$$

where $\phi = (u, w, S, S_1)^T$ is a matrix column, N is a number of grid points.

In [8-10] it is shown that if in finite-difference scheme the following quadratic quantities:

$$EK = \frac{1}{N} \sum_{n=1}^{N} (u_n^2 + w_n^2)/2, \tag{72}$$

$$\Omega^2 = \frac{1}{N} \sum_{n=1}^{N} (\frac{\partial w_n}{\partial x} - \frac{\partial u_n}{\partial z})^2, \tag{73}$$

are maintained over time period then nonlinear instability does not occur. In each experiment we were examining maintenance of (70) and (72) quantities during period of prognosis. Calculation results show that relative error of Ω^2 did not exceed 1.73% in average within physical 72 hours, and that of the EK did not exceed 4.3% in the same time-period. This fact indicates that numerical solution was stable (comparatively big relative error of EK quantity can be attributed to limited sizes of rather than exposure of nonlinear instability).

As numerical calculations showed, $\left|\overline{S}_1\right|$ variation during prognosis period with both classical and nonlocal boundary conditions for of each three cases were nearly of the same character. But the difference between the results of calculations were more distinctly revealed, when so, in order to prevent the Fig.2 from being overloaded, we adduce the calculation results received only in this case:

Fig.2.

As we can see from Fig.2, under classical boundary conditions $\left|\overline{S}_1\right|$ variation has more instable, increasing character (line a). And under nonlocal boundary conditions its variation is in the vicinity of initial deviation after first 6 hours.

Prognosis fields analysis shows us that in case of classical boundary conditions, in the vicinity of boundary Γ_3 unreal increase of concentration values took place, which was not observed in case of nonlocal boundary conditions.

The obtained results allow us to conclude, that while integrating system (34) ,(41)-(44), describing adverse substances migration in GTC, non-local boundary conditions should be used.

References

1. Berlyand, M.E. *Predication and Regulation of Air Pollution.* (Russian) Leningrad, Gidrometeoizdat, 1985, 10-20.
2. Marchuk, G.I. *Mathematical modeling for the problem of the environment.* (Russian) Moscow, "Nauka", 1982.
3. Davitashvili, T.; Khvedelidze, Z. *Numerical model of the Radioactive Products Transference and Fall out for the Caucasian Region.*Bulletin of the Georgian Academy of Sciences, 155(1997), no.1, 58-61.
4. Gordeziani, D. *On the Method of Solving one class of Non-local Boundary Value Problem.* (Russian) Tbilisi, Tbilisi University Press, 1981.
5. Gordeziani, D.; Gordeziani, N.; Avalishvili, G. *On one class of non-local problems for partial differential equations.* Bulletin of the Georgian Academy of Sciences, Tbilisi, 157(1998), no.3.
6. Gordeziani, E. *On one non-local initial-boundary problem solution.* Diploma work for the bachelorship, Tbilisi State University, 1998.
7. Gordeziani, D. Samarski, A. *Some problems of thermoplasticity of the plates and a total approximation method.* Complex analysis and application. Proceedings devoted to the memory of the 70 years anniversary of Acad. Vekua, M. 1978, 173-186.
8. Belov, P.N. *Numerical Methods of Weather predication.* (Russian) Leningrad, Gidrometeoizdat, 1975, 53-62.
9. Davitashvili, T.; Kvedelidze, Z. *Numerical model of the Long-Range transport radioactive products including orography and precipitation.* Georgian symposium for project development and conversion. Tbilisi. 1995, 117-119.
10. Davitashvili, T. *The influence of orography in numerical weather prediction models.* Proceedings of Vekua Institute of Applied Mathematics, Tbilisi, 25(1988), 118-131.