PSEUDODIFFERENTIAL CALCULUS ON MANIFOLDS WITH SINGULAR POINTS

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(Recieved: December 11, 1997)

Abstract

In the pseudodifferential calculus on manifolds with singularities there appear operators in terms of Fourier and Mellin transforms. This gives rise to abstract Fourier-Laplace transforms. We describe pseudodifferential operators and Sobolev spaces with respect to such transforms. It is indicated how these techniques may be used to get an index formula on manifolds with one-point singularities.

Key words and phrases: Fourier-Laplace transform, pseudodifferential operators, index.

AMS subject classification: primary: 35S05; secondary: 46E40, 58G10.

Introduction

The concept of the conification of an operator algebra was introduced in Schulze [Sch91]. It applied earlier results of [Sch90]. Roughly speaking, the conification is a pseudodifferential calculus along the cone axis $\bar{\mathbb{R}}_+$, based on the Mellin transform, with operator-valued symbols taking their values in the given operator algebra. The essential key words in this context are Fuchs-type operators, conormal symbols, Mellin quantization, kernel cut-off, meromorphic operator functions, weighted cone Sobolev spaces, discrete and continuous asymptotics, Green operators. In particular, given a closed compact manifold X, the conification of the algebra of pseudodifferential operators on X gives rise to a cone algebra on the stretched cone $\mathcal{C} = \bar{\mathbb{R}}_+ \times X$.

Let us comment in this connection on the role of the Mellin transform which is the relevant integral transform here, whenever a new cone axis $\bar{\mathbb{R}}_+ \ni t$ appears. We could always substitute the diffeomorphism $r = \log t$ of

[1]Supported by the Deutsche Forschungsgemeinschaft.

$\mathbb{R}_+ \to \mathbb{R}$ and pass to the Fourier transform in the new coordinate. However in the original t-variable the calculus needs the Mellin and the Fourier transform at the same time. The link between the Mellin and the Fourier description of operators by means of Mellin operator conventions belongs to the essential technical points of the calculus. Thus, the coordinate change by $r = \log t$ would destroy this relation, and it seems in fact much more natural to employ the Mellin transform in its classical form, though the elements of the Mellin pseudodifferential calculus have to be established and accepted as a tool.

The advantage of using the Mellin transform in the analysis on manifolds with conical singularities lies in the fact that it "quantizes" the covariable z as the Fuchs-type derivative $-t\,d/dt$ which is the only characteristic component in a local basis of vector fields at the singular point. The basic geometric ingredient is the "germ" of the diffeomorphism $t \mapsto \log t$ at $t = 0$ (or, more precisely, the behavior of this diffeomorphism in an infinitesimal interval $(0, \varepsilon)$, where $\varepsilon > 0$).

More generally, consider an arbitrary diffeomorphism $r = \delta(t)$ of the half-axis $T = \mathbb{R}_+$ onto the whole axis $\mathbb{R}$, with $\delta'(t) > 0$ for $t \in T$. We associate to δ an isomorphism $\boldsymbol{F} : L^2(T, dm) \to L^2(\mathbb{R})$, where $dm = 2\pi\,\delta'(t)\,dt$, by changing the variable in the Fourier transform. Then, we study the integral transform $\boldsymbol{F}$ for complex values of the covariable, too.

Similar considerations apply to the multidimensional case provided that the mapping δ under consideration does not mix up the variables.

The integral transform so obtained is easily verified to "quantize" the covariable z as the derivative $\boldsymbol{D} = \frac{1}{\delta'(t)}\frac{1}{i}\,d/dt$ which keeps an information on the geometric nature of the singular point in question via the "germ" of the diffeomorphism $t \mapsto \delta(t)$ at $t = 0$. It is therefore to be expected that $\boldsymbol{F}$ is the relevant integral transform in the analysis on manifolds with one-point singularities, such as "conical points", power-like and exponential "cusps", etc.

In the literature there exists a number of directions studying the analysis on manifolds with one-point singularities under different aspects (cf., in particular, Plamenevskii [Pla89], Levendorskii [Lev93], Maz'ya, Kozlov and Rossmann [MKR97], et al.). Melrose and Nistor [MN96] defined a "cusp" calculus on a manifold with boundary by use of blow up techniques. They described the Hochschild homology of this algebra and various of its ideals and so deduced a pseudodifferential generalization of the *Atiyah-Patodi-Singer Index Theorem*. We also mention the paper of Schulze, Sternin and Shatalov [SSS96] where an operator algebra on manifolds with power-like cusps is constructed by use of non-commutative analysis.

The aim of the present work is to develop the calculus of pseudodifferential operators on a manifold with one-point singularities by use of the integral transform $\boldsymbol{F}$. Thus, our approach is similar in spirit to the cone calculus of [Sch91].

We will restrict our attention to the local algebra at a singular point, for we use the standard pseudodifferential operators away from the singularity. In this situation, we introduce the notion of ellipticity and establish the Fredholm property of elliptic operators. We then apply *Fedosov*'s *techniques* to derive an index formula for elliptic operators of order zero (cf. Fedosov and Schulze [FS96]).

1 *Abstract Fourier-Laplace Transform*

1.1 *Fourier transform*

Until further notice we assume that $T = (a, b)$ is an arbitrary interval in the real axis and $r = \delta(t)$ is a diffeomorphism of T onto $\mathbb{R}$. To be specific, consider the case where $\delta'(t) > 0$ for $t \in T$.

It is easy to see that, given any $u, v \in C^\infty_{comp}(T)$, we have

$$\left(u \circ \delta^{-1}, v \circ \delta^{-1}\right)_{L^2(\mathbb{R})} = \frac{1}{2\pi}\,(u, v)_{L^2(T,dm)}\,, \tag{1.1.1}$$

where $dm\,(t) = 2\pi\,|\delta'(t)|\,dt$. Hence it follows that $u \circ \delta^{-1}$ is of class $L^2(\mathbb{R})$, for any $u \in L^2(T, dm)$.

Thus, we may define the *Fourier transform* on $L^2(T, dm)$ (denoted by $\boldsymbol{F}$ in contrast to the usual Fourier transform $\mathcal{F}$) so as to make the following diagram commutative:

$$\begin{array}{ccc} L^2(T,dm) & & \\ \Big\downarrow (\delta^{-1})^{\mathrm{Ast}} & \searrow \boldsymbol{F} & \\ L^2(\mathbb{R}) & \stackrel{\mathcal{F}}{\to} & L^2(\mathbb{R}). \end{array}$$

In other words,

$$\begin{aligned} \boldsymbol{F}u\,(\tau) &= \int_{\mathbb{R}} e^{-i\tau r}\, u\left(\delta^{-1}(r)\right) dr \\ &= \frac{1}{2\pi} \int_T e^{-i\tau\delta(t)}\, u(t)\, dm\,(t), \qquad \tau \in \mathbb{R}, \end{aligned}$$

for any $u \in L^2(T, dm)$.

The main property of the Fourier transform so defined is that *Parseval*'s *formula* remains valid.

Lemma 1.1.1 *If $u, v \in L^2(T, dm)$, then*

$$(\boldsymbol{F}u, \boldsymbol{F}v)_{L^2(\mathbb{R})} = (u, v)_{L^2(T,dm)} .$$

Proof. Indeed, applying Parseval's formula yields

$$(\boldsymbol{F}u, \boldsymbol{F}v)_{L^2(\mathbb{R})} = 2\pi \left(u \circ \delta^{-1}, v \circ \delta^{-1}\right)_{L^2(\mathbb{R})} ,$$

which gives the desired conclusion when combined with (1.1.1).

□

Hirschmann [Hir90] defines *regular $\boldsymbol{F}$-transforms* on $L^2(T, dm)$ by requiring the Parseval formula as well as 3 more properties to hold. Then, he proves that any regular $\boldsymbol{F}$-transform is of the form $\boldsymbol{F} = \mathcal{F} \circ \left(\delta^{-1}\right)^*$, with δ a diffeomorphism of $T \to \mathbb{R}$.

1.2 *Fourier-Laplace transform*

If $u \in \mathcal{E}'(T)$, then $\boldsymbol{F}u\,(\tau)$ is also well-defined for every complex value $\tau \in \mathbb{C}$. In this way we obtain what will be referred to as the *Fourier-Laplace transform*. Namely,

$$\begin{aligned} \boldsymbol{F}u\,(z) &= \frac{1}{2\pi} \int_T e^{-iz\delta(t)}\, u(t)\, dm\,(t) \\ &= \boldsymbol{F}\left(e^{v\delta(t)}u\right)(\tau), \qquad z = \tau + iv \in \mathbb{C}, \end{aligned} \tag{1.2.1}$$

where we first assume $u \in C^\infty_{comp}(T)$.

Obviously, $\boldsymbol{F}u\,(z)$ is an entire function in the complex plane and the restriction of $\boldsymbol{F}u\,(z)$ to the real axis coincides with the Fourier transform of u (i.e., $\boldsymbol{F}u\,(\tau)$).

Given a $\gamma \in \mathbb{R}$, set $\Gamma_\gamma = \{z \in \mathbb{C} : \operatorname{im} z = \gamma\}$ and consider the "weighted Fourier transform" $\boldsymbol{F}_\gamma u = \boldsymbol{F}u\,|_{\Gamma_\gamma}$.

Lemma 1.2.1 *For any $\gamma \in \mathbb{R}$, the transform $\boldsymbol{F}_\gamma$ extends by continuity to a unitary isomorphism*

$$\boldsymbol{F}_\gamma :\, L^2(T, e^{2\gamma\delta}dm) \overset{\cong}{\to} L^2(\Gamma_\gamma). \tag{1.2.2}$$

Proof. If $u \in C^\infty_{comp}(T)$, then

$$\sup_{z\in\Gamma_\gamma} (1 + |z|)^N\, |\boldsymbol{F}_\gamma u\,(z)| < \infty$$

for all $N = 0, 1, \ldots$. Now, applying Lemma 1.1.1 gives

$$\begin{aligned} (\boldsymbol{F}_\gamma u, \boldsymbol{F}_\gamma v)_{L^2(\Gamma_\gamma)} &= \left(\boldsymbol{F}(e^{\gamma\delta}u), \boldsymbol{F}(e^{\gamma\delta}v)\right)_{L^2(\mathbb{R})} \\ &= \left(e^{\gamma\delta}u, e^{\gamma\delta}v\right)_{L^2(T,dm)} \\ &= (u, v)_{L^2(T,e^{2\gamma\delta}dm)} \end{aligned}$$

whenever $u, v \in C^\infty_{comp}(T)$. Hence (1.2.2) is an easy consequence of the relation between $\boldsymbol{F}$ and the Fourier transform on $\mathbb{R}$ via the substitution $r = \delta(t)$.

□

1.3 *Inversion formula*

The following is actually an equivalent formulation of the *Fourier inversion formula.*

Lemma 1.3.1 *The inverse of* (1.2.2) *is given by the formula*

$$\boldsymbol{F}^{-1}_\gamma f\,(t) = \frac{1}{2\pi} \int_{\Gamma_\gamma} e^{i\delta(t)z}\, f(z)\, dz.$$

Proof. Indeed, given any $u \in L^2(T, e^{2\gamma\delta}dm)$, we have

$$\begin{aligned} \boldsymbol{F}_\gamma u\,(\tau + i\gamma) &= \boldsymbol{F}_{t\mapsto\tau}\left(e^{\gamma\delta}u\right) \\ &= \mathcal{F}_{r\mapsto\tau}\left(e^{\gamma r}\, u(\delta^{-1}(r))\right), \end{aligned}$$

whence by Fourier's inversion formula

$$\begin{aligned} &\boldsymbol{F}^{-1}_\gamma\left(\boldsymbol{F}_\gamma u\right)(t) \\ &= e^{-\gamma\delta(t)}\, \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta(t)\tau} \mathcal{F}_{r\mapsto\tau}\left(e^{\gamma r} u(\delta^{-1}(r))\right) d\tau \\ &= u. \end{aligned}$$

This proves the lemma.

□

1.4 *Properties*

The Fourier-Laplace transform on $L^2(T, dm)$ is related to the derivative

$$\boldsymbol{D} = \frac{1}{\delta'(t)}\,\frac{1}{i}\, d/dt$$

in the same manner as the usual Fourier-Laplace transform to the derivative $D = \frac{1}{i}\, d/dt$.

Lemma 1.4.1 *For any $u \in C^{\infty}_{comp}(T)$, it follows that*

$$\boldsymbol{F}\,(\boldsymbol{D}u)\,(z) = z\,\boldsymbol{F}u\,(z).$$

Proof. Indeed, integrating by parts yields

$$\begin{aligned} \boldsymbol{F}_{t\mapsto z}\,(\boldsymbol{D}u) &= \int_T e^{-iz\delta(t)} \left(\frac{1}{\delta'(t)} D_t u\right) |\delta'(t)| dt \\ &= \operatorname{sgn}(\delta') \int_T \left(-D_t\, e^{-iz\delta(t)}\right) u\, dt \\ &= z\,\boldsymbol{F}_{t\mapsto z}(u), \end{aligned}$$

as desired.

□

It is therefore to be expected that the Fourier-Laplace transform $\boldsymbol{F}$ will prove to be useful by studying "totally characteristic" differential operators, i.e., those of the form $A = \sum_{j=0}^{m} A_j(t)\boldsymbol{D}^j$ with $A_j(t)$ smooth functions on T.

Yet another basic property of the Fourier-Laplace transform is that multiplication of u by the weight function $e^{\gamma\delta(t)}$ is interpreted under (1.2.1) as the displacement of the reference line by $i\gamma$.

Lemma 1.4.2 *For any $\gamma \in \mathbb{R}$, we have*

$$\boldsymbol{F}(e^{\gamma\delta}u)\,(z) = \boldsymbol{F}(u)\,(z + i\gamma), \qquad z \in \mathbb{C}.$$

Proof. Indeed,

$$\begin{aligned} \boldsymbol{F}(e^{\gamma\delta}u)\,(z) &= \int_T e^{-iz\delta(t)} \left(e^{\gamma\delta(t)}u(t)\right) dm\,(t) \\ &= \boldsymbol{F}(u)\,(z + i\gamma), \end{aligned}$$

as desired.

□

1.5 *Sobolev spaces*

We want to define weighted Sobolev spaces of functions on T, based on the Fourier-Laplace transform. Lemmas 1.4.1 and 1.4.2 sheds some light on how we have to begin with.

Definition 1.5.1 *Let $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$. Denote by $\mathcal{H}^{s,\gamma}(\bar{T})$ the set of all distributions u on T whose derivatives up to order s are locally integrable with respect to the measure dm and satisfy*

$$\|u\|_{\mathcal{H}^{s,\gamma}(\bar{T})} := \left(\int_T e^{-2\gamma\delta(t)} \sum_{j=0}^{s} \left|\boldsymbol{D}^j u(t)\right|^2\, dm\right)^{\frac{1}{2}} < \infty.$$

For integer $s < 0$, we could define the space $\mathcal{H}^{s,\gamma}(\bar{T})$ by duality and then, for fractional s, by complex interpolation. However we derive a direct description of the space $\mathcal{H}^{s,\gamma}(\bar{T})$ so obtained, from the following lemma.

Lemma 1.5.2 *Suppose $s \in \mathbb{Z}_+$. Then*

$$\|u\|_{\mathcal{H}^{s,\gamma}(\bar{T})} \sim \left(\int_{\Gamma_{-\gamma}} \left(1 + |z|^2\right)^s |\boldsymbol{F}u\,(z)|^2 dz \right)^{\frac{1}{2}}, \qquad u \in \mathcal{H}^{s,\gamma}(\bar{T}),$$

where the equivalence of two norms means that their ratio is bounded uniformly in u both from below and above by positive constants.

Proof. For the proof, we rewrite the norm $\|\cdot\|^2_{\mathcal{H}^{s,\gamma}(\bar{T})}$ by using the Fourier-Laplace transform. Namely, Lemmas 1.2.1 and 1.4.1 imply

$$\begin{aligned}
\|u\|^2_{\mathcal{H}^{s,\gamma}(\bar{T})} &= \sum_{j=0}^{s} \int_T \left|\boldsymbol{D}^j u(t)\right|^2 e^{-2\gamma\delta(t)}\, dm \\
&= \sum_{j=0}^{s} \int_{\Gamma_{-\gamma}} \left|\boldsymbol{F}_{t\to z}\boldsymbol{D}^j u\right|^2 dz \\
&= \sum_{j=0}^{s} \int_{\Gamma_{-\gamma}} \left|z^j\, \boldsymbol{F}_{t\mapsto z} u\right|^2 dz.
\end{aligned}$$

Since

$$\frac{1}{s!}\left(1 + |z|^2\right)^s \le \sum_{j=0}^{s} \left|z^j\right|^2 \le \left(1 + |z|^2\right)^s,$$

we conclude that

$$\frac{1}{s!}\int_{\Gamma_{-\gamma}} \left(1 + |z|^2\right)^s |\boldsymbol{F}u\,(z)|^2 dz \le \|u\|^2_{\mathcal{H}^{s,\gamma}(\bar{T})} \le \int_{\Gamma_{-\gamma}} \left(1 + |z|^2\right)^s |\boldsymbol{F}u\,(z)|^2 dz.$$

This is precisely the assertion of the lemma.

□

Since $r = \delta(t)$ is a monotone function, there is no embedding $\mathcal{H}^{s'',\gamma''}(\bar{T}) \hookrightarrow \mathcal{H}^{s',\gamma'}(\bar{T})$ for $s' \le s''$, $\gamma' \le \gamma''$. However, we get an embedding theorem if we cut off the functions in question at one of the bounds of T.

Lemma 1.5.3 *Let $r = \delta(t)$ be a monotone increasing mapping of an interval $T = (a, b)$ onto $\mathbb{R}$ and let $\omega \in C^\infty_{loc}(T)$ be equal to 1 close to a and 0 close to b. Then,*

$$\omega\ \mathcal{H}^{s'',\gamma''}(\bar{T}) \hookrightarrow \mathcal{H}^{s',\gamma'}(\bar{T})$$

for all $s' \le s''$, $\gamma' \le \gamma''$.

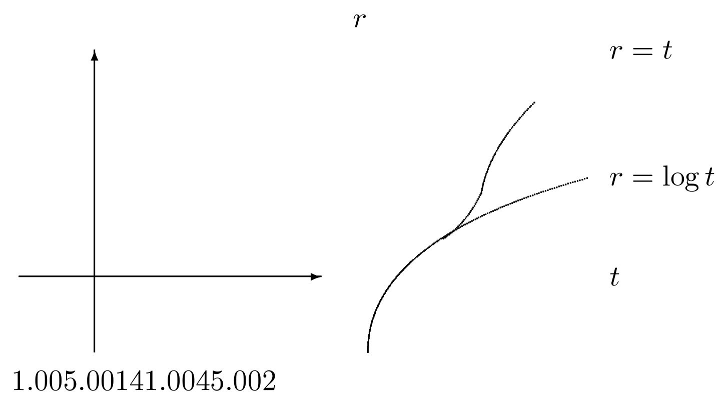


Fig. 1.1: The graph of $r = \delta(t)$**.**

Proof. The proof is straightforward.

$\square$

The following example was intended as an attempt to describe the weighted Sobolev spaces $H^{s,\gamma}(\bar{T}\times X)$ of Schulze [Sch91] in terms of a global integral transform.

Example 1.5.4 If $T = \mathbb{R}_+$ and $\delta(t) = \log t$, then

$$\begin{aligned} \boldsymbol{F}u\,(z) &= \int_0^\infty t^{-iz}\,u(t)\,\frac{dt}{t} \\ &= \mathcal{M}u\,(-iz), \qquad z \in \mathbb{C}, \end{aligned}$$

$\mathcal{M}$ being the Mellin transform. The corresponding spaces $\mathcal{H}^{s,\gamma}(\bar{\mathbb{R}}_+)$ have proved to be extremely useful in the cone theory (cf. Schulze [Sch91]).

On the other hand, if $T = \mathbb{R}$ and $\delta(t) = t$, then $\boldsymbol{F} = \mathcal{F}$ is the usual Fourier-Laplace transform on the line. Then, $\mathcal{H}^{s,0}(\mathbb{R})$ are the classical Sobolev spaces.

We now combine these examples by considering a diffeomorphism $r = \delta(t)$ of $\mathbb{R}_+ \to \mathbb{R}$, such that

$$\delta(t) = \begin{cases} \log t & \text{close to} \quad t = 0, \\ t & \text{close to} \quad t = \infty, \end{cases}$$

see Fig. 1.1.

Then, $\mathcal{H}^{s,0}(\mathbb{R}_+)$ are isomorphic (as normed spaces) to the spaces $H^{s,0}(\bar{\mathbb{R}}_+)$ widely used in the cone theory (cf. Schulze [Sch91]).

$\square$

1.6 *Pseudodifferential operators*

Lemmas 1.2.1 and 1.4.1 imply that $\boldsymbol{F}\boldsymbol{D} = z\,\boldsymbol{F}$, defined, for instance, on functions $u \in L^2(T, dm)$ with $\boldsymbol{D}u \in L^2(T, m)$. This leads to the notion of *pseudodifferential operators* with respect to the transform $\boldsymbol{F}$ via

$$\begin{aligned}\mathrm{op}_{F,\gamma}(a)\,u\,(t) &= (\boldsymbol{F}_{-\gamma})^{-1}_{z\mapsto t}\,\boldsymbol{F}_{t'\mapsto z}\left(a(t,t',z)u(t')\right) \\ &= \frac{1}{(2\pi)^2}\int_{\Gamma_{-\gamma}} dz \int_T e^{i(\delta(t)-\delta(t'))z}\,a(t,t',z)u(t')\,dm(t'), \quad (1.6.1)\end{aligned}$$

where $a \in C^\infty_{loc}(T \times T \times \Gamma_{-\gamma})$ (see Hirschmann [Hir90]).
If $a(t,t',z)$ is independent of t', then this reduces to

$$\mathrm{op}_{F,\gamma}(a)\,u\,(t) = (\boldsymbol{F}_{-\gamma})^{-1}_{z\mapsto t}\,a(t,z)\,\boldsymbol{F}_{t\mapsto z}u, \quad \text{for} \quad u \in C^\infty_{comp}(T).$$

2 *Elements of Pseudodifferential Calculus*

2.1 *The Laplace-Beltrami operator on a manifold with cusps*

Consider the surface

$$S = \{z = (z_1, z_2, z_3) : \, f(z_3) = \sqrt{z_1^2 + z_2^2},\, z_3 \geq 0\}$$

in $\mathbb{R}^3$ which is obtained by revolving the curve $z_1 = f(z_3)$ around the axis Oz_3.

We assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive C^∞ function on the half-line $\mathbb{R}_+$, with $f(0+) = 0$. Then S has a singularity at the origin unless $f'(0+) = \infty$. We will restrict our discussion to the case of singular points by requiring $\int_0^{t_0} \frac{d\theta}{f(\theta)}$ to be infinite (see Fig. 2.1).

In particular, if $f(t) = t^p$, $p \geq 1$, then the origin is a *conical point*, if $p = 1$, and a *cusp*, if $p > 1$.

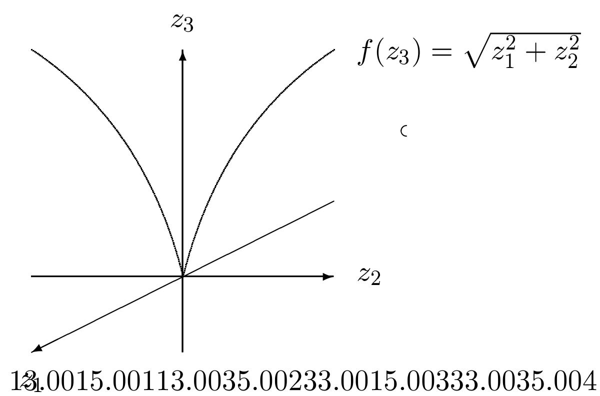
Let us parametrize the surface by

$$\begin{cases} z_1 &= f(t)\cos\phi, \\ z_2 &= f(t)\sin\phi, \\ z_3 &= t, \end{cases}$$

where ϕ runs over $[0, 2\pi)$ and $t \in [0, \infty)$.

Then, the *Riemannian metric* on the smooth part of S is given in the coordinates (t, ϕ) by

$$dz_1^2 + dz_2^2 + dz_3^2 = \left(1 + (f'(t))^2\right)dt^2 + (f(t))^2\,d\phi^2$$



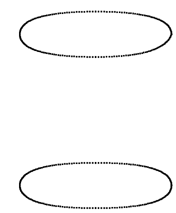


Fig. 2.1: A surface with a singular point at the origin.

or, in *tensor form*,

$$\begin{aligned} g &= (g_{ij})_{\substack{i=1,2\\ j=1,2}} \\ &= \begin{pmatrix} 1+(f'(t))^2 & 0 \\ 0 & (f(t))^2 \end{pmatrix}. \end{aligned}$$

Note that this metric is degenerate at the singular point.

Denote by Δ the *Laplace-Beltrami operator* on the smooth part of S with respect to the metric g. In order to compute Δ, we note that the inverse of the matrix (g_{ij}) is the matrix

$$\left(g^{ij}\right) = \begin{pmatrix} \frac{1}{1+(f'(t))^2} & 0 \\ 0 & \frac{1}{(f(t))^2} \end{pmatrix}.$$

Then, a trivial verification shows that the Laplace-Beltrami operator is given in the coordinates (t, ϕ) as

$$\begin{aligned} \Delta &= \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{2} \partial_i \sqrt{\det g}\, g^{ij}\, \partial_j \\ &= g^{11}\partial_1^2 + \frac{1}{\sqrt{\det g}}\, \partial_1 \left(\sqrt{\det g}\, g^{11}\right) \partial_1 + g^{22}\partial_2^2 \\ &= \frac{1}{g_{22}} \left(\frac{g_{22}}{g_{11}}\, \partial_1^2 + \sqrt{\frac{g_{22}}{g_{11}}} \left(\partial_1 \sqrt{\frac{g_{22}}{g_{11}}}\right) \partial_1 + \partial_2^2 \right), \end{aligned}$$

where $\partial_1 = \partial/\partial t$ and $\partial_2 = \partial/\partial \phi$.

Since

$$\begin{aligned} g_{22}\partial_1^2 &= (\sqrt{g_{22}}\partial_1)^2 - (\partial_1\sqrt{g_{22}})\,(\sqrt{g_{22}}\partial_1)\,, \\ \sqrt{\frac{g_{22}}{g_{11}}}\left(\partial_1\sqrt{\frac{g_{22}}{g_{11}}}\right)\partial_1 &= \frac{\partial\sqrt{g_{22}}}{g_{11}}\,(\sqrt{g_{22}}\partial_1) + \sqrt{\frac{g_{22}}{g_{11}}}\left(\partial_1\frac{1}{\sqrt{g_{11}}}\right)(\sqrt{g_{22}}\partial_1)\,, \end{aligned}$$

we may rewrite this as

$$\begin{aligned} \Delta &= \frac{1}{g_{22}}\left(\frac{1}{g_{11}}\,(\sqrt{g_{22}}\partial_1)^2 + \sqrt{\frac{g_{22}}{g_{11}}}\left(\partial_1\frac{1}{\sqrt{g_{11}}}\right)(\sqrt{g_{22}}\partial_1) + \partial_2^2\right) \\ &= \frac{1}{(f(t))^2}\left(\frac{1}{1+(f'(t))^2}\,(f(t)\,\partial/\partial t)^2 - \frac{f(t)f'(t)f''(t)}{(1+(f'(t))^2)^2}\,(f(t)\,\partial/\partial t) + \Delta_\phi\right) \end{aligned}$$

where Δ_ϕ is the Laplace-Beltrami operator on the unit circle (the *cross-section* of S close to the singular point).

We have thus arrived at a totally characteristic differential operator related to the mapping $r = \delta(t)$ of $\mathbb{R}_+ \to \mathbb{R}$, where

$$\delta(t) = \int_{t_0}^{t} \frac{d\theta}{f(\theta)}$$

with any fixed $t_0 > 0$.

The important point to note here is the form of δ. Namely, since $\delta'(t) = \frac{1}{f(t)}$ is positive for all $t > 0$, it follows that the function $r = \delta(t)$ is monotonically increasing and so one-to-one. However, the image of $\mathbb{R}_+$ by this function may be different from $\mathbb{R}$. While $\delta(0) = \int_{t_0}^{0} \frac{d\theta}{f(\theta)} = -\infty$, the case of power-like cusps $f(t) = t^p$, $p > 1$, gives $\delta(t) = \frac{1}{1-p}\left(\frac{1}{t^{p-1}} - \frac{1}{t_0^{p-1}}\right)$ and so the image of $\mathbb{R}_+$ by δ is $\left(-\infty, \frac{1}{p-1}\frac{1}{t_0^{p-1}}\right)$. Thus, the Fourier-Laplace transform of Section 1.2 needs making more precise to apply in this situation. Namely, what is merely important for us is the behavior of the mapping δ close to the singular point $t = 0$. Thus, we may correct the mapping $r = \delta(t)$ away from an infinitesimal interval $(0, \varepsilon)$, $\varepsilon > 0$, in order to arrive at a diffeomorphism $r = \tilde{\delta}(t)$ of the half-axis $\mathbb{R}_+$ onto the whole axis $\mathbb{R}$.

2.2 *Typical differential operators on manifolds with singular points*

When *blown up*, a manifold with singular points has locally the form of a cylinder $\mathcal{C} = \bar{T} \times X$ close to a singular point, where $T = \mathbb{R}_+$ and X is a smooth compact manifold of dimension n without boundary. We also call $\mathcal{C}$ the *stretched cone*. Moreover, the "push-forwards" of the vector fields on

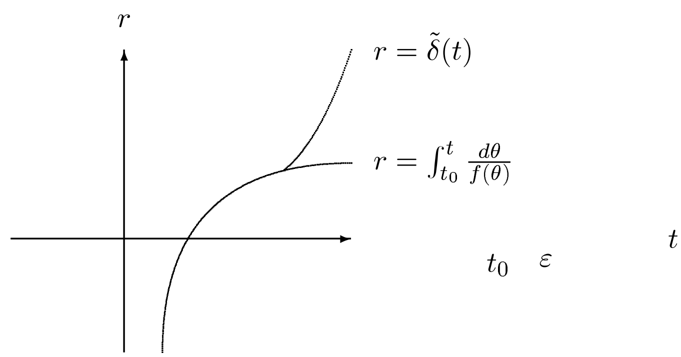


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Fig. 2.2: A modified diffeomorphism $r = \tilde{\delta}(t)$**.**

the manifold, which originate with the singular point, splits into a *totally characteristic* derivative $\boldsymbol{D} = \frac{1}{\delta'(t)} \frac{1}{i}\, d/dt$ along the half-axis T and smooth vector fields along the base X.

The local algebra of pseudodifferential operators close to the singular point is completely determined by the "germ" of $r = \delta(t)$ at $t = 0$. Our basic assumption is the following: the "germ" of δ at $t = 0$ can be represented by a smooth diffeomorphism of T onto $\mathbb{R}$, with a positive derivative. We continue to write $r = \delta(t)$ for this representative.

Clearly, the form of $r = \delta(t)$ in an infinitesimal neighborhood of $t = 0$ depends on the geometric nature of the singular point in question. In particular, if $\delta(t) = \log t$ near $t = 0$, then the singular point is a conical point. On the other hand, if $\delta(t) = \frac{1}{1-p} \frac{1}{t^{p-1}}$ close to $t = 0$, with $p > 1$, then the singular point is a power-like cusp. Yet another case is $\delta(t) = -e^{\frac{1}{t}}$ for $t \in (0, \varepsilon)$, that corresponds to exponential cusps, etc.

By the above said, when constructing a local algebra close to the singular point, we have to begin with *typical* differential equations on $\mathcal{C}$ which are of the form

$$(\delta'(t))^m \sum_{j=0}^{m} A_j(t)\, \boldsymbol{D}^j u(t) = f(t), \quad t > 0, \tag{2.2.1}$$

where $A_j(t) \in C^\infty_{loc}(\mathbb{R}_+, \mathrm{Diff}^{m-j}(X))$, $j = 0, 1, \ldots, m$. We did not specify the variable x in (2.2.1), keeping in mind that, for fixed t, the value $u(t)$ is a distribution on X and $A_j(t)$ act as differential operators with respect to the x-variables.

The relevant weighted Sobolev spaces in a neighborhood of the singular point are defined as follows.

Fix a family of order reductions $\Lambda^s(\lambda)$, $s \in \mathbb{R}$, on X depending on a parameter $\lambda \in \mathbb{R}$. Now, for $s, \gamma \in \mathbb{R}$, we let $\mathcal{H}^{s,\gamma}(\mathcal{C})$ be the completion of $C^\infty_{comp}(\mathbb{R}_+ \times X)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{C})} = \left(\int_{\Gamma_{-\gamma}} \|\Lambda^s(\Re z)\, \boldsymbol{F}u\,(z)\|^2_{L^2(X)}\, dz \right)^{\frac{1}{2}}. \tag{2.2.2}$$

Lemma 2.2.1 *A function $u \in \mathcal{D}'(\mathbb{R}_+ \times X)$ belongs to the space $\mathcal{H}^{s,\gamma}(\mathcal{C})$ if and only if the "pull-back" $t^*u = u(\delta^{-1}(\log \varrho))$ of u under the diffeomorphism $\varrho \mapsto \delta^{-1}(\log \varrho)$ of $\mathbb{R}_+$ belongs to the corresponding cone weighted Sobolev space.*

Proof. Indeed, if $\varrho = e^{\delta(t)}$, then an easy computation shows that

$$\begin{array}{rcl} \frac{1}{i}\frac{1}{\delta'(t)}\frac{d}{dt} & = & \frac{1}{i}\varrho\frac{d}{d\varrho}, \\ d\delta(t) & = & \frac{d\varrho}{\varrho}, \end{array} \tag{2.2.3}$$

whence

$$\boldsymbol{F}u\,(z) = \mathcal{M}\,(t^*u)\,(-iz), \qquad z \in \mathbb{C},$$

$\mathcal{M}$ being the Mellin transform (cf. Example 1.5.4). This gives the desired conclusion when substituted into (2.2.2).

□

Since the derivative $\delta'(t)$ is different from zero for $t > 0$, equation (2.2.1) is equivalent to the equation

$$Au(t) = (\delta'(t))^{-m}\, f(t), \qquad t > 0,$$

where

$$A = \sum_{j=0}^{m} A_j(t)\, \boldsymbol{D}^j. \tag{2.2.4}$$

From what has already been proved in Sections 1.4 and 1.6, it follows that the operators (2.2.4) behave properly in the scale (2.2.2). Hence, when one treats typical differential equations in a neighborhood of the singular point, there naturally appears an additional scale of norms

$$\|u\|_{\mathcal{H}^{s,\gamma,\mu}(\mathcal{C})} = \|(\delta')^\mu\, u\|_{\mathcal{H}^{s,\gamma}(\mathcal{C})}, \tag{2.2.5}$$

with parameters $s, \gamma, \mu \in \mathbb{R}$.

Lemma 2.2.2 *Let $\omega \in C^\infty_{comp}(\bar{\mathbb{R}}_+)$. Then, for each real numbers $s' \leq s''$, $\gamma' \leq \gamma''$ and $\mu' \leq \mu''$, we have a continuous embedding*

$$\omega\ \mathcal{H}^{s'',\gamma'',\mu''}(\mathcal{C}) \hookrightarrow \mathcal{H}^{s',\gamma',\mu'}(\mathcal{C}).$$

Proof. This is obvious because of (2.2.2) and (2.2.5).

□

We now return to the "transcedental" change of variables $\varrho = e^{\delta(t)}$ used in the proof of Lemma 2.2.1. The first equality of (2.2.3) shows that the "pull-back" of the operator A under the diffeomorphism $\varrho \mapsto \delta^{-1}(\log \varrho)$ of $\mathbb{R}_+$ is the differential operator

$$t^\sharp A = \sum_{j=0}^{m} t^* A_j(\varrho) \left(\frac{1}{i} \varrho \frac{d}{d\varrho} \right)^j , \tag{2.2.6}$$

so that $t^*(Au) = t^\sharp A\ (t^* u)$. Thus, we deduce that the operator A transforms, under the change of variables $t = \delta^{-1}(\log \varrho)$, into a *Fuchs-type* operator $t^\sharp A$. Conversely, each Fuchs-type operator on $\mathbb{R}_+ \times X$ transforms, under the change of variables $\varrho = e^{\delta(t)}$, into an operator of the form (2.2.4).

It is worth pointing out that $t = \delta^{-1}(\log \varrho)$ is a homeomorphism of the closed semi-axis $\bar{\mathbb{R}}_+$, with the inverse $\varrho = e^{\delta(t)}$. Therefore, the coefficients $t^* A_j\ (\varrho) = A_j(\delta^{-1}(\log \varrho))$ of $t^\sharp A$ are continuous up to $\varrho = 0$ if and only if so are the coefficients of A. However, as the change of variables $t = \delta^{-1}(\log \varrho)$ is not smooth up to $\varrho = 0$ in general, the coefficients of $t^\sharp A$ need not be smooth up to $\varrho = 0$, even if so are the coefficients of A.

Another way of stating this observation is to say that topologically all the one-point singularities are equivalent. However, having fixed a geometric type of the singular point, we are allowed to use only those homeomorphisms of $\bar{\mathbb{R}}_+$ close to $t = 0$ which preserve the geometric structure of the singularity.

2.3 *Kernel cut-off*

The Mellin calculus in the form developed by Schulze [Sch91,Sch94] gives also a general framework for the analysis on manifolds with arbitrary one-point singularities. The basic idea is the following. Outside the singular point, one uses the standard pseudodifferential calculus and the standard Sobolev spaces. Near the singularity, however, the analysis relies on the operators constructed by use of the transform $\boldsymbol{F}$, and the spaces $\mathcal{H}^{s,\gamma}(\mathcal{C})$.

More precisely, one considers the operators (2.3.1) on the semiaxis whose symbols take their values in the algebra of all pseudodifferential operators on X.

In the sequel, let $m, \gamma \in \mathbb{R}$ be fixed. Given $a \in C^\infty_{loc}(\mathbb{R}_+ \times \mathbb{R}_+, \Psi^m(X; \Gamma_{-\gamma}))$, we shall write $a = a(t, t', z)$, where $z = \tau - i\gamma$ indicates the variable in $\Gamma_{-\gamma}$. For t, t', z fixed, this $a(t, t', z)$ is a pseudodifferential operator on X.

Definition 2.3.1 *Suppose that* $a \in C^\infty_{loc}(\mathbb{R}_+ \times \mathbb{R}_+, \Psi^m(X; \Gamma_{-\gamma}))$. *The* *operator* $\mathrm{op}_{F,\gamma}(a)$ *with the symbol* a *on* $C^\infty_{comp}(\mathbb{R}_+ \times X) = C^\infty_{comp}(\mathbb{R}_+, C^\infty(X))$ *is*

$$\mathrm{op}_{F,\gamma}(a)u\,(t) = \frac{1}{(2\pi)^2}\int_{\Gamma_{-\gamma}} dz \int_0^\infty e^{i(\delta(t)-\delta(t'))z}\, a(t,t',z)u(t')\,dm(t'). \tag{2.3.1}$$

The right-hand side of (2.3.1) has to be understood as an iterated integral. We did not specify the variable x in (2.3.1), understanding that, for fixed t', the value $u(t')$ is in $C^\infty(X)$ and that $a(t,t',z)$ acts as a pseudodifferential operator with respect to the x-variables.

If X consists of one point, then (2.3.1) reduces to what has already been defined in Section 1.6. In general, we have

$$\mathrm{op}_{F,\gamma+\beta}(a) = e^{\beta\delta(t)}\,\mathrm{op}_{F,\gamma}(a(t,t',z-i\beta))\,e^{-\beta\delta(t)}, \tag{2.3.2}$$

which is due to Lemma 1.4.2.

Like pseudodifferential double symbols, the double symbols in (2.3.1) are not uniquely determined.

Example 2.3.2 *It is immediate from integration by parts in (2.3.1) that*

$$\mathrm{op}_{F,\gamma}\left((i\,\partial/\partial z)^j a\right) = \mathrm{op}_{F,\gamma}\left((\delta(t)-\delta(t'))^j a\right).$$

□

Given any $a \in C^\infty_{loc}(\mathbb{R}_+ \times \mathbb{R}_+, \Psi^m(X;\Gamma_{-\gamma}))$, we will have a continuous map

$$\mathrm{op}_{F,\gamma}(a) : C^\infty_{comp}(\mathbb{R}_+ \times X) \to C^\infty_{loc}(\mathbb{R}_+ \times X).$$

Smoothness of a up to zero yields the continuity of $\mathrm{op}_{F,\gamma}(a)$ on the weighted Sobolev spaces. The preceding Example 2.3.2, however, shows that the smoothness is not necessary.

Proposition 2.3.3 *Let* $a \in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^m(X;\Gamma_{-\gamma}))$. *For each* $s \in \mathbb{R}$ *and* $\omega_1, \omega_2 \in C^\infty_{comp}(\bar{\mathbb{R}}_+)$, *there is a continuous extension*

$$\omega_1\,\mathrm{op}_{F,\gamma}(a)\,\omega_2 : \mathcal{H}^{s,\gamma}(\mathcal{C}) \to \mathcal{H}^{s-m,\gamma}(\mathcal{C}).$$

Proof. See Schulze [Sch91, 1.2.3].

□

We also mention the following results. The first of them shows that, just as in the case of pseudodifferential operators, one has *asymptotic summation* of symbols.

Proposition 2.3.4 *Let $(m_j)_{j=1,2,\dots}$ be a sequence in $\mathbb{R}$ tending to $-\infty$, let $a_j \in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^{m_j}(X;\Gamma_{-\gamma}))$, and let $m = \max m_j$. Then there exists a symbol $a \in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^m(X;\Gamma_{-\gamma}))$ with $a \sim \sum_{j=1}^\infty a_j$, i.e., for any $N \in \mathbb{Z}_+$ there is a J such that $a - \sum_{j=1}^J a_j \in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^{m-N}(X;\Gamma_{-\gamma}))$. Moreover, a is unique modulo $C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^{-\infty}(X;\Gamma_{-\gamma}))$.*

Proof. See Schulze [Sch91, 1.2.4].

□

Moreover, one obtains smoothing operators by a special analytic procedure that Schulze [Sch91] calls "kernel cut-off."

For the standard pseudodifferential operators, we may calculate the *Schwartz kernels* in terms of the amplitude functions. For operators (2.3.1), we can analogously write

$$\mathrm{op}_{F,\gamma}(a)u\,(t) = \frac{1}{2\pi}\int_0^\infty k(a)\left(t,t',\delta^{-1}(\delta(t)-\delta(t'))\right)u(t')\,dm(t')$$

whenever $u \in C^\infty_{comp}(\mathbb{R}_+, C^\infty(X))$, where

$$k(a)\,(t,t',\varsigma) = \boldsymbol{F}^{-1}_{-\gamma_{z\mapsto\varsigma}} a(t,t',z)$$

is interpreted in the distributional sense. Obviously, $k(a)$ is a C^∞ function of $(t,t') \in \mathbb{R}_+ \times \mathbb{R}_+$ with values in $\mathcal{D}'(\mathbb{R}_+, \Psi^m(X))$. For fixed t and t', the singular support of $k(a)\,(t,t',\varsigma)$ is contained in the only point $\varsigma_0 \in \mathbb{R}_+$ where $\delta(\varsigma_0) = 0$ (cf. Schulze [Sch91, 1.2.4]). Moreover, we have $a(t,t',z) = \boldsymbol{F}_{-\gamma_{\varsigma\mapsto z}} k(a)\,(t,t',z)$.

Proposition 2.3.5 *Assume that $\omega \in C^\infty_{comp}(\mathbb{R}_+)$ is a cut-off function close to ς_0 (i.e., $\omega(\varsigma) \equiv 1$ near $\varsigma = \varsigma_0$). Given any $a \in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^m(X;\Gamma_{-\gamma}))$, let*

$$\begin{aligned} a_1(t,t',z) &= \boldsymbol{F}_{\varsigma\mapsto z}\left(\omega(\varsigma)\boldsymbol{F}^{-1}_{-\gamma_{z\mapsto\varsigma}} a(t,t',z)\right), \\ a_2(t,t',z) &= \boldsymbol{F}_{\varsigma\mapsto z}\left((1-\omega(\varsigma))\boldsymbol{F}^{-1}_{-\gamma_{z\mapsto\varsigma}} a(t,t',z)\right). \end{aligned}$$

Then

$$\begin{aligned} a_1 &\in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^m(X;\Gamma_{-\gamma})), \\ a_2 &\in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^{-\infty}(X;\Gamma_{-\gamma})). \end{aligned}$$

Proof. See Schulze [Sch91, 1.2.4].

□

2.4 *Algebras without asymptotics*

We are now able to describe an *algebra without asymptotics* on a manifold M with singular points. We shall not attempt any discussion of the rigorous definition of such a manifold. The important point to note here is the form of transition diffeomorphisms close to a singular point - they have to preserve the geometric type of the singularity. We shall instead deal from the very beginning with the "stretched object" associated to M.

Proposition 2.4.1 *For any manifold M with singular points S there is a smooth manifold with boundary $\mathcal{M}$ such that:*
1) $M \setminus S$ is diffeomorphic to $\mathcal{M} \setminus \partial\mathcal{M}$; and
2) there is a neighborhood O of S in M and a collar neighborhood $\mathcal{O} \simeq \partial\mathcal{M} \times [0,1)$ of $\partial\mathcal{M}$ in $\mathcal{M}$ such that $O \setminus S$ is diffeomorphic to $\partial\mathcal{M} \times (0,1)$.

Proof. We construct $\mathcal{M}$ by replacing, for every one-point singularity $p \in S$, a neighborhood O of p by $[0,1) \times X$ via gluing with any one of the diffeomorphisms $O \setminus \{p\} \to (0,1) \times X$ of which equivalence classes determine the structure of M close to p. We even get $\partial\mathcal{M} = \cup_{p \in S} X_p$, the subscript p pointing to the dependence of X on p.

□

In the sequel we tacitly assume that M has only one singular point; this does not include any restriction of generality.

We begin with definition of weighted Sobolev spaces $H^{s,\gamma}(\mathcal{M})$ (cf. Sections 1.5, 2.3). We shall say that a function or distribution is supported *close to the boundary of* $\mathcal{M}$ if it vanishes outside the part of $\mathcal{M}$ that is identified with $[0,1) \times X$. Fix a smooth function ω on $\mathcal{M}$ which is supported close to the boundary and equal to 1 in a smaller neighborhood of the boundary. Given a distribution $u \in \mathcal{D}'(\mathring{\mathcal{M}})$, we can write it as $u = u_1 + u_2$ with $u_1 = \omega u$ supported close to the boundary and $u_2 = (1-\omega)u$ supported away from the boundary. We shall say that $u \in H^{s,\gamma}(\mathcal{M})$, provided that $u_1 \in \mathcal{H}^{s,\gamma}(\mathcal{C})$ and $u_2 \in H^s_{loc}(\mathring{\mathcal{M}})$. It is easy to see that this definition is independent of the particular choice of ω. We can topologize $H^{s,\gamma}(\mathcal{M})$ as a Hilbert space, using the Hilbert space structures on $\mathcal{H}^{s,\gamma}(\mathcal{C})$ and $H^s(\mathbb{R}^{1+n})$.

In the sequel, we use the notion of a *weight datum* of an algebra without asymptotics. By such a datum is meant any couple $w = (\gamma, \beta)$ of numbers $\gamma, \beta \in \mathbb{R}$.

For a weight datum $w = (\gamma, \beta)$, denote by $\mathrm{Alg}^{-\infty}(\mathcal{M}, w)$ the set of all operators $S_F : C^\infty_{comp}(\mathring{\mathcal{M}}) \to \mathcal{D}'(\mathring{\mathcal{M}})$ such that, for all $s \in \mathbb{R}$, there is a continuous extension $S_F : H^{s,\gamma}(\mathcal{M}) \to H^{\infty,\beta}(\mathcal{M})$.

Given any m and weight datum $w = (\gamma, \beta)$, let $\mathrm{Alg}^m(\mathcal{M}, w)$ be the space of all operators $A : C^\infty_{comp}(\mathring{\mathcal{M}}) \to \mathcal{D}'(\mathring{\mathcal{M}})$ of the form $A = A_F + A_{\mathcal{F}} + S_F$, where

A_F is an operator based on the transform $\boldsymbol{F}$ close to the boundary, i.e., there are $\varphi_0, \psi_0 \in C^\infty(\mathcal{M})$ supported close to the boundary of $\mathcal{M}$, and a symbol $a \in C^\infty_{loc}(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+, \Psi^m(X; \Gamma_{-\gamma}))$ such that $A_F = e^{(\beta-\gamma)\delta(t)}\, \varphi_0 \operatorname{op}_{F,\gamma}(a)\, \psi_0$;

$A_{\mathcal{F}}$ is a pseudodifferential operator supported away from the boundary, i.e., there are functions $\varphi_\infty, \psi_\infty$ vanishing in a neighborhood of the boundary of $\mathcal{M}$, and a symbol $a \in \mathcal{S}^m(\mathring{\mathcal{M}})$ such that $A_{\mathcal{F}} = \varphi_\infty \operatorname{op}_{\mathcal{F}}(a)\, \psi_\infty$;

S_F is an operator in $\mathrm{Alg}^{-\infty}(\mathcal{M}, w)$.

The collection of all the spaces $\mathrm{Alg}^m(\mathcal{M}, w)$ with $m, \gamma, \beta \in \mathbb{R}$ is the algebra without asymptotics. With the help of Proposition 2.3.3. it is easy to see that an operator $A \in \mathrm{Alg}^m(\mathcal{M}, w)$ induces a continuous mapping

$$A : H^{s,\gamma}(\mathcal{M}) \to H^{s-m,\beta}(\mathcal{M})$$

for any $s \in \mathbb{R}$.

It is not so trivial that the algebra without asymptotics is an "algebra" in the sense that, for all $m_1, m_2, \gamma \in \mathbb{R}$, the composition of operators induces a continuous multiplication

$$\mathrm{Alg}^{m_2}(\mathcal{M}, w_2) \times \mathrm{Alg}^{m_1}(\mathcal{M}, w_1) \to \mathrm{Alg}^{m_1+m_2}(\mathcal{M}, w),$$

where $w_1 = (\gamma, \beta)$, $w_2 = (\beta, \alpha)$ and $w = (\gamma, \alpha)$.

It follows from the mapping properties that the operators in $\mathrm{Alg}^{-\infty}(\mathcal{M}, w)$ form an "ideal" in the sense that the above multiplication restricts to continuous maps

$$\begin{array}{l}\mathrm{Alg}^{-\infty}(\mathcal{M}, w_2) \times \mathrm{Alg}^{m_1}(\mathcal{M}, w_1) \to \mathrm{Alg}^{-\infty}(\mathcal{M}, w),\\ \mathrm{Alg}^{m_2}(\mathcal{M}, w_2) \times \mathrm{Alg}^{-\infty}(\mathcal{M}, w_1) \to \mathrm{Alg}^{-\infty}(\mathcal{M}, w),\end{array}$$

w_j being as above.

Remark 2.4.2 *If* $m = 0$, *then* $\mathrm{Alg}^0(\mathcal{M}, w)$ *is an algebra in the usual sense, and* $\mathrm{Alg}^{-\infty}(\mathcal{M}, w)$ *is an ideal in this algebra.*

Finally, the important point to note here is the following relation between the operators based on the transform $\boldsymbol{F}$, and the usual pseudodifferential operators.

Proposition 2.4.3 *Suppose* ψ_F *is an operator as above. Then, for any functions* $\varphi_\infty, \psi_\infty \in C^\infty_{comp}(\mathring{\mathcal{M}})$, *there is an operator* $\psi_{\mathcal{F}} \in \Psi^m(\mathring{\mathcal{M}})$ *supported in the interior of* $\mathcal{M}$ *such that* $\varphi_\infty\, \psi_F\, \psi_\infty = \psi_{\mathcal{F}}$.

Proof. The reader did certainly recognize that this assertion is none other than the change of variables on the space of usual pseudodifferential operators.

□

Under the hypotheses of Proposition 2.4.3, if moreover the supports of φ_∞ and ψ_∞ are disjoint, then $\varphi_\infty \psi_F \psi_\infty \in \mathrm{Alg}^{-\infty}(\mathcal{M}, w)$.

From what has been said at the end of Section 2.2, it may be concluded that the algebra $\mathrm{Alg}^m(\mathcal{M}, w)$ is an *extension* of the cone algebra in the sense that, close to the singular points, the pull-back of the cone algebra under the mapping $t \mapsto \varrho = e^{\delta(t)}$ is a *proper* subalgebra of $\mathrm{Alg}^m(\mathcal{M}, w)$. However, the norm closures of both algebras coincide because each operator corresponding to the conical setting can be approximated by operators with "coefficients" constant near singular points (cf. Mantlik [Man95]).

2.5 *Asymptotics*

In general, asymptotic expansions of solutions to equation (2.2.1) in a neighborhood of the singular point seem to be controlled by the scale of norms (2.2.5), i.e., the remainders, when cut off, lie in the spaces $\mathcal{H}^{s,\gamma,\mu}(\mathcal{C})$ with μ large enough.

In the last two sections of this chapter we will restrict the discussion to the situation when the asymptotic expansions of solutions may be controlled by the scale (2.2.2). Roughly speaking, this corresponds to the case where the standard parametrix construction for elliptic operators A leads not only to a gain in the power of $(\delta'(t))^{-1}$ but also to a gain in the power of $e^{\delta(t)}$. To ensure this, we need an additional assumption on smoothness of the coefficients of A close to $t = 0$, namely, that $A_j(\delta^{-1}(\log \varrho))$ are C^∞ up to $\varrho = 0$, for $j = 0, 1, \ldots, m$. This assumption is satisfied, in particular, if all the coefficients A_j are constant in a small interval $[0, \epsilon)$, $\epsilon > 0$.

To begin with, we make more precise the definition of the spaces $H^{s,\gamma}(\mathcal{C})$ on the infinite stretched cone $\mathcal{C} = \bar{\mathbb{R}}_+ \times X$. The analysis on $\mathcal{C}$ employs the transform $\boldsymbol{F}$ and weights only near the base $t = 0$. The weight factor $e^{-2\gamma\delta(t)}$ in Definition 1.5.1 affects the space also for $t \to \infty$. It is advantageous to introduce another variant of spaces on $\mathcal{C}$ that refers to the Fourier-Laplace transform and to weight factors only near $t = 0$. The idea is to multiply $\mathcal{H}^{s,\gamma}(\mathcal{C})$ by a cut-off function ω and then to add $(1-\omega)\,\mathcal{H}^s(\mathcal{C})$, where $\mathcal{H}^s(\mathcal{C})$ is the usual Sobolev space on $\mathcal{C}$ properly interpreted [2]. Thus, for $s, \gamma \in \mathbb{R}$, we set

$$H^{s,\gamma}(\mathcal{C}) = \omega\, \mathcal{H}^{s,\gamma}(\mathcal{C}) + (1-\omega)\, \mathcal{H}^s(\mathcal{C}),$$

[2]The proper definition of $\mathcal{H}^s(\mathcal{C})$ should perhaps take into account the behavior of $r = \delta(t)$ for $t \to \infty$.

with ω a fixed cut-off function on $\bar{\mathbb{R}}_+$. We topologize $H^{s,\gamma}(\mathcal{C})$ by the norm

$$\|u\|_{H^{s,\gamma}(\mathcal{C})} = \inf_{u=\omega u_1+(1-\omega)u_2} \left(\|\omega u_1\|_{\mathcal{H}^{s,\gamma}(\mathcal{C})} + \|(1-\omega)u_2\|_{\mathcal{H}^{s}(\mathcal{C})}\right).$$

It is easy to check that the space $H^{s,\gamma}(\mathcal{C})$ is independent of the particular choice of the cut-off function ω up to an equivalent norm. Moreover, the topology of $H^{s,\gamma}(\mathcal{C})$ is still induced by a Hilbert inner product [3].

Having disposed of this preliminary step, we can now return to spaces with asymptotics. The following assertion is of basic interest in the analysis of the conormal asymptotics of distributions on $\mathcal{C}$ for $t \to 0$.

Lemma 2.5.1 *Let $\omega \in C^\infty_{comp}(\bar{\mathbb{R}}_+)$ be a cut-off function with respect to the origin, and let $p \in \mathbb{C}$, $\mu \in \mathbb{Z}_+$. Then the Fourier-Laplace transform of the function $\omega(t)\, e^{ip\delta(t)}\, (\delta(t))^\mu$ extends to a meromorphic function in the whole complex plane with exactly one pole, of multiplicity $\mu+1$, at p.*

Proof. We first prove a reduced form of the lemma, namely, assume that both p and μ are zero.

For $\operatorname{im} z > 0$, write

$$\boldsymbol{F}\omega\,(z) = \frac{1}{z}\, u(z),$$

where $u(z) = \boldsymbol{F}\,(\boldsymbol{D}\omega)\,(z)$. Since $\boldsymbol{D}\omega \in C^\infty_{comp}(\mathbb{R}_+)$, we conclude that u is an entire function of $z \in \mathbb{C}$. Moreover,

$$\begin{aligned} u(0) &= \frac{1}{i} \int_0^\infty \frac{d\omega}{dt}\, dt \\ &= \frac{-1}{i}\, \omega(0) \\ &= i. \end{aligned}$$

Hence it follows that $\frac{1}{z}\, u(z)$ is a meromorphic function with a single pole located at $z = 0$ and of multiplicity $\mu = 1$.

For the general case, take $z \in \mathbb{C}$ with $\operatorname{im}(z-p) > 0$. An easy computation shows that

$$\begin{aligned} &\boldsymbol{F}_{t\mapsto z}\left(\omega(t)\, e^{ip\delta(t)}\, (\delta(t))^\mu\right) \\ &= \frac{1}{2\pi} \int_0^\infty e^{-iz\delta(t)} \left(\omega(t)\, e^{ip\delta(t)}\, (\delta(t))^\mu\right) dm(t) \\ &= (i\,\partial/\partial z)^\mu\, \boldsymbol{F}\omega\,(z-p) \\ &= (i\,\partial/\partial z)^\mu \left(\frac{1}{z-p}\, u(z-p)\right). \end{aligned}$$

[3] In the same way, starting with $\mathcal{H}^{s,\gamma,\mu}(\mathcal{C})$, we can define the spaces $H^{s,\gamma,\mu}(\mathcal{C})$ for all $s, \gamma, \mu \in \mathbb{R}$.

By the above, the right-hand side here is a meromorphic function with a single pole located at $z = p$ and of multiplicity $\mu + 1$. Hence the lemma follows.

□

The proof above gives more, namely, given any point $p \in \mathbb{C}$ and excision function $\chi(z)$ of p, the function $\chi \boldsymbol{F} \left(\omega(t)\, e^{ip\delta(t)}\, (\delta(t))^{\mu}\right)$ is rapidly decreasing on each weight line $\Gamma_{-\gamma}$, uniformly in γ on finite intervals.

Remark 2.5.2 *It is easily seen from the proof that the difference*

$$\boldsymbol{F}_{t \mapsto z} \left(\omega(t)\, e^{ip\delta(t)}\, (\delta(t))^{\mu}\right) - \frac{(-i)^{\mu}}{\mu!} \frac{1}{(z-p)^{\mu+1}}$$

extends to an entire function.

Fix a cut-off function $\omega \in C^{\infty}_{comp}(\bar{\mathbb{R}}_+)$ with respect to $t = 0$. Given a point $p \in \mathbb{C}$, it follows from Lemma 2.2.1 that the asymptotics

$$u(t,x) = \omega(t)\, e^{ip\delta(t)}\, (\delta(t))^{\mu}\; u_1(x)$$

lies in $H^{s,\gamma}(\mathcal{C})$, for each $\mu \in \mathbb{Z}_+$ and $u_1 \in H^s(X)$, if and only if $\operatorname{im} p < -\gamma$. We want to introduce subspaces $H^{s,\gamma}_{as}(\mathcal{C})$ consisting of functions $u \in H^{s,\gamma}(\mathcal{C})$ which have a gain in the weight up to elements of some finite-dimensional subspace of asymptotics. To make this more precise we give the following definitions.

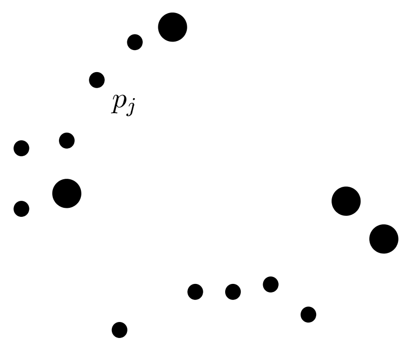
A *weight datum* $w = (\gamma, (-l, 0])$ consists of a number $\gamma \in \mathbb{R}$ and an interval $(-l, 0]$ on the real axis. We consider *finite* weight intervals (i.e., those with $0 < l < \infty$) as well as the infinite one (i.e., $(-\infty, 0]$).

By an asymptotic type associated with the weight datum w is meant any collection $as = ((p_j, \mu_j, \Sigma_j))_{j=0,1,\dots,J}$, where

- (p_j) are complex numbers in the strip $-\gamma - l < \operatorname{im} p_j < -\gamma$;
- (μ_j) are non-negative integers; and
- (Σ_j) are finite-dimensional subspaces of $C^{\infty}(X)$.

For the infinite weight interval, the value $J = \infty$ is also admissible. However, we assume that each strip $\{c' \le \operatorname{im} z \le c''\}$ contains only a finite number of the points p_j. Thus, a condition on (p_j) is $\operatorname{im} p_j \to -\infty$ when $j \to \infty$.

Definition 2.5.3 *Given a weight datum w, we denote by* $\operatorname{As}(w)$ *the set of all asymptotic types associated with w. Asymptotic types of such a kind are called discrete.*



24.0010.00134.0015.002 34.0015.00339.0015.004 39.0015.00544.0016.006 44.001

Fig. 2.3: A *carrier* **of an asymptotics for the weight datum** $w = (\gamma, (-l, 0])$**.**

By the above said, when working with the spaces $H^{s,\gamma}(\mathcal{C})$, we have to consider weight data $w = (\gamma, (-l, 0])$. Fix such a datum.

Given any asymptotic type $as \in \mathrm{As}\,(w)$, we denote by $\mathcal{A}_{as}$ the finite-dimensional space spanned by the functions

$$\Big(\omega(t)\, e^{ip_j\delta(t)}\, (\delta(t))^{\mu}\, c_{j\mu}(x)\Big)_{\substack{j=0,1,\ldots,J \\ \mu=0,1,\ldots,\mu_j}},$$

where $c_{j\mu} \in \Sigma_j$. (The cut-off function $\omega(t)$ is kept fixed.)

From what has already been proved it follows that $\mathcal{A}_{as} \subset H^{s,\gamma}(\mathcal{C})$, for all $s \in \mathbb{R}$, and $\mathcal{A}_{as} \cap H^{s,\gamma+l-0}(\mathcal{C}) = \emptyset$, where $H^{s,\gamma+l-0}(\mathcal{C}) = \cap_{\epsilon>0} H^{s,\gamma+l-\epsilon}(\mathcal{C})$. The elements of $H^{s,\gamma+l-0}(\mathcal{C})$ may be regarded as being flat of order $l-0$ relative to the weight γ.

We endow $\mathcal{A}_{as}$ with the natural topology, and $H^{s,\gamma+l-0}(\mathcal{C})$ with the topology of projective limit of Hilbert spaces.

Definition 2.5.4 *For* $s, \gamma \in \mathbb{R}$ *and* $as \in \mathrm{As}\,(w)$*, let*

$$H^{s,\gamma}_{as}(\mathcal{C}) = \mathcal{A}_{as} + H^{s,\gamma+l-0}(\mathcal{C}).$$

We make $H^{s,\gamma}_{as}(\mathcal{C})$ a Fréchet space by giving it the topology of the sum of Fréchet spaces. Obviously, $H^{s,\gamma}_{as}(\mathcal{C})$ is independent of the particular choice of the cut-off function ω up to an equivalent topology.

Remark 2.5.5 *Spaces with asymptotics are also well-defined on a manifold with one-point singularities if we either keep coordinates fixed or else interpret the subscript 'as' associated with an asymptotic type 'as' as an equivalence class of possible asymptotic types.*

We finish this section by a brief discussion of operator-valued meromorphic functions. Namely, for a discrete set σ in the complex plane, we define meromorphic functions $h(z) \in \mathcal{A}(\mathbb{C} \setminus \sigma, \Psi^m_{cl}(X))$ whose values are classical pseudodifferential operators on X.

The space $\Psi^m_{cl}(X)$ is endowed with its natural Freéchet topology, and by $\mathcal{A}(\mathbb{C} \setminus \sigma, \mathcal{L})$ we mean the space of $\mathcal{L}$-valued holomorphic functions on $\mathbb{C} \setminus \sigma$. Clearly, $\mathcal{A}(\mathbb{C} \setminus \sigma, \Psi^m_{cl}(X))$ has a natural Fréchet structure, again.

Denote the points of σ by $(p_j)_{j=0,1,\ldots}$. We will assume that each horizontal strip $\{c' \leq \operatorname{im} z \leq c''\}$ contains only a finite number of points in σ.

Furthermore, let us fix a sequence (μ_j) of positive integers and a sequence $(\mathcal{L}_j)$ of finite-dimensional subspaces of operators of finite range in $\Psi^{-\infty}(X)$.

Every collection $as = ((p_j, \mu_j, \mathcal{L}_j))_{j=0,1,\ldots}$ will be called a (discrete) *asymptotic type* for $\boldsymbol{F}$-symbols.

Now $\mathcal{M}_{as}(\mathbb{C}, \Psi^m_{cl}(X))$ denotes the subspace of $\mathcal{A}(\mathbb{C} \setminus \sigma, \Psi^m_{cl}(X))$ consisting of all functions $h(z)$ such that

- for every σ-excision function $\chi(z)$ (i.e., $\chi \in C^\infty_{loc}(\mathbb{C})$ is equal to 0 near σ and 1 away from a neighborhood of σ) we have $\chi\, h|_{\Gamma_{-\gamma}} \in \Psi^m_{cl}(X; \Gamma_{-\gamma})$ uniformly in γ on finite intervals in $\mathbb{R}$;

- $h(z)$ is meromorphic with poles at p_j of multiplicities $\mu_j + 1$, and the Laurent expansion at p_j is $h(z) = \sum_{\mu=0}^{\mu_j} l_j\,(z - p_j)^{-(\mu+1)} + h_j(z)$ with $l_j \in \mathcal{L}_j$ and $h_j(z) \in \mathcal{A}(O, \Psi^m_{cl}(X))$ for some neighborhood O of p_j.

The space $\mathcal{M}_{as}(\mathbb{C}, \Psi^m_{cl}(X))$ has a natural Fréchet topology. We will write it simply $\mathcal{M}(\mathbb{C}, \Psi^m_{cl}(X))$ when $\sigma = \emptyset$.

2.6 *Algebras with asymptotics*

Until further notice, we restrict our attention to a coordinate patch on X with variable x and covariable ξ.

For $m \in \mathbb{R}$ and a weight datum $w = (\gamma, \beta)$, we denote by $e^{(\beta-\gamma)\delta}\mathrm{S}^m({}^bT^*\mathcal{C})$ the subspace of $\mathrm{S}^m(\mathring{\mathcal{C}} \times \mathbb{R}^{1+n})$ consisting of symbols of the form

$$a(t, x, \tau, \xi) = e^{(\beta-\gamma)\delta(t)}\, \tilde{a}\left(t, x, \frac{1}{\delta'(t)}\tau, \xi\right), \tag{2.6.1}$$

where $\tilde{a}(t,x,\tilde{\tau},\xi) \in \mathrm{S}^m(\mathcal{C} \times \mathbb{R}^{1+n})$.

Since the *weight factor* $e^{(\beta-\gamma)\delta(t)}$ is fixed for every m, we can endow each space $e^{(\beta-\gamma)\delta}\mathrm{S}^m({}^bT^*\mathcal{C})$ with a canonical Fréchet topology. The elements of $e^{(\beta-\gamma)\delta}\mathrm{S}^m({}^bT^*\mathcal{C})$ are called *degenerate symbols* of order m.

Let $e^{(\beta-\gamma)\delta}\mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$ stand for the subspace of $e^{(\beta-\gamma)\delta}\mathrm{S}^m({}^bT^*\mathcal{C})$ induced by $\mathrm{S}^m_{cl}(\mathcal{C} \times \mathbb{R}^{1+n})$ in the same way.

As usually, we set

$$\mathrm{S}^{-\infty}({}^bT^*\mathcal{C}, w) = \cap_m \, \mathrm{S}^m_{cl}({}^bT^*\mathcal{C}).$$

By definition, each symbol $a \in e^{(\beta-\gamma)\delta}\mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$ can be written in the form (2.6.1), where $\tilde{a} \in \mathrm{S}^m(\mathcal{C} \times \mathbb{R}^{1+n})$. Write $\tilde{a} \sim \sum_j \chi \, \tilde{a}_{m-j}$, with χ an excision function and $\tilde{a}_{m-j} \in C^\infty_{loc}(\mathcal{C} \times \mathbb{R}^{1+n})$ homogeneous of degree $m-j$.

Definition 2.6.1 *Given an $a \in e^{(\beta-\gamma)\delta}\mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$, by the principal inner symbol of a is meant*

$$\sigma^m_{\mathcal{F}}(a)(t,x,\tau,\xi) = e^{(\beta-\gamma)\delta(t)} \, \tilde{a}_m\left(t, x, \frac{1}{\delta'(t)}\tau, \xi\right), \quad t > 0.$$

Since $\delta(t) \to -\infty$ as $t \to 0$, the principal inner symbol itself cannot control the behavior of a at $t=0$. For this purpose, we may invoke the component $\tilde{a}_m(t,x,\tilde{\tau},\xi)$ because it is well-defined up to $t=0$ and captures the behavior of $\sigma^m_{\mathcal{F}}(a)$ away from $t=0$.

A symbol $a \in e^{(\beta-\gamma)\delta}\mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$ is said to be *elliptic* if $\tilde{a}_m(t,x,\tilde{\tau},\xi) \neq 0$ for all $(t,x,\tilde{\tau},\xi) \in \mathcal{C} \times (\mathbb{R}^{1+n} \setminus \{0\})$. The important point to emphasize here is that the ellipticity of a degenerate symbol subtends the non-singularity of $\tilde{a}_m$ *up to* $t=0$. By the above, every elliptic symbol $a \in e^{(\beta-\gamma)\delta}\mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$ is also elliptic in the sense of symbol algebra on the interior of $\mathcal{C}$. Therefore, a has a *Leibniz inverse* in this algebra, which is unique modulo $\mathrm{S}^{-\infty}(\mathring{\mathcal{C}} \times \mathbb{R}^{1+n})$. The crucial fact is that we can ensure the existence of a Leibniz inverse within the class $e^{(\gamma-\beta)\delta}\mathrm{S}^{-m}_{cl}({}^bT^*\mathcal{C})$.

Under ellipticity, we want to associate with the Leibniz inverse of a a continuous mapping between weighted Sobolev spaces on $\mathcal{C}$. We first demonstrate these techniques by example of those degenerate symbols which are polynomial in τ.

Example 2.6.2 Let A be of form (2.2.4). We have $A = \mathrm{op}_{\mathcal{F}}(a)$ with a unique symbol $a \in \mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$, for each weight data $w = (\gamma,\gamma)$. In this case,

$$\sigma^m_{\mathcal{F}}(a)(t,x,\tau,\xi) = \sum_{j=0}^{m} \sigma^{m-j}_{\mathcal{F}}(A_j)(t,x,\xi) \left(\frac{1}{\delta'(t)}\,\tau\right)^j. \tag{2.6.2}$$

Fix a cut-off function ω on $\bar{\mathbb{R}}_+$, so that $\omega(t) = 1$ for $t \le a$ and $\omega(t) = 0$ for $t \ge A$, where $0 < a < A < \infty$. Then $\varphi_0 = \omega$ and $\varphi_\infty = 1 - \omega$ give the partition of unity on the semiaxis subordinated to the covering $I_0 = [0, 2A)$, $I_\infty = (\frac{1}{2}a, \infty)$. We now choose $\psi_\nu \in C^\infty_{loc}(\bar{\mathbb{R}}_+)$ $(\nu = 0, \infty)$ such that $\operatorname{supp}\psi_\nu \subset I_\nu$ and $\psi_\nu = 1$ near $\operatorname{supp}\varphi_\nu$. Since the operator A is *local*, it is a simple matter to see that $A = A_0 + A_\infty$, where $A_0 = \varphi_0 \operatorname{op}_{\mathcal{F}}(a)\, \psi_0$, $A_\infty = \varphi_\infty \operatorname{op}_{\mathcal{F}}(a)\, \psi_\infty$. The operator A_∞ is "supported" away from the singularity $t = 0$, so it extends to a continuous linear operator $H^{s,\gamma}(\mathcal{C}) \to H^{s-m,\gamma}(\mathcal{C})$, for each $s, \gamma \in \mathbb{R}$, provided that the coefficients A_j are independent of t for $t > 0$ large enough. The next task is to rewrite A as a pseudodifferential operator with respect to the transform $\boldsymbol{F}$, thus making A_0 more prepared to act in the weighted Sobolev spaces close to $t = 0$. However, a trivial verification shows that, for every $\gamma \in \mathbb{R}$, we have $A = \operatorname{op}_{F,\gamma}(h)$ on $C^\infty_{comp}(\mathbb{R}_+ \times X)$, where

$$h(t, z) = \sum_{j=0}^{m} A_j(t)\, z^j. \tag{2.6.3}$$

Thus, the operator $A_1 = \varphi_0 \operatorname{op}_{F,\gamma}(h)\, \psi_0$ extends to a continuous linear operator of $H^{s,\gamma}(\mathcal{C}) \to H^{s-m,\gamma}(\mathcal{C})$, for each $s, \gamma \in \mathbb{R}$.

□

For arbitrary degenerate symbols, it is no longer possible to obtain a precise representation just as in Example 2.6.2. When studying arbitrary symbols we had to look for such a representation only modulo "smoothing" operators.

To each symbol $a \in e^{(\beta-\gamma)\delta} \mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$, we may assign a pseudodifferential operator $A = \operatorname{op}_{\mathcal{F}}(a)$ which is well-defined on distributions supported in the interior of $\mathcal{C}$.

Fix an open covering $I_0 \cup I_\infty$ of the semiaxis $\bar{\mathbb{R}}_+$, where $I_0 = [0, 2A)$ and $I_\infty = (\frac{1}{2}a, \infty)$ $(0 < a < A < \infty)$. Let $(\varphi_\nu)_{\nu=0,\infty}$ be a partition of unity on $\bar{\mathbb{R}}_+$ subordinated to this covering, and let $\psi_\nu \in C^\infty_{loc}(\bar{\mathbb{R}}_+)$ satisfy $\operatorname{supp}\psi_\nu \subset I_\nu$ and $\psi_\nu = 1$ in a neighborhood of $\operatorname{supp}\varphi_\nu$.

Since the operator A is *pseudo-local*, we see at once that $A = A_0 + A_\infty$ modulo smoothing operators in $\mathring{\mathcal{C}}$, where

$$\begin{aligned} A_0 &= \varphi_0 \operatorname{op}_{\mathcal{F}}(a)\, \psi_0, \\ A_\infty &= \varphi_\infty \operatorname{op}_{\mathcal{F}}(a)\, \psi_\infty. \end{aligned}$$

Our next goal is to find a suitable reformulation of the operator A_0 in terms of the transform $\boldsymbol{F}$.

Theorem 2.6.3 *Given any* $a \in e^{(\beta-\gamma)\delta}\mathrm{S}^m_{cl}({}^bT^*\mathcal{C})$*, there exists an* $\boldsymbol{F}$*-symbol* $h(t,z) \in C^\infty_{loc}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^m_{cl}(X)))$ *such that, for each* $\gamma \in \mathbb{R}$*, we have*

$$\mathrm{op}_{\mathcal{F}}(a) = e^{(\beta-\gamma)\delta(t)}\,\mathrm{op}_{F,\gamma}(h) \quad \left(\mathrm{mod} \;\; \Psi^{-\infty}(\mathbb{R}_+ \times X)\right). \tag{2.6.4}$$

Proof. Cf. Egorov and Schulze [ES96, 8.1.3].

□

Summarizing, we have

$$\mathrm{op}_{\mathcal{F}}(a) = e^{(\beta-\gamma)\delta(t)}\,\varphi_0\,\mathrm{op}_{F,\gamma}(h)\,\psi_0 + \varphi_\infty\,\mathrm{op}_{\mathcal{F}}(a)\,\psi_\infty \quad \left(\mathrm{mod} \;\; \Psi^{-\infty}(\mathbb{R}_+ \times X)\right), \tag{2.6.5}$$

for each $\gamma \in \mathbb{R}$. Moreover, Proposition 2.3.3, when combined with boundedness properties of pseudodifferential operators in Sobolev spaces, shows that the right-hand side of (2.6.5) extends to a continuous linear operator of $H^{s,\gamma}(\mathcal{C}) \to H^{s-m,\beta}(\mathcal{C})$, for each $s \in \mathbb{R}$, provided a is independent of t for $t > 0$ large enough.

We are now in a position to describe our algebra with asymptotics on a manifold with singular points. By the above, we are interested in a simple edition of this algebra based on the spaces with asymptotics of Section 2.5. In order to get asymptotic results, it is necessary to put some restrictions on the symbols in question. The requirement on a is that $\tilde{a}(\delta^{-1}(\log \varrho), x, \tilde{\tau}, \xi)$ is smooth up to $\varrho = 0$ (cf. (2.6.1)). Such is the case, in particular, if $\tilde{a}$ is independent of t close to $t = 0$.

The algebra with asymptotics on a manifold with singular points starts with operators of the form (2.6.5), where the Fourier symbol $a(t,x,\tau,\xi)$ and the $\boldsymbol{F}$-symbol $h(t,z)$ are *compatible* in the sense of Theorem 2.6.3. Our operator convention (2.6.4) does produce at once smoothing errors of different kind under changing $h(t,z)$ or the cut-off functions. It is worth pointing out that if we add to $h(t,z)$ a symbol $h_0(t,z) \in C^\infty_{loc}(\bar{\mathbb{R}}_+, \mathcal{M}_{as}(\mathbb{C}, \Psi^{-\infty}_{cl}(X)))$ whose poles do not meet the reference line $\Gamma_{-\gamma}$, then the equality (2.6.4) is still true. In this way we obtain what we shall call the *smoothing* $\boldsymbol{F}$*-operators*, by analogy with smoothing Mellin operators (cf. Schulze [Sch94, 1.4.3]). Since the parametrix constructions for the typical elliptic differential operators (2.2.4) lead to smoothing $\boldsymbol{F}$-operators, it is adequate to have them from the very beginning in the class.

To describe more precisely the smoothing $\boldsymbol{F}$-operators, we need the notion of a *weight datum* for an algebra with asymptotics. By such a datum we mean any triple $w = (\gamma, \beta, (-l, 0])$ consisting of real numbers $\gamma, \beta \in \mathbb{R}$ and a weight interval $(-l, 0]$, $l > 0$.

The smoothing $\boldsymbol{F}$-operators under a weight datum $w = (\gamma, \beta, (-l, 0])$,

$l = 1, 2, \ldots$, are defined to be operators of the form

$$S_F = e^{(\beta-\gamma)\delta(t)} \sum_{j=0}^{l-1} e^{j\delta(t)} \operatorname{op}_{F,\gamma_j}(h_j), \tag{2.6.6}$$

with $h_j(z) \in \mathcal{M}_{as_j}(\mathbb{C}, \Psi^{-\infty}(X))$, where $h_j(z)$ has no poles on the weight line $\Gamma_{-\gamma_j}$ and $\gamma - j \le \gamma_j \le \gamma$ for all $j = 0, 1, \ldots, l-1$. The latter two conditions ensure, by Proposition 2.3.3, that $\varphi_0 \, S_F \psi_0$ extends to a continuous linear mapping of $H^{s,\gamma}(\mathcal{C}) \to H^{\infty,\beta}(\mathcal{C})$, for each $s \in \mathbb{R}$.

Remark 2.6.4 *The absence in (2.6.5) of smoothing* $\boldsymbol{F}$*-operators with meromorphic symbols allows actions on spaces with arbitrary weights* $\gamma \in \mathbb{R}$*, in contrast to what we obtain by adding an operator of the form (2.6.6), which contains meromorphic ingredients and hence natural restrictions on the weights.*

The only point remaining concerns the smoothing errors produced by (2.6.4) under changing the partition of unity $(\varphi_\nu)_{\nu=0,\infty}$ on the half-line $\mathbb{R}_+$. These are known as the *Green operators* and defined via their mapping properties.

For each operator $A \in \mathcal{L}(H^{s,\gamma}(\mathcal{C}), H^{t,\beta}(\mathcal{C}))$, we can define the transpose A' as an element of $\mathcal{L}(H^{-t,-\beta}(\mathcal{C}), H^{-s,-\gamma}(\mathcal{C}))$ via the non-degenerate pairings $H^{s,\gamma}(\mathcal{C}) \times H^{-s,-\gamma}(\mathcal{C}) \to \mathbb{C}$ induced by the inner product in $H^{0,0}(\mathcal{C})$. Namely, we require $(A\,u, \bar{g})_{H^{0,0}(\mathcal{C})} = (u, \bar{A'g})_{H^{0,0}(\mathcal{C})}$ to hold for all $u, g \in C^\infty_{comp}(\mathcal{C})$.

Since we are again aimed at the analysis near $t = 0$, we shall replace $H^{\infty,\beta}_{as'}(\mathcal{C})$ and $H^{\infty,-\gamma}_{as''}(\mathcal{C})$ by subspaces $\mathcal{S}^{\beta}_{as'}(\mathcal{C})$ and $\mathcal{S}^{-\gamma}_{as''}(\mathcal{C})$ respectively, where

$$\begin{aligned} \mathcal{S}^{\beta}_{as'}(\mathcal{C}) &= \omega \, H^{\infty,\beta}_{as'}(\mathcal{C}) + (1-\omega)\, \mathcal{S}(\mathcal{C}), \\ \mathcal{S}^{-\gamma}_{as''}(\mathcal{C}) &= \omega \, H^{\infty,-\gamma}_{as''}(\mathcal{C}) + (1-\omega)\, \mathcal{S}(\mathcal{C}). \end{aligned}$$

Here $\omega(t)$ is a cut-off function and $\mathcal{S}(\mathcal{C}) = \mathcal{S}(\bar{\mathbb{R}}_+, C^\infty(X))$. It is easily seen that these new spaces are independent of the concrete choice of ω.

Definition 2.6.5 *An operator* $G \in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{C}), H^{\infty,\beta}(\mathcal{C}))$ *is said to be a Green operator with respect to a weight datum* $w = (\gamma, \beta, (-l, 0])$*, if there are asymptotic types* $as' \in \operatorname{As}(\beta, (-l, 0])$ *and* $as'' \in \operatorname{As}(-\gamma, (-l, 0])$ *such that*

$$\begin{aligned} G &\in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{C}), \mathcal{S}^{\beta}_{as'}(\mathcal{C})), \\ G' &\in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,-\beta}(\mathcal{C}), \mathcal{S}^{-\gamma}_{as''}(\mathcal{C})). \end{aligned}$$

These operators can be characterized in the following way: G is a Green operator with asymptotic types as' and as'' if and only if G is an integral operator with a kernel in $\mathcal{S}^{\beta}_{as'}(\mathcal{C}) \hat{\otimes}_\pi \mathcal{S}^{-\gamma}_{as''}(\mathcal{C})$.

The difference between a smoothing $\boldsymbol{F}$-operator and a Green operator is that the former preserves the asymptotics of the argument function and adds specific ones, whereas the latter forget the original asymptotics and produces new ones. The reason for taking $l-1$ as the upper summation bound in (2.6.6) is that $e^{(\beta-\gamma)\delta(t)}\, e^{j\delta(t)}\, \mathrm{op}_{F,\gamma_j}(h_j)$ is a Green operator with respect to a weight datum $w=(\gamma,\beta,(-l,0])$, provided that $j \geq l$.

We leave it to the reader to carry over the above local results to the whole manifold X by using a familiar argument invoking a partition of unity on X.

To complete the construction of the algebra with asymptotics on a stretched manifold $\mathcal{M}$, we can argue just as in Section 2.3. Given any $m \in \mathbb{R}$ and weight datum $w=(\gamma,\beta,(-l,0])$, let $\mathrm{Alg}^m(\mathcal{M},w)$ stand for the space of all operators $A:\, C^\infty_{comp}(\overset{\circ}{\mathcal{M}}) \to \mathcal{D}'(\overset{\circ}{\mathcal{M}})$ of the form

$$A = \varphi_0\, A_F\, \psi_0 + \varphi_\infty\, A_{\mathcal{F}}\, \psi_\infty + \varphi_0\, S_F\, \psi_0 + G, \tag{2.6.7}$$

where

A_F is an operator based on the transform $\boldsymbol{F}$ close to the boundary, as on the right-hand side of (2.6.4);

$A_{\mathcal{F}}$ is a pseudodifferential operator of order m in the interior of $\mathcal{M}$, which differs from A_F, close to the boundary, by a smoothing operator;

S_F is a smoothing $\boldsymbol{F}$-operator close to the boundary, as in (2.6.6); and

G is a Green operator with respect to the weight data w, defined via its mapping properties.

The collection of all the spaces $\mathrm{Alg}^m(\mathcal{M},w)$, with $m \in \mathbb{R}$ and weight data $w=(\gamma,\beta,(-l,0])$, is the algebra with asymptotics. As follows, $H^{s,\gamma}_{as}(\mathcal{M})$ is an adequate choice of *domains* for the operators in this algebra.

Proposition 2.6.6 *If* $A \in \mathrm{Alg}^m(\mathcal{M},w)$*, then for each asymptotic type* $as' \in \mathrm{As}\,(\gamma,(-l,0])$ *there is an asymptotic type* $as'' \in \mathrm{As}\,(\beta,(-l,0])$ *such that* A *has a continuous extension*

$$A:\, H^{s,\gamma}_{as'}(\mathcal{M}) \to H^{s-m,\beta}_{as''}(\mathcal{M}) \tag{2.6.8}$$

for any $s \in \mathbb{R}$.

Proof. Cf. Theorem 1.4.42 in Schulze [Sch94].

□

To see that the algebra with asymptotics is an "algebra" in the sense of Section 2.3, we refer the reader to Schulze [Sch94, 1.2]. It is worth mentioning that the operators $\varphi_0\, S_F\, \psi_0 + G$ form an "ideal" in this algebra.

By the above, if the mapping $t \mapsto \varrho = e^{\delta(t)}$ is C^∞ up to $t = 0$, then the algebra $\mathrm{Alg}^m(\mathcal{M}, w)$ just constructed is an *extension* of the cone algebra with discrete asymptotics (cf. Schulze [Sch94, 1.2]) in the sense that, close to the singular points, the pull-back of the latter under the mapping ϱ is a *proper* subalgebra of the former. As mentioned, the norm closures of the two coincide.

The symbol $\sigma^m_{\mathcal{F}}(A)$ controls the interior ellipticity of operator (2.6.7). However, there is yet another symbolic level given by the principal *conormal symbol* $\sigma_F(A)$ (elsewhere this is referred to as the *operator pencil*). For the typical differential operators (2.2.4), this is defined by $\sigma_F(A)\,(z) = \sum_{j=0}^m A_j(0) z^j$. For the operators A on the right-hand side of (2.6.5), we have $\sigma_F(A)\,(z) = h(0, z)$. In the general case (2.6.7), also the component $h_0(z)$ of S_F invests to the principal conormal symbol, thus giving $\sigma_F(A)\,(z) = h(0, z) + h_0(z)$. Given any fixed $z \in \Gamma_{-\gamma}$, the principal conormal symbol is a pseudodifferential operator along the base X, of order m. In this way we obtain a family of operators $\sigma_F(A)\,(z) : H^s(X) \to H^{s-m}(X)$, $s \in \mathbb{R}$, parametrized by $z \in \Gamma_{-\gamma}$.

Definition 2.6.7 *An operator $A \in \mathrm{Alg}^m(\mathcal{M}, w)$ is said to be elliptic if:*
1) the interior symbol of A is elliptic up to $t = 0$;
2) $\sigma_F(A)\,(z) : H^s(X) \to H^{s-m}(X)$ is an isomorphism for each $z \in \Gamma_{-\gamma}$ and $s \in \mathbb{R}$.

It follows from the elliptic theory on closed compact manifolds that, in condition 2), we may require $\sigma_F(A)\,(z)$ to be an isomorphism for any *one* $s \in \mathbb{R}$.

The key result in the algebra $\mathrm{Alg}^m(\mathcal{M}, w)$ can be then formulated as follows.

Theorem 2.6.8 *If $A \in \mathrm{Alg}^m(\mathcal{M}, w)$ is elliptic, then for each asymptotic type $as' \in \mathrm{As}\,(\gamma, (-l, 0])$ there is an asymptotic type $as'' \in \mathrm{As}\,(\beta, (-l, 0])$ such that the operator (2.6.8) is Fredholm.*

Proof. See Schulze [Sch94, 1.2.2].

□

No attempt has been made here to extend Theorem 2.6.8 to arbitrary 'cusp' operators, i.e., those without additional restrictions on the regularity of symbols near conical points. It is worth pointing out that the 'ellipticity condition' of a 'cusp' operator A is necessary and sufficient in order that A induce a Fredholm operator $H^{s,\gamma}(\mathcal{M}) \to H^{s-m,\beta}(\mathcal{M})$ for any one $s \in \mathbb{R}$ (and hence for all real s). This is the topic of our next paper (cf. [RST97]). We just mention that the Fredholm property in weighted Sobolev spaces without asymptotics holds for all elliptic 'cusp' operators with symbols slowly varying close to singular points (in particular, for operators whose symbols are smooth up to $t = 0$). On the other hand, the class of symbols under study is sufficient for the purposes of index theory because a familiar argument shows that each elliptic 'cusp' operator with coefficients smooth up to $t = 0$ is homotopic in the class of elliptic operators to an operator with coefficients constant in a neighborhood of each conical point.

3 *Index formula*

3.1 *Overview*

Let $r = \delta(t)$ be a diffeomorphism of an interval $T = (a, b)$ onto the whole axis $\mathbb{R}$. We assume that $\delta'(t) > 0$ for all $t \in T$.

In this chapter we consider a special case of $\boldsymbol{F}$-pseudodifferential operators on the stretched cone $\mathcal{C} = \bar{T} \times X$, whose base X is a smooth compact manifold of dimension n without boundary. These operators have the form

$$(Au)(t) = \left(\frac{1}{2\pi}\right)^2 \int_\Gamma dz \int_T e^{i(\delta(t)-\delta(t'))z}\, a(t,z)\, u(t')\, dm(t'), \qquad (3.1.1)$$

when defined on functions $u \in C^\infty_{comp}(T, C^\infty(X))$. The weight line Γ may be any horizontal line $\Gamma_{-\gamma} = \{z \in \mathbb{C} : \operatorname{im} z = -\gamma\}$ in the complex plane. We may assume without loss of generality that Γ coincides with the real axis Γ_0 (cf. (2.3.1)).

The operator-valued symbol $a(t, z)$ is assumed to satisfy the following conditions:

- $a(t, z) \in C^\infty_{loc}(T, \Psi^m_{cl}(X; \Gamma))$ is "sufficiently" smooth up to the endpoint $t = a$ of T in the sense that $a(\delta^{-1}(\log \varrho), z)$ is C^∞ up to $\varrho = 0$;

- $a(t, z)$ is independent of t close to the endpoint $t = b$ of T, more precisely, $a(t, z) = a(b-, z)$ for $t \in (C, b)$, with $a < C < b$;

- for $t \in [a, c)$, with $a < c < C$, the symbol $a(t, z)$ admits an analytic continuation to some strip $\{z \in \mathbb{C} : |\operatorname{im} z + \gamma| < \varepsilon\}$ and on each line $\Gamma_{-\alpha}$ it is a parameter-dependent pseudodifferential operator of order m on X, uniformly in $\alpha \in [\gamma - \epsilon, \gamma + \epsilon]$, $\epsilon < \varepsilon$.

By the above, operators (3.1.1) are of great importance for the calculus of pseudodifferential operators on manifolds with singular points. Here we restrict our attention to the model case where the singular manifold is an infinite stretched cone and the operator may be written globally via the $\boldsymbol{F}$-transform. Since the symbol behaves well close to the endpoints of T, it follows from Proposition 2.3.3 that A extends to a continuous linear operator $A_s : \mathcal{H}^{s,\gamma}(\mathcal{C}) \to \mathcal{H}^{s-m,\gamma}(\mathcal{C})$, for each $s \in \mathbb{R}$.

Throughout this chapter we assume that $m \le 0$. In particular, we need the following concept of elliptic operators of order 0 (cf. Definition 2.6.7).

Definition 3.1.1 *An operator A of order 0 is called elliptic if its symbol satisfies the following conditions:*

1) for each $t \in T$, the symbol $a(t,z)$ is a parameter-dependent elliptic operator on X with parameter $z \in \Gamma$;

2) $a(t,z)$ is invertible for each $t \in [a,c)$ and each z in the strip $\{z \in \mathbb{C} : |\operatorname{im} z + \gamma| < \varepsilon\}$; and

3) $a(b-,z) = 1$, where 1 stands for the identity operator on X.

When compared with Definition 2.6.7, this includes an additional assumption 3) which is connected with the *exit condition* on the infinite stretched cone $\mathcal{C}$ (cf. Schulze [Sch94, 1.2.3]).

We prove in Section 3.5 that ellipticity implies the existence of a *parametrix* for A, i.e., an inverse up to smoothing operators of *trace class*. The important point to note here is the form of the parametrix, which is again an elliptic operator of order 0. Thus, the *kernel* and the *cokernel* of the operator A_s are actually independent of s. Moreover, a familiar argument of functional analysis yields that the mapping A_s is Fredholm for each $s \in \mathbb{R}$. Hence, for every elliptic pseudodifferential operator A of order zero, we may define its *index* via $\operatorname{ind} A = \dim \ker A_s - \dim \operatorname{coker} A_s$.

In order to evaluate the index of A, a basic observation is that the ellipticity conditions imply that the Fredholm family $a(t,z)$, parametrized by $(t,z) \in T \times \Gamma$, is *trivial* outside a compact set in $T \times \Gamma$ [4] (see Fig. 3.1). Therefore it defines an *index bundle* $\operatorname{ind} a \in K_{comp}(T \times \Gamma)$, where K_{comp} means K-functor with compact support (cf. Atiyah and Singer [AS71]). The *Chern character* of this bundle is represented by a closed differential from of compact support, and we prove the following result.

Theorem 3.1.2 *For any elliptic operator (3.1.1), we have*

$$\operatorname{ind} A = \iint_{T \times \Gamma} \operatorname{ch}(\operatorname{ind} a). \tag{3.1.2}$$

The proof of (3.1.2) follows the scheme developed in Fedosov [Fed74]. It consists in comparing three expressions:

- *analytical index*

$$\operatorname{ind} A = \operatorname{tr}(1 - PA) - \operatorname{tr}(1 - AP),$$

where P is a parametrix of A up to a *trace class* operator;

[4] That is, $a(t,z)$ is invertible.

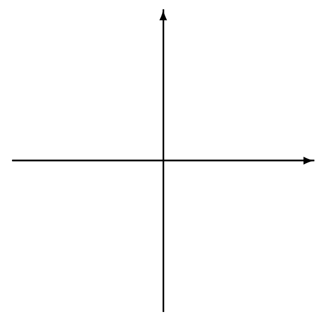


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Fig. 3.1: The domain of invertibility of an elliptic symbol.

- *algebraic index*

$$\operatorname{ind} a = \operatorname{tr}(1 - p \circ a) - \operatorname{tr}(1 - a \circ p),$$

  where p is a formal complete symbol of P and $\circ$ means a composition of formal complete symbols (*Leibniz product*); and

- *topological index* given by the right-hand side of (3.1.2).

The most important step is transition from the analytical index to the algebraic one, or, using the terminology of Fedosov [Fed74], the *Theorem on a Regularized Trace of Product*. The transition from the algebraic index to the topological one is based on the machinery developed by Fedosov [Fed78]. Namely, we prove that

$$\operatorname{ind} a = \frac{1}{2\pi i} \iint_{T\times\Gamma} \operatorname{tr}\left(p_0 da \wedge p_0 da + dp_0 \wedge da\right), \qquad (3.1.3)$$

where $p_0(t,z)$ is a point-wise parametrix for $a(t,z)$, such that both $1 - p_0 a$ and $1 - a p_0$ are trace class operators, for any $(t,z) \in T \times \Gamma$, and $p_0 = a^{-1}$ outside a compact subset of $T \times \Gamma$. Then, (3.1.2) is a consequence of the fact that the integrand in (3.1.3) represents the Chern character of $\operatorname{ind} a$ in terms of a and p_0 (cf. Fedosov [Fed78]).

Let us mention some particular cases of Theorem 3.1.2. If $T = \mathbb{R}$, $\delta(t) = t$ and $a(t,z) = 1$ away from a finite interval in $\mathbb{R}$, then (3.1.2) follows from the Atiyah-Singer theorem on the index of a family of elliptic operators (cf. [AS71]) and the Atiyah-Singer Index Theorem (cf. [AS68a, AS68b]).

This formula goes back to the work of Luke [Luk72]. Yet another case corresponds to $T = \mathbb{R}_+$ and $\delta(t) = \log t$ (i.e., conical singularities). In this situation the Atiyah-Singer theorem for families is no longer applicable, and formula (3.1.2) is due to Fedosov and Schulze [FS96] (cf. also Rosenblum [Ros96] for another proof).

3.2 *Trace estimates for remainders*

In the sequel, we need a special order reduction $\Lambda^s(\lambda)$ on X which admits an analytic extension in λ to the strip $|\operatorname{im}\lambda + \gamma| < 1$ and in s to the whole complex plane $\mathbb{C}$. To construct such an order reduction, we consider the function

$$(1+\zeta^2)^{\frac{s}{2}} = e^{\frac{s}{2}\log(1+\zeta^2)}$$

for $|\operatorname{im}\zeta| < 1$ and $s \in \mathbb{C}$, assuming that the branch of $\log(1+\zeta^2)$ in the strip $|\operatorname{im}\zeta| < 1$ is real at $\zeta \in \Gamma_0$. Since $1+\zeta^2 = 1 + (\Re\zeta)^2 - (\operatorname{im}\zeta)^2 + 2i\,\Re\zeta\operatorname{im}\zeta$, the function $(1+\zeta^2)^{\frac{s}{2}}$ is well-defined and holomorphic in ζ, belonging to the strip $|\operatorname{im}\zeta| < 1$, and in $s \in \mathbb{C}$.

Lemma 3.2.1 *For any $R > 0$ there exists a constant c, depending only on R, such that*

$$|(1+\zeta^2)^{\frac{s}{2}}| \le c\,(1+|\zeta|^2)^{\frac{\Re s}{2}}$$

whenever $|\operatorname{im}\zeta| < 1$ and $|\operatorname{im} s| \le R$.

Proof. Indeed, since $\arg(1+\zeta^2) < \pi$ for all ζ in the strip $|\operatorname{im}\zeta| < 1$, we get

$$\begin{aligned} |(1+\zeta^2)^{\frac{s}{2}}| &= e^{\frac{\Re s}{2}\log|1+\zeta^2| - \frac{\operatorname{im} s}{2}\arg(1+\zeta^2)} \\ &\le e^{\frac{R}{2}\pi}\,(1+|\zeta|^2)^{\frac{\Re s}{2}}, \end{aligned}$$

as desired.

□

Now, letting Δ denote the Laplace-Beltrami operator on X, we set

$$\Lambda^s(z) = (1 + (z+i\gamma)^2 - \Delta)^{\frac{s}{2}} \qquad (3.2.1)$$

for $|\operatorname{im} z + \gamma| < 1$ and $s \in \mathbb{C}$. A complex power is understood in the sense of elliptic theory (cf. Seeley [See67]). This family is holomorphic in z and s belonging to the mentioned strips.

Let us return to the symbols $a(t,z)$ of operators (3.1.1) under consideration. In what follows, we tacitly assume that these symbols are "sufficiently" smooth up to $t = a$ and independent of t close to the endpoint b of T.

Lemma 3.2.2 *Let $a(t,z) \in C^\infty_{loc}(T, \Psi^m(X;\Gamma))$, where $m \le 0$. Suppose that $a(b-,z) = 0$. Then*

$$\|\boldsymbol{F}_{t\mapsto\zeta} a(t,z)\|_{\mathcal{L}(L^2(X))} \le c\,(1+|\zeta|^2)^{-\frac{j}{2}}\,(1+|z|^2)^{\frac{m}{2}}, \quad j \in \mathbb{Z}_+,$$

for $z \in \Gamma$ and $\zeta \in \Gamma_{-\beta}$, $\beta < 0$, with c a constant depending only on j and β.

Proof. The integral

$$\begin{aligned} \boldsymbol{F}_{t\mapsto\zeta} a(t,z) &= \frac{1}{2\pi}\int_T e^{-i\zeta\delta(t)} a(t,z)\, dm(t) \\ &= \int_0^\infty \varrho^{-i\zeta}\, a(\delta^{-1}(\log\varrho), z)\, \frac{d\varrho}{\varrho} \end{aligned}$$

converges in the upper half-plane $\operatorname{im}\zeta > 0$. Moreover, for $\operatorname{im}\zeta > 0$, we have

$$\boldsymbol{F}_{t\mapsto\zeta} a(t,z) = \frac{1}{\zeta^j}\,\boldsymbol{F}_{t\mapsto\zeta}(\boldsymbol{D}^j a(t,z)), \quad j = 1,2,\ldots,$$

where $\boldsymbol{F}_{t\mapsto\zeta}(\boldsymbol{D}^j a(t,z))$ is holomorphic for $\operatorname{im}\zeta > 0$.

Thus, if $\zeta \in \Gamma_{-\beta}$ with $\beta < 0$, then

$$\begin{aligned} &\|\boldsymbol{F}_{t\mapsto\zeta} a(t,z)\|_{\mathcal{L}(L^2(X))} \\ &\quad \le |\zeta|^{-j} \int_0^{e^{\delta(C)}} \varrho^{-\beta}\, \|\boldsymbol{D}^j a(\delta^{-1}(\log\varrho), z)\|_{\mathcal{L}(L^2(X))}\, \frac{d\varrho}{\varrho}, \end{aligned} \qquad (3.2.2)$$

where $C \in T$ is such that $a(t,z) = 0$ for $t > C$. We now invoke the condition that, for fixed $t \in T$, the derivative $\boldsymbol{D}^j a(\delta^{-1}(\log\varrho), z)$ is a parameter-dependent pseudodifferential operator of order $m \le 0$ on X. Hence it follows that

$$\|\boldsymbol{D}^j a(\delta^{-1}(\log\varrho), z)\|_{\mathcal{L}(L^2(X))} \le const\,(j, C)\ (1+|z|^2)^{\frac{m}{2}}$$

for all $\varrho \in (0, e^{\delta(C)}]$ and $z \in \Gamma$ (cf. Shubin [Shu87]). Substituting this estimate into (3.2.2s), we obtain the desired conclusion.

□

A slight change in the proof actually shows that if $m < -\frac{n}{2}$, then $\boldsymbol{F}_{t\mapsto\zeta} a(t,z)$ is a *Hilbert-Schmidt operator* in $L^2(X)$ and its Hilbert-Schmidt norm $\|\cdot\|_2$ satisfies

$$\|\boldsymbol{F}_{t\mapsto\zeta} a(t,z)\|_2 \le const\,(j,\beta)\ (1+|\zeta|^2)^{-\frac{j}{2}}\,(1+|z|^2)^{\frac{m+\frac{n}{2}}{2}}, \quad j \in \mathbb{Z}_+, \qquad (3.2.3)$$

for all $z \in \Gamma$ and $\zeta \in \Gamma_{-\beta}$, $\beta < 0$.

Finally, if $m < -n$, then the operator $\boldsymbol{F}_{t\mapsto\zeta} a(t,z)$ is of trace class in $L^2(X)$ and its trace norm $\|\cdot\|_1$ satisfies an estimate

$$\|\boldsymbol{F}_{t\mapsto\zeta} a(t,z)\|_1 \leq const\,(j,\beta)\ (1+|\zeta|^2)^{-\frac{j}{2}}\,(1+|z|^2)^{\frac{m+n}{2}}, \quad j \in \mathbb{Z}_+, \tag{3.2.4}$$

whenever $z \in \Gamma$ and $\zeta \in \Gamma_{-\beta}$, $\beta < 0$.

Given any two symbols $a(t,z)$ and $p(t,z)$ in $C^\infty_{loc}(T, \Psi^{\cdot}(X;\Gamma))$, we set

$$p \circ a|_K = \sum_{k=0}^{K-1} \frac{1}{k!} (\partial/\partial z)^k p(t,z)\, \boldsymbol{D}^k a(t,z), \quad K = 1,2,\ldots \tag{3.2.5}$$

(in this way we obtain what is known as the *Leibniz product*).

We will write an operator $\mathrm{op}_{F,\gamma}(a)$ simply $\mathrm{op}\,(a)$ when no confusion can arise. With these notations the main result of this section is as follows.

Theorem 3.2.3 *Let*

$$\begin{array}{ll} a(t,z) \in C^\infty_{loc}(T, \Psi^m(X;\Gamma)), & m \leq 0, \\ p(t,z) \in C^\infty_{loc}(T, \Psi^\varpi(X;\Gamma)), & \varpi \leq 0, \end{array}$$

and let both $a(t,z)$ and $p(t,z)$ vanish at $t=b$. Then, for K large enough, the operator $R_K = \mathrm{op}\,(p)\ \mathrm{op}\,(a) - \mathrm{op}\,(p \circ a|_K)$ as an operator in the space $\mathcal{H}^{0,\gamma}(\mathcal{C})$ is of trace class.

Proof. Choose a partition of unity (ϕ_a, ϕ_i, ϕ_b) on T in such a way that

$$\begin{array}{llll} \phi_a = 1 & \text{on} & (a,c'), & \mathrm{supp}\,\phi_a \subset (a,c''); \\ & & & \mathrm{supp}\,\phi_i \subset (c',c); \\ \phi_b = 1 & \text{on} & (c,b), & \mathrm{supp}\,\phi_b \subset (c''',b), \end{array}$$

where $a < c' < c'' < c''' < c < b$. It follows that the supports of ϕ_a and ϕ_b do not meet each other (see Fig. 3.2).

Then any operator $\mathrm{op}\,(a)$ may be represented as the sum

$$\mathrm{op}\,(a) = \sum_\nu\ \mathrm{op}\,(\phi_\nu a),$$

the symbols $\phi_a(t)\,a(t,z)$ and $\phi_i(t)\,a(t,z)$ being holomorphic in z belonging to the strip $\{z \in \mathbb{C} :\ |\mathrm{im}\, z + \gamma| < \varepsilon\}$. Hence it follows that the operator R_K is the sum

$$R_K = \sum_{\mu,\nu}\ \mathrm{op}\,(\phi_\mu p)\ \mathrm{op}\,(\phi_\nu a) - \mathrm{op}\,((\phi_\mu p) \circ (\phi_\nu a)|_K), \tag{3.2.6}$$

and we consider several cases according to the values of μ, ν.

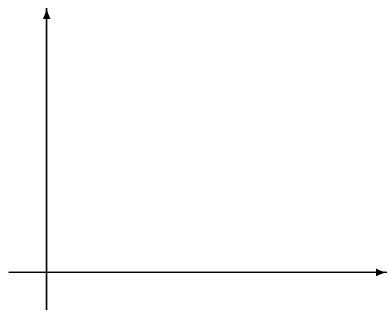

Fig. 3.2: The partition of unity (ϕ_a, ϕ_i, ϕ_b) on T.

Case 1 $(\mu, \nu \neq a)$ In this case the supports of symbols $\phi_\mu p$ and $\phi_\nu a$ are bounded away from $t = 0$. The $\boldsymbol{F}$-calculus of such operators on the interval T may be reduced to the usual Fourier calculus of pseudodifferential operators on the whole real axis by the change of variables $t = \delta^{-1}(r)$. Indeed, equality (3.1.1) transforms to

$$Au\,(\delta^{-1}(r)) = \frac{1}{2\pi} \int_\Gamma dz \int_{-\infty}^{\infty} e^{i(r-r')z}\, a(\delta^{-1}(r), z)\, u(\delta^{-1}(r'))\, dr', \quad r \in \mathbb{R},$$

and the Leibniz product of two symbols becomes

$$p \circ a|_K = \sum_{k=0}^{K-1} \frac{1}{k!} (\partial/\partial z)^k p(\delta^{-1}(r), z)\, (\frac{1}{i}\, \partial/\partial r)^k a(\delta^{-1}(r), z),$$

which is the usual composition rule for Fourier pseudodifferential operators. The symbols

$$\begin{array}{c} \phi_\mu(\delta^{-1}(r))\, p(\delta^{-1}(r), z), \\ \phi_\nu(\delta^{-1}(r))\, a(\delta^{-1}(r), z) \end{array}$$

have compact supports in r, so the theorem follows from the usual calculus of pseudodifferential operators (cf. Fedosov [Fed74]).

Case 2 $(\mu = b, \nu = a)$ In this case $(\phi_\mu p) \circ (\phi_\nu a)|_K = 0$, for the supports of ϕ_a and ϕ_b do not meet each other. Consequently, we need to prove that the operator

$$\mathrm{op}\,(\phi_b p)\ \mathrm{op}\,(\phi_a a) = \mathrm{op}\,(\phi_b p)\ \phi_a\, \mathrm{op}\,(a)$$

is of trace class [5].

[5] This is referred to as the pseudolocality property.

The operator $\operatorname{op}(a)$ is bounded in $\mathcal{H}^{0,\gamma}(\mathcal{C})$, for its order m is non-positive, and hence it suffices to prove that $\operatorname{op}(\phi_b p)\ \phi_a$ belongs to trace class. To this end, we represent it as a composition

$$\mathcal{H}^{0,\gamma}(\mathcal{C}) \xrightarrow{\mathrm{P}} \mathcal{H}^{0,\gamma+\epsilon}(\mathcal{C}) \hookrightarrow \mathcal{H}^{0,\gamma}(\mathcal{C}), \tag{3.2.7}$$

with some integer $s > \frac{n+1}{2}$ and some $\epsilon > 0$, and show that both operators in (3.2.7) are Hilbert-Schmidt operators. The second operator is an embedding.

The following lemma is a kind of the *Rellich Embedding Theorem* for the spaces $\mathcal{H}^{s,\gamma}(\mathcal{C})$.

Lemma 3.2.4 *If $s > \frac{n+1}{2}$ and $\epsilon > 0$, then the embedding*

$$\mathcal{H}^{0,\gamma+\epsilon}_{(a,A]}(\mathcal{C}) \hookrightarrow \mathcal{H}^{0,\gamma}(\mathcal{C}),$$

when considered on the subspace of functions $u \in \mathcal{H}^{s,\gamma+\epsilon}(\mathcal{C})$ whose supports belong to an interval $(a,A] \subset T$, is a Hilbert-Schmidt operator.

Proof. Suppose first $u \in C^\infty_{loc}(T, C^\infty(X))$ has a compact support in the interval $(a,A]$. Since

$$\begin{aligned} |z^j\, e^{iz\delta(A)}\, \boldsymbol{F}u(z)| &= \left| \frac{1}{2\pi} \int_a^A e^{iz(\delta(A)-\delta(t))}\, \boldsymbol{D}^j u(t)\, dm(t) \right| \\ &\leq \frac{1}{2\pi} \int_a^A e^{\beta(\delta(A)-\delta(t))}\, |\boldsymbol{D}^j u(t)|\, dm(t) \quad \text{for} \quad \operatorname{im} z \geq -\beta, \end{aligned}$$

we deduce that $e^{iz\delta(A)}\, \boldsymbol{F}u(z)$ is an entire function rapidly decreasing in each half-plane $\operatorname{im} z \geq -\beta$, where $\beta \in \mathbb{R}$.

The norm of $u(t)$ in the Sobolev space $\mathcal{H}^{s,\gamma+\epsilon}(\mathcal{C})$ is equal to the L^2-norm of $\Lambda^s(z)\, \boldsymbol{F}u(z)$ on the line $\Gamma_{-\gamma-\epsilon}$. Here $\Lambda^s(z)$ is the order-reducing family given by (3.2.1). The norm of $u(t)$ in $\mathcal{H}^{0,\gamma}(\mathcal{C})$ is equal to the L^2-norm of the restriction of $\boldsymbol{F}u(z)$ to the line $\Gamma_{-\gamma}$. Applying the Cauchy formula to the function $e^{iz\delta(A)}\, \boldsymbol{F}u(z)$ in the half-plane $\operatorname{im} z \geq -\gamma - \epsilon$, we see that the restrictions of $\boldsymbol{F}u(z)$ to $\Gamma_{-\gamma-\epsilon}$ and $\Gamma_{-\gamma}$ are connected by the Cauchy integral

$$\boldsymbol{F}u(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_{-\gamma-\epsilon}} \frac{e^{i(z-\zeta)\delta(A)}}{z-\zeta}\, \boldsymbol{F}u(z)\, dz, \tag{3.2.8}$$

for $\zeta \in \Gamma_{-\gamma}$.

Setting $v = \boldsymbol{F}^{-1}_{-\gamma-\epsilon}\, \Lambda^s(z)\, \boldsymbol{F}u$, we rewrite (3.2.8) in the form

$$\boldsymbol{F}u(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_{-\gamma-\epsilon}} \frac{e^{i(z-\zeta)\delta(A)}}{z-\zeta}\, \Lambda^{-s}(z)\, \boldsymbol{F}v(z)\, dz, \quad \zeta \in \Gamma_{-\gamma}.$$

This operator acts between L^2-spaces on the lines $\Gamma_{-\gamma-\epsilon}$ and $\Gamma_{-\gamma}$ and its Hilbert-Schmidt norm is equal to the L^2-norm of its Schwartz kernel

$$K(\zeta, z) = \frac{e^{i(z-\zeta)\delta(A)}}{z-\zeta}\, \Lambda^{-s}(z)$$

which is an operator-valued function.

For $s > \frac{n}{2}$, we may estimate the Hilbert-Schmidt norm of the operator $\Lambda^{-s}(z)$ in $L^2(X)$ by

$$\|\Lambda^{-s}(z)\|_2 \le |1 + (z + i\gamma)^2|^{\frac{-s+\frac{n}{2}}{2}}, \quad z \in \Gamma_{-\gamma-\epsilon}$$

(cf. (3.2.3)). Hence it follows that

$$\|K(\zeta, z)\|_2^2 \le \frac{e^{2\epsilon\delta(A)}}{(\Re\zeta - \Re z)^2 + \epsilon^2}\, |1 + (\Re z - i\epsilon)^2|^{-s+\frac{n}{2}}$$

for $\zeta \in \Gamma_{-\gamma}$ and $z \in \Gamma_{-\gamma-\epsilon}$. Integrating this over $\Re\zeta$ and $\Re z$ and using that $-s + \frac{n}{2} < -\frac{1}{2}$, we obtain that the L^2-norm of $K(\zeta, z)$ is finite, whence the lemma follows.

□

To prove that the operator P in (3.2.7) is a Hilbert-Schmidt operator, we use the fact that the product $\phi_b(t)\,\phi_a(t')$ vanishes on the diagonal $t = t'$. So, writing

$$e^{i(\delta(t)-\delta(t'))z} = (\delta(t) - \delta(t'))^{-K} (-i\,\partial/\partial z)^K e^{i(\delta(t)-\delta(t'))z}$$

for any $K \in \mathbb{Z}_+$ and integrating by parts, we represent P as an integral operator

$$Pu\,(t) = \frac{1}{2\pi} \int_\Gamma dz \int_T \times\, e^{i(\delta(t)-\delta(t'))z}\, \phi_b(t)\phi_a(t')\, \frac{(i\partial/\partial z)^K p(t, z)}{(\delta(t) - \delta(t'))^K}\, \sqrt{\delta'(t')} e^{\gamma\delta(t')}\; v(t')dt',$$

where $v = \sqrt{\delta'} e^{-\gamma\delta} u$ belongs to $L^2(T, L^2(X))$ if $u \in \mathcal{H}^{0,\gamma}(\mathcal{C})$. The inclusion $Pu \in \mathcal{H}^{s,\gamma+\epsilon}(\mathcal{C})$ for a non-negative integer s means that, given any $j = 0, 1, \ldots, s$ and any differential operator A_j of order $s - j$ on X, we have

$$\sqrt{\delta'} e^{-(\gamma+\epsilon)\delta}\, A_j \boldsymbol{D}^j\, Pu \in L^2(T, L^2(X)).$$

This operator, when acting on $v = \sqrt{\delta'} e^{-\gamma\delta} u$, has the Schwartz kernel

$$K(t, t') = \frac{1}{2\pi} \int_\Gamma \sqrt{\delta'(t)} e^{-(\gamma+\epsilon)\delta(t)} \times \boldsymbol{D}_t^j \left(e^{i(\delta(t)-\delta(t'))z}\, \phi_b(t)\phi_a(t')\, \frac{A_j\, (i\partial/\partial z)^K p(t, z)}{(\delta(t) - \delta(t'))^K} \right) \sqrt{\delta'(t')} e^{\gamma\delta(t')} dz, \tag{3.2.9}$$

and we are going to estimate the L^2-norm of this kernel. More precisely, our objective is to prove that

$$\iint_{T\times T} \|K(t,t')\|_2^2 \; dtdt' < \infty,$$

the integration being in fact over $t \in (c''', C)$ and $t' \in (a, c'')$.

To do this, we observe that

$$\boldsymbol{D}_t^j \, e^{i(\delta(t)-\delta(t'))z} = z^j \;\; e^{i(\delta(t)-\delta(t'))z}$$

and

$$|e^{i(\delta(t)-\delta(t'))z}| = e^{\gamma(\delta(t)-\delta(t'))}$$

for $z \in \Gamma$. Moreover,

$$\left\| \boldsymbol{D}_t^j \left(\phi_b(t)\, A_j\, (i\partial/\partial z)^K p(t,z)\right)\right\|_2 \le const\,(j,C)\;(1+|z|^2)^{\frac{\varpi-K+\frac{n}{2}}{2}},$$

the constant being independent of $t \in [c''', C]$ (cf. (3.2.3)). Taking into account that $\phi_b(t)\,\phi_a(t')$ vanishes in a neighborhood of the diagonal $t = t'$, we may estimate the Hilbert-Schmidt norm of the integrand in (3.2.9) as

$$\begin{aligned}
&\sqrt{\delta'(t)}\, e^{-\epsilon\delta(t)}\, |z|^j \, \frac{(1+|z|^2)^{\frac{\varpi-K+\frac{n}{2}}{2}}}{(1+|\delta(t)-\delta(t')|)^K}\, \sqrt{\delta'(t')} \\
&\quad \le \sqrt{\delta'(t)}\, e^{-\epsilon\delta(t)}\, \frac{(1+|z|^2)^{\frac{\varpi-K+\frac{n}{2}+s}{2}}}{(1+|\delta(t)-\delta(t')|)^K}\, \sqrt{\delta'(t')} \qquad (3.2.10)
\end{aligned}$$

up to a constant, uniformly in $z \in \Gamma$, $(t,t') \in (c''', C) \times (a, c'')$ and $j \le s$.

If K is large enough, then the integral of (3.2.10) over $z \in \Gamma$ converges and we obtain

$$\|K(t,t')\|_2 \le const\,(K)\;\sqrt{\delta'(t)}\, e^{-\epsilon\delta(t)}\, \frac{1}{(1+|\delta(t)-\delta(t')|)^K}\, \sqrt{\delta'(t')}$$

for all $(t,t') \in (c''', C) \times (a, c'')$. Thus, for the L^2-norm of $\|K(t,t')\|_2$ we obtain an estimate

$$\begin{aligned}
&\iint_{T\times T} \|K(t,t')\|_2^2 \; dtdt' \\
&\quad \le const\,(K)\; \int_{c'''}^{C} d\delta(t) \int_a^{c''} \frac{e^{-2\epsilon\delta(t)}}{(1+|\delta(t)-\delta(t')|)^{2K}}\, d\delta(t') \\
&\quad = const\,(K)\; \int_{\delta(c''')}^{\delta(C)} dr \int_{-\infty}^{\delta(c'')} \frac{e^{-2\epsilon r}}{(1+|r-r'|)^{2K}}\, dr' \\
&\quad < \infty,
\end{aligned}$$

provided that $K \geq 1$. This proves *Case 2*.

Case 3 $(\mu, \nu = a, i)$ This is the most difficult case. Here we will make use of the fact that both $\phi_\mu(t)\, p(t,z)$ and $\phi_\nu(t)\, a(t,z)$ are holomorphic functions in the strip $\{z \in \mathbb{C} : |\mathrm{im}\, z + \gamma| < \epsilon\}$. To shorten notation, we omit the factors $\phi_\mu(t)$ and $\phi_\nu(t)$ including them into $p(t,z)$ and $a(t,z)$.

For $u \in C^\infty_{comp}(T, C^\infty(X))$, we have

$$\begin{aligned} Au\,(t) &= \mathrm{op}\,(a)u\,(t) \\ &= \frac{1}{2\pi} \int_{\Gamma_{-\alpha}} e^{i\delta(t)z}\, a(t,z)\, \boldsymbol{F}u(z)\, dz, \end{aligned}$$

where $\Gamma_{-\alpha}$ may be any horizontal line with $\gamma - \varepsilon < \alpha < \gamma + \varepsilon$. Given any $\zeta \in \mathbb{C}$ with $\mathrm{im}\, \zeta > -\alpha$, it follows that

$$\begin{aligned} \boldsymbol{F}_{t\mapsto\zeta} Au &= \frac{1}{2\pi} \int_{\Gamma_{-\alpha}} \left(\frac{1}{2\pi} \int_T e^{-i(\zeta - z)\delta(t)}\, a(t,z)\, dm(t) \right) \boldsymbol{F}u(z)\, dz \\ &= \frac{1}{2\pi} \int_{\Gamma_{-\alpha}} \boldsymbol{F}a(\zeta - z, z)\, \boldsymbol{F}u(z)\, dz \end{aligned}$$

(cf. Lemma 3.2.2). In much the same way for $\boldsymbol{F}(PAu)$ we obtain

$$\boldsymbol{F}_{t\mapsto w} PAu = \frac{1}{(2\pi)^2} \int_{\Gamma_{-\beta}} d\zeta \int_{\Gamma_{-\alpha}} \boldsymbol{F}p(w-\zeta, \zeta)\, \boldsymbol{F}a(\zeta - z, z)\, \boldsymbol{F}u(z)\, dz, \tag{3.2.11}$$

with $\mathrm{im}\, w > -\beta > -\alpha$. Analysis similar to that in the proof of Lemma 3.2.2 shows that the integral in (3.2.11) converges.

Now, using the Taylor formula, we write

$$\boldsymbol{F}p(w-\zeta,\zeta) = \sum_{k=0}^{K-1} \frac{1}{k!} (\partial/\partial z)^k \boldsymbol{F}p\,(w-\zeta, z)\ (\zeta - z)^k + R_K \boldsymbol{F}p\ (\zeta - z)^K, \tag{3.2.12}$$

where

$$R_K \boldsymbol{F}p\,(w-\zeta,\zeta,z) = \int_0^1 \frac{(1-\theta)^{K-1}}{(K-1)!} \boldsymbol{F}(\partial/\partial z)^K p\,(w-\zeta, z + \theta(\zeta - z))\, d\theta. \tag{3.2.13}$$

By Lemma 1.4.1, the regular terms in (3.2.12) after substitution into (3.2.11) and integration over ζ give

$$\begin{aligned} &\frac{1}{2\pi} \int_{\Gamma_{-\beta}} (\partial/\partial z)^k \boldsymbol{F}p\,(w-\zeta,z)\,(\zeta - z)^k\, \boldsymbol{F}a(\zeta - z, z)\, d\zeta \\ &\quad = \frac{1}{2\pi} \int_{\Gamma_{-\beta}} \boldsymbol{F}(\partial/\partial z)^k p\,(w-\zeta,z)\, \boldsymbol{F}\boldsymbol{D}^k a(\zeta - z, z)\, d\zeta \\ &\quad = \boldsymbol{F}_{t\mapsto w-z} \Big((\partial/\partial z)^k p(t,z)\, \boldsymbol{D}^k a(t,z) \Big), \end{aligned}$$

thus resulting in $\operatorname{op}(p \circ a|_K)u$. Hence the operator R_K corresponds to the remainder term in (3.2.12), i.e.,

$$\boldsymbol{F}_{t\mapsto w} R_K u = \frac{1}{(2\pi)^2} \int_{\Gamma_{-\beta}} d\zeta \int_{\Gamma_{-\alpha}} R_K \boldsymbol{F} p\,(w-\zeta,\zeta,z)\, \boldsymbol{F}\boldsymbol{D}^K a\,(\zeta-z,z)\, \boldsymbol{F}u(z)\, dz. \tag{3.2.14}$$

If $K \geq 1$, then the function $\boldsymbol{F}\boldsymbol{D}^K a\,(\zeta - z, z)$ is holomorphic in ζ belonging to the half-plane $\operatorname{im}\zeta > -\alpha - 1$, for $a(\delta^{-1}(\log \varrho), z)$ is C^∞ up to $\varrho = 0$ (cf. (3.2.2)). Hence it follows that the integration line $\Gamma_{-\beta}$ may be shifted arbitrarily within the strip $|\operatorname{im}\zeta + \gamma| < \varepsilon$. Thus, the assumption $\operatorname{im} w > -\beta > -\alpha$ is needed no longer, and the only requirement remaining is $\operatorname{im} w > -\beta$. It will be convenient to take

$$-\beta < \operatorname{im} w \leq -\alpha. \tag{3.2.15}$$

To prove that (3.2.14) belongs to the trace class, we again represent it as the composition of two Hilbert-Schmidt operators

$$\mathcal{H}^{0,\gamma}(\mathcal{C}) \stackrel{\mathrm{R_K}}{\to} \mathcal{H}^{s,\gamma+\epsilon}(\mathcal{C}) \hookrightarrow \mathcal{H}^{0,\gamma}(\mathcal{C})$$

for some $s > \frac{n+1}{2}$, which may be taken as an even integer, and some $\epsilon > 0$. The second operator in this sequence is an embedding. By Lemma 3.2.4, the weight gain ϵ is necessary to have a Hilbert-Schmidt embedding. Now, we choose $\alpha = \gamma$, $\operatorname{im} w = -\gamma - \epsilon$ and $\gamma + \epsilon < \beta < \gamma + \varepsilon$ in (3.2.15).

Lemma 3.2.5 *Suppose s is a non-negative even integer. Then the operator*

$$R_K : \mathcal{H}^{0,\gamma}(\mathcal{C}) \to \mathcal{H}^{s,\gamma+\epsilon}(\mathcal{C})$$

is a Hilbert-Schmidt operator provided K is sufficiently large.

Proof. The assumption that s is an even integer serves only to simplify the proof. Using representation (3.2.14) for R_K, we have to evaluate the Hilbert-Schmidt norm of the operator between L^2-spaces on the lines $\Gamma_{-\gamma}$ and $\Gamma_{-\gamma-\epsilon}$, whose Schwartz kernel is

$$K(w,z) = \frac{1}{(2\pi)^2} \int_{\Gamma_{-\beta}} \Lambda^s(w)\, R_K \boldsymbol{F} p\,(w-\zeta,\zeta,z)\, \boldsymbol{F}\boldsymbol{D}^K a\,(\zeta - z, z)\, d\zeta.$$

To estimate the Hilbert-Schmidt norm of the integrand, we invoke Lemma 3.2.2. Namely, we have

$$\begin{aligned}
&\|\Lambda^s(w)\, R_K \boldsymbol{F} p\,(w-\zeta,\zeta,z)\, \boldsymbol{F}\boldsymbol{D}^K a\,(\zeta-z,z)\|_2 \\
&\quad \leq \|\Lambda^s(w)\, R_K \boldsymbol{F} p\,(w-\zeta,\zeta,z)\|_2\ \|\boldsymbol{F}\boldsymbol{D}^K a\,(\zeta-z,z)\|_{\mathcal{L}(L^2(X))} \\
&\quad \leq \mathit{const}\,(j)\ (1+|\zeta-z|^2)^{-\frac{j}{2}}\ \|\Lambda^s(w)\, R_K \boldsymbol{F} p\,(w-\zeta,\zeta,z)\|_2
\end{aligned}$$

with arbitrary large j.

In view of (3.2.13) our next goal is to evaluate the norm

$$\|\Lambda^s(w)\, \boldsymbol{F}(\partial/\partial z)^K p\,(w-\zeta, z+\theta(\zeta-z))\|_2. \qquad (3.2.16)$$

Denoting $z+\theta(\zeta-z)$ by υ and putting $s=2N$, $N\in\mathbb{Z}_+$, we may rewrite $\Lambda^s(w)$ by the binomial formula as

$$\begin{aligned}\Lambda^s(w) &= (1+(\upsilon+i\gamma)^2-\Delta-(\upsilon+i\gamma)^2+(w+i\gamma)^2)^N\\ &= \sum_{i+j+k=N}(-1)^j\,\frac{N!}{i!j!k!}\,\Lambda^{2i}(\upsilon)\,(\upsilon+i\gamma)^{2j}\,(w+i\gamma)^{2k}.\end{aligned}$$

Finally, applying inequality (3.2.3) for the operator $\Lambda^{2i}(\upsilon)\,\boldsymbol{F}(\partial/\partial z)^K p\,(w-\zeta,\upsilon)$ of order $\varpi-K+2i$ yields

$$\|\Lambda^{2i}(\upsilon)\,\boldsymbol{F}(\partial/\partial z)^K p\,(w-\zeta,\upsilon)\|_2 \le const\,(j)\;(1+|w-\zeta|^2)^{-\frac{j}{2}}\;(1+|\upsilon|^2)^{\frac{\varpi-K+2i+\frac{n}{2}}{2}}$$

provided $\varpi-K+2i+\frac{n}{2}<0$, with j an arbitrary positive integer. This enables us to derive a rough estimate of (3.2.16), namely

$$\begin{aligned}&\|\Lambda^s(w)\,\boldsymbol{F}(\partial/\partial z)^K p\,(w-\zeta,z+\theta(\zeta-z))\|_2\\ &\quad\le const\,(j,s,\gamma)\;(1+|w-\zeta|^2)^{-\frac{j}{2}}\,(1+|\upsilon|^2)^{\frac{\varpi-K+s+\frac{n}{2}}{2}}\;(1+|w|^2)^{\frac{s}{2}}\end{aligned}$$

where $\varpi-K+s+\frac{n}{2}$ is supposed to be negative.

We next write w as $w=(w-\zeta)+(\zeta-z)+z$ and apply the binomial formula to arrive at

$$(1+|w|^2)^{\frac{s}{2}} \le const\,(s)\;(1+|w-\zeta|^2)^{\frac{s}{2}}\,(1+|\zeta-z|^2)^{\frac{s}{2}}\,(1+|z|^2)^{\frac{s}{2}}.$$

These crude estimates result in the following estimate for the kernel $K(w,z)$

$$\begin{aligned}\|K(w,z)\|_2 &\le const\,(i,j,s,\gamma)\;\int_0^1 d\theta\\ &\times\int_{\Gamma_{-\beta}}\frac{(1+|z|^2)^{\frac{s}{2}}}{(1+|w-\zeta|^2)^{\frac{i}{2}}(1+|\zeta-z|^2)^{\frac{j}{2}}(1+|z+\theta(\zeta-z)|^2)^{\frac{k}{2}}}\,d\zeta,\end{aligned} \qquad (3.2.17)$$

where i, j are arbitrary positive integers and $k>0$ may be made larger in magnitude at the expense of K.

The needed Hilbert-Schmidt norm is

$$\int_{\Gamma_{-\gamma-\epsilon}} dw\int_{\Gamma_{-\gamma}}\|K(w,z)\|_2^2\,dz.$$

Taking i large enough, we may integrate over $w \in \Gamma_{-\gamma-\epsilon}$ thus arriving at the following estimate of the above integral

$$\int_0^1 d\theta \int_{\Gamma_{-\gamma}} dz \int_{\Gamma_{-\beta}} \frac{(1+|z|^2)^s}{(1+|\zeta-z|^2)^j(1+|z+\theta(\zeta-z)|^2)^k}\, d\zeta,$$

up to an unessential constant.

We first consider the region $|\zeta - z| < \frac{|z|}{2}$. Then

$$\begin{array}{rcl} \frac{1}{(1+|\zeta-z|^2)^j} & \le & 1, \\ \frac{1}{(1+|z+\theta(\zeta-z)|^2)^k} & \le & \frac{2^{2k}}{(1+|z|^2)^k} \end{array}$$

and

$$\int_{\Gamma_{-\gamma}} dz \int_{\zeta\in\Gamma_{-\beta}:\,|\zeta-z|<\frac{|z|}{2}} \frac{1}{(1+|z|^2)^{k-s}}\, d\zeta \le \int_{\Gamma_{-\gamma}} \frac{|z|}{(1+|z|^2)^{k-s}}\, dz,$$

which is convergent for k large enough. On the other hand, for $|\zeta - z| \ge \frac{|z|}{2}$, we estimate

$$\frac{1}{(1+|z+\theta(\zeta-z)|^2)^k} \le 1$$

and

$$\begin{array}{l} \displaystyle\int_{\Gamma_{-\gamma}} dz \int_{\zeta\in\Gamma_{-\beta}:\,|\zeta-z|\ge\frac{|z|}{2}} \frac{(1+|z|^2)^s}{(1+|\zeta-z|^2)^j}\, d\zeta \\ \displaystyle\quad \le 2\int_{\Gamma_{-\gamma}} (1+|z|^2)^s\, dz \int_{\frac{|z|}{2}}^{\infty} \frac{1}{(1+(\gamma-\beta)^2+\vartheta^2)^j}\, d\vartheta \\ \displaystyle\quad \le 2^{2j-1} \int_{\Gamma_{-\gamma}} \frac{1}{(1+|z|^2)^{j-1-s}}\, dz \int_{\frac{|z|}{2}}^{\infty} \frac{d\vartheta}{1+\vartheta^2}, \end{array}$$

which converges if j is large enough. It follows that the Hilbert-Schmidt norm of R_K is finite.

□

To complete the proof, it remains to observe that the functions of the type $R_K u$ are supported in a finite interval $(0, A] \subset T$ because $p \circ a|_K$ vanishes for t close to b, independently of z. Thus, Lemma 3.2.4 may be applied implying that R_K is of trace class.

Case 4 ($\mu = a, \nu = b$) Here we have a pseudolocality property similar to *Case 2*, but the proof runs in a slightly different way. Again we have $(\phi_a p) \circ (\phi_b a)|_K = 0$, so we need to show that the operator $\mathrm{op}\,(\phi_a p)\,\phi_b \mathrm{op}\,(a)$ is of trace class. As the symbol $a(t,z)$ vanishes for t close to the endpoint b of T, we may assume that ϕ_b has a compact support in the interval T. Since $\mathrm{op}\,(a)$ is bounded in $\mathcal{H}^{0,\gamma}(\mathcal{C})$, it suffices to prove that $\mathrm{op}\,(\phi_a p)\,\phi_b$ belongs to the trace class. But the multiplication operator $u \mapsto \phi_b u$ may be regarded as an $\boldsymbol{F}$-pseudodifferential operator with a holomorphic symbol, so we are under the assumptions of *Case 3*. This proves the theorem.

□

3.3 *Regularized trace of product*

Given any two operators $A = \mathrm{op}\,(a)$ and $P = \mathrm{op}\,(p)$ with

$$\begin{array}{ll} a(t,z) \in C^\infty_{loc}(T, \Psi^m(X;\Gamma)), & m \le 0, \\ p(t,z) \in C^\infty_{loc}(T, \Psi^\varpi(X;\Gamma)), & \varpi \le 0, \end{array}$$

satisfying $a(b-,z) = 0$ and $p(b-,z) = 0$, we define the *regularized trace* of the product PA by

$$\mathrm{tr}_\mathrm{K}\,\mathrm{PA} = \mathrm{tr}\,\left(\mathrm{PA} - \mathrm{op}\,(\mathrm{p} \circ \mathrm{a}|_\mathrm{K})\right). \qquad (3.3.1)$$

Theorem 3.2.3 shows that the regularized trace of PA does exist if the number K is sufficiently large.

Theorem 3.3.1 *The regularized trace of the product is independent of the order, that is*

$$\mathrm{tr}_\mathrm{K}\,\mathrm{PA} = \mathrm{tr}_\mathrm{K}\,\mathrm{AP}. \qquad (3.3.2)$$

Proof. We are going to consider several cases corresponding to those listed in the proof of Theorem 3.2.3.

Case 1 $(\mu, \nu \neq a)$ The assertion reduces to the theorem on the regularized trace of the product of Fourier pseudodifferential operators (cf. Fedosov [Fed74]).

Case 2 $(\mu = b, \nu = a)$ or *Case 4* $(\mu = a, \nu = b)$ For $P = \mathrm{op}\,(\phi_\mu p)$ and $A = \mathrm{op}\,(\phi_\nu a)$, we deduce by Theorem 3.2.3 that both PA and AP are of trace class. *Lidskii*'s *Theorem* now shows that the traces of PA and AP coincide.

Case 3 $(\mu, \nu = a, i)$ Using (3.2.14) with $\alpha = \gamma$ and $-\beta < \mathrm{im}\, w = -\alpha$ (cf. (3.2.15)), we get

$$\begin{aligned} \mathrm{tr}_\mathrm{K}\,\mathrm{PA} &= \mathrm{tr}\,R_K \\ &= \frac{1}{(2\pi)^2} \int_{\Gamma_{-\beta}} d\zeta \int_{\Gamma_{-\gamma}} \mathrm{tr}\,R_K \boldsymbol{F} p\,(z-\zeta,\zeta,z)\,\boldsymbol{F}\boldsymbol{D}^K a\,(\zeta - z, z) dz, \end{aligned} \qquad (3.3.3)$$

the second equality being a consequence of $\mathrm{tr}\,\mathrm{R_K} = \mathrm{tr}\,\boldsymbol{F}_{-\gamma}\mathrm{R_K}\boldsymbol{F}^{-1}_{-\gamma}$. The function

$$R_K \boldsymbol{F} p\,(z-\zeta,\zeta,z) = \frac{1}{z-\zeta} R_K \boldsymbol{F}\boldsymbol{D} p\,(z-\zeta,\zeta,z)$$

has a pole of the first order at $\zeta = z$ (for $p(\delta^{-1}(\log \varrho), z)$ is smooth up to $\varrho = 0$). So, we may move the lines $\Gamma_{-\gamma}$ and $\Gamma_{-\beta}$ within the strips $\{z \in \mathbb{C} : |\mathrm{im}\, z + \gamma| < \varepsilon\}$ and $\{z \in \mathbb{C} : |\mathrm{im}\, \zeta + \gamma| < \varepsilon\}$, provided that $-\beta$ remains smaller than $-\gamma$.

Moreover, we may shift $\Gamma_{-\beta}$ crossing $\Gamma_{-\gamma}$, but then we must take into account the residue at $\zeta = z$. It is equal to

$$\frac{1}{2\pi i} \int_{\Gamma_{-\gamma}} \mathrm{tr}\, \mathrm{R_K} \boldsymbol{FD} \mathrm{p}\,(0, \mathrm{z}, \mathrm{z})\, \boldsymbol{FD}^{\mathrm{K}} \mathrm{a}\,(0, \mathrm{z})\, \mathrm{dz}.$$

By (3.2.13), we have

$$R_K \boldsymbol{FD} p\,(0, z, z) = \frac{1}{K!} \boldsymbol{FD} (\partial/\partial z)^K p\,(0, z),$$

so the residue is equal to

$$\frac{1}{2\pi i} \int_{\Gamma_{-\gamma}} \mathrm{tr}\, \frac{1}{\mathrm{K}!} \boldsymbol{FD} (\partial/\partial \mathrm{z})^{\mathrm{K}} \mathrm{p}\,(0, \mathrm{z})\, \boldsymbol{FD}^{\mathrm{K}} \mathrm{a}\,(0, \mathrm{z})\, \mathrm{dz}.$$

However, for $K > 1$, this integral vanishes because

$$\begin{aligned} \boldsymbol{FD}^K a\,(0, z) &= \int_T \frac{1}{i} \frac{\partial}{\partial t} \boldsymbol{D}^{K-1} a(t, z)\, dt \\ &= \frac{1}{i} \left(\boldsymbol{D}^{K-1} a(b-, z) - \boldsymbol{D}^{K-1} a(a+, z) \right) \\ &= 0. \end{aligned}$$

Hence it follows that, for $K > 1$, integral (3.3.3) does not depend on the position of the lines $\Gamma_{-\gamma}$ and $\Gamma_{-\beta}$ within the strips $\{z \in \mathbb{C} : |\mathrm{im}\, z + \gamma| < \varepsilon\}$ and $\{z \in \mathbb{C} : |\mathrm{im}\, \zeta + \gamma| < \varepsilon\}$. A similar assertion holds for the integral

$$\mathrm{tr_K\, AP} = \frac{1}{(2\pi)^2} \int_{\Gamma_{-\beta}} \mathrm{d}\zeta \int_{\Gamma_{-\gamma}} \mathrm{tr}\, \mathrm{R_K} \boldsymbol{F} \mathrm{a}\,(\mathrm{z} - \zeta, \zeta, \mathrm{z})\, \boldsymbol{FD}^{\mathrm{K}} \mathrm{p}\,(\zeta - \mathrm{z}, \mathrm{z})\, \mathrm{dz}. \tag{3.3.4}$$

Our goal is to prove that (3.3.3) and (3.3.4) are equal. To this end, we consider the families

$$\begin{aligned} a_s(t, z) &= a(t, z)\, \Lambda^s(z), \\ p_s(t, z) &= p(t, z)\, \Lambda^s(z), \end{aligned}$$

where s is a complex parameter ranging in the half-strip

$$\{s \in \mathbb{C} : \Re s \le 0,\ |\mathrm{im}\, s| \le R\},$$

$R > 0$. Let A_s and P_s be the corresponding operators on the stretched cone $\mathcal{C}$. The estimates used in the proof of Theorem 3.2.3 show that both

$\mathrm{tr_K\,P_sA_s}$ and $\mathrm{tr_K\,A_sP_s}$ are holomorphic functions of s belonging to the above half-strip. Thus, it is sufficient to prove the equality

$$\mathrm{tr_K\,P_sA_s} = \mathrm{tr_K\,A_sP_s}$$

for $\Re s$ close to $-\infty$.

We are reduced to verifying (3.3.2) for operators A and P of sufficiently large negative order. For this purpose, we write

$$\frac{1}{z-\zeta}\, R_K \boldsymbol{FD}p\,(z-\zeta,\zeta,z)$$
$$= \frac{-1}{(\zeta-z)^{K+1}}\left(\boldsymbol{FD}p\,(z-\zeta,\zeta) - \sum_{k=0}^{K-1}\frac{1}{k!}\,\boldsymbol{FD}(\partial/\partial z)^k p\,(z-\zeta,z)\,(\zeta-z)^k\right)$$

and then

$$\mathrm{tr_K\,PA} = -\frac{1}{(2\pi)^2}\int_{\Gamma_{-\beta}}\mathrm{d}\zeta\int_{\Gamma_{-\gamma}}\frac{\mathrm{tr}\,\boldsymbol{FD}\mathrm{p}\,(\mathrm{z}-\zeta,\zeta)\,\boldsymbol{FD}\mathrm{a}\,(\zeta-\mathrm{z},\mathrm{z})}{(\zeta-\mathrm{z})^2}\mathrm{dz}$$
$$+\sum_{k=0}^{K-1}\frac{1}{k!}\frac{1}{(2\pi)^2}\int_{\Gamma_{-\beta}}d\zeta\int_{\Gamma_{-\gamma}}\frac{\mathrm{tr}\,\boldsymbol{FD}(\partial/\partial \mathrm{z})^{\mathrm{k}}\mathrm{p}(\mathrm{z}-\zeta,\mathrm{z})(\zeta-\mathrm{z})^{\mathrm{k}}\boldsymbol{FD}\mathrm{a}(\zeta-\mathrm{z},\mathrm{z})}{(\zeta-z)^2}dz. \tag{3.3.5}$$

Each summand in (3.3.5) makes sense if both a and p have large negative orders and $-\beta < -\gamma$ are fixed.

Interchanging the variables z and ζ in the first integral and using the equality $\mathrm{tr}\,\boldsymbol{FD}\mathrm{p}\,\boldsymbol{FD}\mathrm{a} = \mathrm{tr}\,\boldsymbol{FD}\mathrm{a}\,\boldsymbol{FD}\mathrm{p}$ for trace class operators, we obtain

$$-\frac{1}{(2\pi)^2}\int_{\Gamma_{-\gamma}}d\zeta\int_{\Gamma_{-\beta}}\frac{\mathrm{tr}\,\boldsymbol{FD}\mathrm{a}\,(\mathrm{z}-\zeta,\zeta)\,\boldsymbol{FD}\mathrm{p}\,(\zeta-\mathrm{z},\mathrm{z})}{(\zeta-z)^2}dz. \tag{3.3.6}$$

The remaining summands in the right-hand side of (3.3.5) may be transformed as follows. We introduce the new coordinate $\upsilon = z-\zeta$ ranging along the line $\Gamma_{-\gamma+\beta}$, thus obtaining

$$\int_{\Gamma_{-\beta}}d\zeta\int_{\Gamma_{-\gamma}}\frac{\mathrm{tr}\,\boldsymbol{FD}(\partial/\partial \mathrm{z})^{\mathrm{k}}\mathrm{p}\,(\mathrm{z}-\zeta,\mathrm{z})\,(\zeta-\mathrm{z})^{\mathrm{k}}\,\boldsymbol{FD}\mathrm{a}\,(\zeta-\mathrm{z},\mathrm{z})}{(\zeta-z)^2}dz$$
$$=(-1)^k\int_{\Gamma_{-\gamma+\beta}}d\upsilon\int_{\Gamma_{-\gamma}}\frac{\mathrm{tr}\,(\partial/\partial \mathrm{z})^{\mathrm{k}}\boldsymbol{FD}\mathrm{p}\,(\upsilon,\mathrm{z})\,\upsilon^{\mathrm{k}}\,\boldsymbol{FD}\mathrm{a}\,(-\upsilon,\mathrm{z})}{\upsilon^2}dz.$$

Now, integrating by parts with respect to z and permutting $\boldsymbol{FD}p$ and $\boldsymbol{FD}p$ under the trace sign, we get

$$\int_{\Gamma_{-\gamma+\beta}}d\upsilon\int_{\Gamma_{-\gamma}}\frac{\mathrm{tr}\,(\partial/\partial \mathrm{z})^{\mathrm{k}}\boldsymbol{FD}\mathrm{a}\,(-\upsilon,\mathrm{z})\,\upsilon^{\mathrm{k}}\,\boldsymbol{FD}\mathrm{p}\,(\upsilon,\mathrm{z})}{\upsilon^2}dz$$

$$= \int_{\Gamma_{-\gamma+(-\gamma+\beta)}} d\zeta \int_{\Gamma_{-\gamma}} \frac{\mathrm{tr}\, \boldsymbol{FD}(\partial/\partial \mathrm{z})^{\mathrm{k}} \mathrm{a}\,(\mathrm{z}-\zeta,\mathrm{z})\,(\zeta-\mathrm{z})^{\mathrm{k}}\, \boldsymbol{FD}\mathrm{p}\,(\zeta-\mathrm{z},\mathrm{z})}{(\zeta-z)^2} dz$$
$$= \int_{\Gamma_{-\gamma}} d\zeta \int_{\Gamma_{-\beta}} \frac{\mathrm{tr}\, \boldsymbol{FD}(\partial/\partial \mathrm{z})^{\mathrm{k}} \mathrm{a}\,(\mathrm{z}-\zeta,\mathrm{z})\,(\zeta-\mathrm{z})^{\mathrm{k}}\, \boldsymbol{FD}\mathrm{p}\,(\zeta-\mathrm{z},\mathrm{z})}{(\zeta-z)^2} dz, \tag{3.3.7}$$

the last equality being obtained by shifting both the lines of integration towards the vector $i(\gamma-\beta)$.

Summing up (3.3.6) and (3.3.7) for $k = 0, 1, \dots, K-1$, we arrive at the equality

$$\mathrm{tr}_{\mathrm{K}}\, \mathrm{PA} = -\frac{1}{(2\pi)^2} \int_{\Gamma_{-\gamma}} \mathrm{d}\zeta \int_{\Gamma_{-\beta}} \frac{\mathrm{tr}\, \mathrm{R}_{\mathrm{K}} \boldsymbol{FD}\mathrm{a}\,(\mathrm{z}-\zeta,\zeta,\mathrm{z})\, \boldsymbol{FD}^{\mathrm{K}}\mathrm{p}\,(\zeta-\mathrm{z},\mathrm{z})}{\zeta-\mathrm{z}}\, \mathrm{dz}.$$

This expression coincides with the corresponding expression (3.3.4) for $\mathrm{tr}_{\mathrm{K}}\, \mathrm{AP}$ except that the lines $\Gamma_{-\gamma}$ and $\Gamma_{-\beta}$ are interchanged. To complete the proof it remains to note that, as we have seen, we may interchange $\Gamma_{-\gamma}$ and $\Gamma_{-\beta}$ not affecting the value of the integral.

□

It is worth pointing out that Theorem 3.3.1 remains valid for all operators A and P of zero order, whose symbols are equal to 1 for $t \in T$ close to the endpoint b of T. Indeed, we can write

$$A = \mathrm{op}\,(\mathrm{a}) + 1,$$
$$P = \mathrm{op}\,(\mathrm{p}) + 1$$

with some symbols $a(t,z)$ and $p(t,z)$ vanishing for $t \in T$ close to b. Then, an easy computation shows that

$$\begin{aligned} \mathrm{tr}_{\mathrm{K}}\, \mathrm{PA} &= \mathrm{tr}_{\mathrm{K}}\ \mathrm{op}\,(\mathrm{p})\, \mathrm{op}\,(\mathrm{a}), \\ \mathrm{tr}_{\mathrm{K}}\, \mathrm{AP} &= \mathrm{tr}_{\mathrm{K}}\ \mathrm{op}\,(\mathrm{a})\, \mathrm{op}\,(\mathrm{p}), \end{aligned}$$

so the desired conclusion follows from Theorem 3.3.1 applied to the operators op (a) and op (p).

3.4 *Algebraic index*

First we introduce an algebra of formal symbols on T, define elliptic symbols and introduce an algebraic index of elliptic elements. Then, constructing a parametrix and applying the theorem on a regularized trace of product, we prove that the analytical and algebraic indices coincide.

A *formal symbol* is a formal power series

$$a(t,z) = \sum_{j=0}^{\infty} h^j a_j(t,z),$$

where the coefficients $a_j(t,z) \in C^\infty_{loc}(T, \Psi^{\mathrm{m-j}}(\mathrm{X};\Gamma))$, $m \le 0$, satisfy the following conditions:

- $(\partial/\partial z)^k\, a_j(t,z) \in C^\infty_{loc}(T, \Psi^{\mathrm{m-j-k}}(\mathrm{X};\Gamma))$ for all $k = 0, 1, \ldots$;
- $a_j(t,z)$ is "sufficiently" smooth up to $t = 0$ in the sense that the symbol $a_j(\delta^{-1}(\log\varrho), z)$ is C^∞ up to $\varrho = 0$;
- for $t \in (C, b)$, $a_0(t,z)$ is independent of t while $a_j(t,z) = 0$ if $j > 0$.

The powers of a formal parameter h serve for ordering the series terms. Let us define the product $\circ$ of two symbols by

$$p \circ a = \sum_{i,j,k=0}^{\infty} h^{i+j+k}\, \frac{1}{k!}\, (\partial/\partial z)^k\, p_i(t,z)\, \boldsymbol{D}^k a_j(t,z).$$

It is easy to check that the formal symbols form an associative algebra with the unit $e(t,z) \equiv 1$ consisting of the leading term only. We denote this algebra by $\mathcal{A}$.

Introduce a *trace ideal* $\mathcal{I}$ in this algebra, consisting of those formal symbols $a(t,z)$ for which $m < -n-1$ and all the components $a_j(t,z)$ vanish at the point $t = a$ and for $t \in (C,b)$. A *trace* for $a \in \mathcal{I}$ is defined by

$$\mathrm{tr\, a} = \sum_{\mathrm{j}=0}^{\infty} \mathrm{h}^{\mathrm{j}-1} \left(\frac{1}{2\pi}\right)^2 \int_\Gamma \mathrm{dz} \int_{\mathrm{T}} \mathrm{tr\, a_j(t,z)\, dm(t)}.$$

This is a formal series with constant coefficients and the exponents of h ranging from -1 to $+\infty$.

Lemma 3.4.1 *If one of the formal symbols p and a belongs to the ideal $\mathcal{I}$, then*

$$\mathrm{tr\, p \circ a = tr\, a \circ p}.$$

Proof. Use integration by parts.

□

A symbol $a \in \mathcal{A}$ is said to be *elliptic* if there exists a symbol p such that both $1 - p\circ a$ and $1 - a \circ p$ belong to $\mathcal{I}$.

Such a symbol p is called a (formal) *parametrix* of a. In particular, for leading terms p_0 and a_0 we obtain

$$\begin{array}{rcl} 1 - p_0 a_0 & \in & \mathcal{I}, \\ 1 - a_0 p_0 & \in & \mathcal{I}. \end{array} \qquad (3.4.1)$$

The following construction is well-known (see for instance Fedosov [Fed95]).

Lemma 3.4.2 *Let there exist a function $p_0(t,z)$ satisfying (3.4.1). Then, for J large enough, the symbol*

$$\begin{aligned} p &= \textstyle\sum_{j=0}^{J}(1-p_0\circ a)^j\circ p_0 \\ &= p_0\circ\textstyle\sum_{j=0}^{J}(1-a\circ p_0)^j, \end{aligned} \tag{3.4.2}$$

the powers being understood with respect to the product $\circ$, is a parametrix of a.

Proof. By direct calculation we have

$$\begin{aligned} 1-p\circ a &= (1-p_0\circ a)^{\circ(J+1)}, \\ 1-a\circ p &= (1-a\circ p_0)^{\circ(J+1)}, \end{aligned}$$

where the exponent $\circ(J+1)$ means the $(J+1)$-th power with respect to the product $\circ$. Clearly, the formal symbols on the right belong to $\mathcal{I}$ if J is large enough.

□

Given any elliptic symbol $a\in\mathcal{A}$, we define the *algebraic index* of a by

$$\operatorname{ind} a = \operatorname{tr}(1-\mathrm{p}\circ\mathrm{a})-\operatorname{tr}(1-\mathrm{a}\circ\mathrm{p}). \tag{3.4.3}$$

By definition, $\operatorname{ind} a$ is a formal power series in h with constant coefficients. It turns out, however, that all the coefficients, with the possible exception of a constant term, vanish, so we can treat it as a number. Moreover, the algebraic index is independent of the particular choice of the formal parametrix p. All these properties are standard consequences of the *stability* of the index.

Lemma 3.4.3 *Suppose $a(\lambda)$ is a family of elliptic symbols in $\mathcal{A}$ and $p(\lambda)$ is a family of formal parametrices for $a(\lambda)$. Then,*

$$\operatorname{tr}(1-\mathrm{p}(\lambda)\circ\mathrm{a}(\lambda))-\operatorname{tr}(1-\mathrm{a}(\lambda)\circ\mathrm{p}(\lambda))$$

is independent of λ.

Proof. An easy computation shows that

$$\begin{aligned} (1-p\circ a)' &= (1-p\circ a)'\circ(p\circ a)+(1-p\circ a)'\circ(1-p\circ a) \\ &= ((1-p\circ a)\circ(p\circ a))'-(1-p\circ a)\circ(p\circ a)'-(p\circ a)'\circ(1-p\circ a), \end{aligned}$$

where "prime" means the derivation in λ. Thus,

$$\operatorname{tr}(1-\mathrm{p}\circ\mathrm{a})' = (\mathrm{d}/\mathrm{d}\lambda)\operatorname{tr}((1-\mathrm{p}\circ\mathrm{a})\circ(\mathrm{p}\circ\mathrm{a}))-2\operatorname{tr}((1-\mathrm{p}\circ\mathrm{a})\circ(\mathrm{p}'\circ\mathrm{a}+\mathrm{p}\circ\mathrm{a}')).$$

Similarly,

$$\mathrm{tr}\,(1-\mathrm{a}\circ\mathrm{p})' = (\mathrm{d}/\mathrm{d}\lambda)\,\mathrm{tr}\,((1-\mathrm{a}\circ\mathrm{p})\circ(\mathrm{a}\circ\mathrm{p})) - 2\,\mathrm{tr}\,((1-\mathrm{a}\circ\mathrm{p})\circ(\mathrm{a}'\circ\mathrm{p}+\mathrm{a}\circ\mathrm{p}')).$$

Since

$$\begin{array}{lclcl}
\mathrm{tr}\,((1-\mathrm{p}\circ\mathrm{a})\circ(\mathrm{p}\circ\mathrm{a})) & = & \mathrm{tr}\,(\mathrm{p}\circ(1-\mathrm{a}\circ\mathrm{p})\circ\mathrm{a}) & = & \mathrm{tr}\,((1-\mathrm{a}\circ\mathrm{p})\circ(\mathrm{a}\circ\mathrm{p})),\\
\mathrm{tr}\,((1-\mathrm{p}\circ\mathrm{a})\circ(\mathrm{p}'\circ\mathrm{a})) & = & \mathrm{tr}\,(\mathrm{a}\circ(1-\mathrm{p}\circ\mathrm{a})\circ\mathrm{p}') & = & \mathrm{tr}\,((1-\mathrm{a}\circ\mathrm{p})\circ(\mathrm{a}\circ\mathrm{p}')),\\
\mathrm{tr}\,((1-\mathrm{p}\circ\mathrm{a})\circ(\mathrm{p}\circ\mathrm{a}')) & = & \mathrm{tr}\,(\mathrm{p}\circ(1-\mathrm{a}\circ\mathrm{p})\circ\mathrm{a}') & = & \mathrm{tr}\,((1-\mathrm{a}\circ\mathrm{p})\circ(\mathrm{a}'\circ\mathrm{p})),
\end{array}$$

both expressions $\mathrm{tr}\,(1-\mathrm{p}\circ\mathrm{a})'$ and $\mathrm{tr}\,(1-\mathrm{a}\circ\mathrm{p})'$ coincide, and the lemma follows.

□

In particular, given two parametrices p_1 and p_2 of the same elliptic symbol a, we consider the linear homotopy $p(\lambda) = (1-\lambda)p_1+\lambda p_2$, $\lambda \in [0,1]$, which gives a family of parametrices. Then, Lemma 3.4.3 implies that the index does not depend on the choice of a parametrix.

Now, for a real $\lambda > 0$, we define a homomorphism $H(\lambda) : \mathcal{A} \to \mathcal{A}$ by

$$H(\lambda)a\,(t,z) = \sum_{j=0}^{\infty} \lambda^j h^j a_j(t,\lambda z).$$

It is a simple matter to see that $H(\lambda)$ is in fact a homomorphism of the algebra $\mathcal{A}$, i.e., $H(\lambda)(p\circ a) = (H(\lambda)p)\circ(H(\lambda)a)$.

Lemma 3.4.4 *If $a \in \mathcal{I}$, then*

$$\mathrm{tr}\,\mathrm{H}(\lambda)\mathrm{a} = \mathrm{H}(\lambda)\ \mathrm{tr}\,\mathrm{a},$$

where $H(\lambda)$ acts on formal series with constant coefficients by replacing h by λh.

Proof. The lemma follows by the change of variables $z \mapsto \frac{z}{\lambda}$. Indeed,

$$\begin{array}{rcl}
\mathrm{tr}\ \mathrm{H}(\lambda)\mathrm{a} & = & \displaystyle\sum_{j=0}^{\infty} \lambda^j h^{j-1} \left(\frac{1}{2\pi}\right)^2 \int_\Gamma dz \int_T \mathrm{tr}\,\mathrm{a_j}(\mathrm{t},\lambda\mathrm{z})\,\mathrm{dm}(\mathrm{t})\\
& = & \displaystyle\sum_{j=0}^{\infty} \lambda^{j-1} h^{j-1} \left(\frac{1}{2\pi}\right)^2 \int_\Gamma dz \int_T \mathrm{tr}\,\mathrm{a_j}(\mathrm{t},\mathrm{z})\,\mathrm{dm}(\mathrm{t})\\
& = & H(\lambda)\ \mathrm{tr}\,\mathrm{a},
\end{array}$$

as required.

□

We are now in a position to show that the algebraic index is actually independent of the parameter h.

Lemma 3.4.5 *The formal series* $\operatorname{ind} a$ *consists of the constant term only.*

Proof. For $\lambda > 0$, we consider the family $a(\lambda) = H(\lambda)a$ of elliptic symbols. Then $p(\lambda) = H(\lambda)p$ is a family of parametrices since

$$\begin{array}{rcll} 1 - H(\lambda)p \circ H(\lambda)a & = & H(\lambda)(1 - p \circ a) & \in \mathcal{I}, \\ 1 - H(\lambda)a \circ H(\lambda)p & = & H(\lambda)(1 - a \circ p) & \in \mathcal{I}. \end{array}$$

Hence

$$\begin{array}{rcl} \operatorname{ind} a(\lambda) & = & \mathrm{tr\,H(\lambda)(1 - p \circ a) - tr\,H(\lambda)(1 - a \circ p)} \\ & = & H(\lambda) \operatorname{ind} a. \end{array}$$

On the other hand, $\operatorname{ind} a(\lambda)$ is independent of λ, by Lemma 3.4.3, which completes the proof.

□

3.5 *Analytical index*

The following result has encountered so often that it can be attributed to the mathematical folk-lore.

Lemma 3.5.1 *A closed densely defined operator* $A : H^0 \to H^1$ *in Hilbert spaces is Fredholm if and only if there exists an operator* $P : H^1 \to H^0$ *such that both* $1 - PA$ *and* $1 - AP$ *are operators of trace class. Moreover,*

$$\operatorname{ind} A = \mathrm{tr\,(1 - PA) - tr\,(1 - AP)}. \tag{3.5.1}$$

Proof. *Necessity.* The equality $\dim \operatorname{coker} A = d$ means that there are elements $f_1, \ldots, f_d$ such that each element $f \in H^1$ can be uniquely written in the form $f = Au + \sum_{j=1}^{d} c_j f_j$, where u is orthogonal to $\ker A$. The operator $\hat{A} : (\ker A)^\perp \oplus \mathbb{C}^d \to H^1$, given by

$$(u, c_1, \ldots, c_d) \mapsto Au + \sum_{j=1}^{d} c_j f_j,$$

is densely defined, closed and has the inverse operator defined on the whole space H^1. By the *Open Mapping Theorem*, the inverse operator $\hat{A}^{-1}$ is bounded. Hence it follows that there is a constant $c > 0$ such that $\|u\|_{H^0} \le c\,\|Au\|_{H^1}$ for each $u \in D(A) \cap (\ker A)^\perp$. Therefore, the range $\operatorname{im} A$ of A is closed and so we have the *orthogonal decomposition* $H^1 = \ker A^* \oplus \operatorname{im} A$, where A^* stands for the adjoint of A.

¿From the *a priori estimate* written as $(A^*Au,u)_{H^0} \le \frac{1}{c^2}(u,u)_{H^0}$, we deduce that the range of A^* is closed. Thus, we have the orthogonal decomposition $H^0 = \ker A \oplus \operatorname{im} A^*$, whence the Fredholm property of A^* is obvious.

Now, we introduce an operator R by being $\hat{A}^{-1}$ on $\operatorname{im} A$ and 0 on $\ker A^*$. Then, $R: H^1 \to H^0$ is a bounded operator and a trivial verification shows that $1 - RA$ and $1 - AR$ are orthogonal projections onto $\ker A$ and $\ker A^*$ respectively.

Finally, from what has already been proved it follows that the spaces $\operatorname{coker} A$ and $\ker A^*$ are isomorphic, whence

$$\begin{aligned} \operatorname{ind} A &= \dim\ker A - \dim\ker A^* \\ &= \operatorname{tr}(1 - \mathrm{RA}) - \operatorname{tr}(1 - \mathrm{AR}), \end{aligned}$$

as required.

Sufficiency. Suppose $P: H^1 \to H^0$ is a bounded operator, such that both $1-PA$ and $1-AP$ are of trace class. Then, PA and AP are Fredholm operators, because they differ from the identity operators by compact operators. Since $\ker A \subset \ker PA$ and $\operatorname{im} A \supset \operatorname{im} AP$, the Fredholm property of A follows.

On the other hand, the equality (3.5.1) is fulfilled for the operator $P = R$ and is independent on the particular choice of P, for the operators $(P-R)A$ and $A(P-R)$ are of trace class and have the same traces. This completes the proof.

□

The operator P is called a *parametrix* (or *regularizer*) of the operator A. This notation is sometimes used for the operators P with the weaker property that both $1 - PA$ and $1 - AP$ are compact.

In the sequel, by the *analytical index* of A we mean the right-hand side of (3.5.1). We are going to compare the analytical and algebraic indices.

Given an operator $A = \mathrm{op(a)}$ of the form (3.1.1), we may treat its symbol $a(t,z)$ as a formal symbol consisting of the leading term only.

Lemma 3.5.2 *If* $A = \mathrm{op(a)}$ *is an elliptic operator of order* 0*, then there exists a formal parametrix for a.*

Proof. The ellipticity conditions listed in Definition 3.1.1 imply that there exists a symbol $p_0(t,z)$ such that both $1 - p_0 a$ and $1 - ap_0$ belong to $\mathcal{I}$.

Indeed, for $t \in T$ close to the endpoints of T, a^{-1} exists by definition. An important point to note here is the so-called *spectral invariance* of the subalgebra of parameter-dependent elliptic operators on X. This implies that if $a(t,z)$ is invertible within the (larger) algebra of pseudodifferential operators on X, then it is also invertible within the subalgebra of parameter-dependent elliptic operators. Thus, there is a segment $[c,C] \subset\subset T$ such that $a^{-1}(t,z)$ is a parameter-dependent elliptic pseudodifferential operator of order zero on X, for all $t \in T$ away from $[c,C]$.

For $t \in [c,C]$, the symbol $a(t,z)$ is parameter-dependent elliptic, with the parameter $z \in \Gamma$. This implies, in particular, that $a(t,z)$ is also invertible for $|z| > R$, provided R is large enough.

As for $t \in [c,C]$ and $|z| \le R$, it follows from the parameter-dependent ellipticity of $a(t,z)$ that this symbol has a parametrix $r(t,z) \in C^\infty_{loc}(T, \Psi^0(\mathrm{X};\Gamma))$ on X.

Now, we pick a C^∞ function ϕ on $T \times \Gamma$ which is equal to 1 in a neighborhood of the rectangle $[c,C] \times [-R,R]$ and vanishes away from a compact subset of $T \times \Gamma$. For $(t,z) \in T \times \Gamma$, define

$$p_0(t,z) = \phi(t,z)\ r(t,z) + (1 - \phi(t,z))\, a^{-1}(t,z). \tag{3.5.2}$$

Then,

$$\begin{aligned} 1 - p_0 a &= \phi\,(1 - ra) \in \mathcal{I}, \\ 1 - a p_0 &= \phi\,(1 - ar) \in \mathcal{I}, \end{aligned}$$

i.e., p_0 satisfies (3.4.1). By Lemma 3.4.2, the function $p_0(t,z)$ may serve as a leading term of the formal parametrix given by (3.4.2). This proves the lemma.

□

Thus, for elliptic operators $A = \mathrm{op(a)}$ of order zero, the algebraic index is well-defined.

To compute the analytical index of A, we need an operator parametrix P inverting A up to trace class operators. To this end, given a formal symbol $p(t,z) = \sum_{j=0}^\infty h^j p_j(t,z)$, we introduce the notation

$$p|_J = \sum_{j=0}^{J-1} p_j, \qquad J = 1, 2, \ldots.$$

Theorem 3.5.3 *Let* $A = \mathrm{op(a)}$ *be an elliptic operator of order zero and let* p *be a formal parametrix (3.4.2) of the symbol* a*. Then, for* J *large enough, the operator* $P = \mathrm{op(p|_J)}$ *is an operator parametrix of* A *and*

$$\begin{aligned} \operatorname{ind} A &= \mathrm{tr}\,(1 - \mathrm{PA}) - \mathrm{tr}\,(1 - \mathrm{AP}) \\ &= \mathrm{tr}\,(1 - \mathrm{p} \circ \mathrm{a}) - \mathrm{tr}\,(1 - \mathrm{a} \circ \mathrm{p}). \end{aligned} \tag{3.5.3}$$

Proof. Denoting $p|_J$ by r and taking N sufficiently large, we obtain

$$\begin{aligned} 1 - \mathrm{op(r)\ op(a)} &= \mathrm{op((1 - r \circ a)|_N) - (op(r)\,op(a) - op(r \circ a|_N))}, \\ 1 - \mathrm{op(a)\ op(r)} &= \mathrm{op((1 - a \circ r)|_N) - (op(a)\,op(r) - op(a \circ r|_N))}. \end{aligned}$$

By Theorems 3.2.3 and 3.3.1, the operators

$$\mathrm{op(r)\,op(a) - op(r \circ a|_N)}, \\ \mathrm{op(a)\,op(r) - op(a \circ r|_N)}$$

are of trace class and their traces are equal.

On the other hand, an easy computation shows that

$$\begin{aligned} r \circ a|_N &= \sum_{k=0}^{N-1} \sum_{j=0}^{J-1} \frac{1}{k!} \left(\partial/\partial z\right)^k p_j \, \boldsymbol{D}^k a, \\ p \circ a|_J &= \sum_{i+k \le J-1} \frac{1}{k!} \left(\partial/\partial z\right)^k p_i \, \boldsymbol{D}^k a, \end{aligned}$$

hence, for $N \geq J$, the difference $\mathrm{op(r \circ a|_N) - op(p \circ a|_J)}$ is a finite sum of terms

$$\mathrm{op}\left((\partial/\partial \mathrm{z})^{\mathrm{k}} \, \mathrm{p_j} \, \boldsymbol{D}^{\mathrm{k}} \mathrm{a} \right),$$

with $j+k \geq N$, $j \leq J-1$ and $k \leq N-1$. If $N > n+1$, then this operator is of trace class since its order is less then $-n-1$ and its symbol vanishes for $t \in T$ away from the segment $[c, C]$.

The same is true for $\mathrm{op(a \circ r|_N) - op(a \circ p|_J)}$, which is the sum of

$$\mathrm{op}\left((\partial/\partial \mathrm{z})^{\mathrm{k}} \, \mathrm{a} \, \boldsymbol{D}^{\mathrm{k}} \mathrm{p_j} \right),$$

with $j+k \geq N$, $j \leq J-1$ and $k \leq N-1$.

Moreover, the traces of such operators are equal. Indeed, integrating by parts yields

$$\begin{aligned} &\int_\Gamma dz \int_T \mathrm{tr}\left((\partial/\partial \mathrm{z})^{\mathrm{k}} \, \mathrm{p_j(t,z)} \, \boldsymbol{D}^{\mathrm{k}} \mathrm{a(t,z)} \right) \mathrm{dm(t)} \\ &\quad = \int_\Gamma dz \int_T \mathrm{tr}\left((\partial/\partial \mathrm{z})^{\mathrm{k}} \, \mathrm{a(t,z)} \, \boldsymbol{D}^{\mathrm{k}} \mathrm{p_j(t,z)} \right) \mathrm{dm(t)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{tr\,(1 - PA) - tr\,(1 - AP)} &= \mathrm{tr\,op(1 - r \circ a)|_N - tr\,op(1 - a \circ r)|_N} \\ &= \mathrm{tr\,op(1 - p \circ a)|_J - tr\,op(1 - a \circ p)|_J} \\ &= \mathrm{tr\,(1 - p \circ a)|_J - tr\,(1 - a \circ p)|_J}, \end{aligned}$$

which is precisely the algebraic index.

□

3.6 *Topological index*

Following Fedosov [Fed74], we introduce yet another algebra which allows one to simplify significantly various calculations with non-commutative differential forms.

We will use a real variable τ instead of $z = \tau + i\gamma$. An element a of our new algebra $\hat{\mathcal{A}}$ is an operator-valued non-homogeneous differential form of even degrees on $T \times \mathbb{R}$. Thus,

$$a(t, \tau) = a_0(t, \tau) + a_1(t, \tau) dt \wedge d\tau, \qquad (3.6.1)$$

where $a_0(t, \tau)$ and $a_1(t, \tau)$ 3are pseudodifferential operators on X of non-positive orders.

A product $\hat{\circ}$ of two elements $p, a \in \hat{\mathcal{A}}$ is defined by

$$p \hat{\circ} a = p \wedge a + \frac{i}{2} dp \wedge da.$$

One immediately checks that this product is associative.

Any function $a(t, \tau) \in C^\infty_{loc}(T, \Psi^m(X; \Gamma))$, $m \le 0$, may be considered as an element of $\hat{\mathcal{A}}$ consisting of 0-component only. Thus, for functions a and p we have three products:

- pa is the usual point-wise operator product of functions;
- $p \circ a = pa + h\ (\partial/\partial\tau)\, p\, \boldsymbol{D} a + \ldots$ is a product in $\mathcal{A}$ as formal symbols; and
- $p \hat{\circ} a = pa + \frac{i}{2} dp \wedge da$ is a product in $\hat{\mathcal{A}}$.

We may also consider the powers of a function a with respect to any of these products, using the notations a^j, $a^{\circ j}$ and $a^{\hat{\circ} j}$ to distinguish the three possibilities.

One can verify a simple rule to pass from the $\circ$-product to the $\hat{\circ}$-product of functions:

- *keep the terms of degree* ≤ 1 *in* h, *then alternate the derivations* $\partial/\partial\tau$ *and* $\boldsymbol{D}$, *and then write* $dt \wedge d\tau$ *instead of* $h \frac{1}{\delta'(t)}$.

This rule remains true for any number of factors $a_1 \circ \ldots \circ a_j$ and $a_1 \hat{\circ} \ldots \hat{\circ} a_j$.

Similarly to $\mathcal{I}$ we introduce a *trace ideal* $\hat{\mathcal{I}}$ in $\hat{\mathcal{A}}$. It consists of forms (3.6.1) where a_0 and a_1 are operators of order $m < -n - 1$ with regard for a parameter $\tau \in \mathbb{R}$, vanishing at the point $t = a$ and for $t \in (C, b)$. For $a \in \hat{\mathcal{I}}$, we define a *trace* by

$$\begin{aligned} \operatorname{tr} a &= \frac{1}{2\pi} \iint_{T \times \mathbb{R}} \operatorname{tr} a \\ &= \frac{1}{2\pi} \iint_{T \times \mathbb{R}} \operatorname{tr} a_1\ dt \wedge d\tau, \end{aligned}$$

the orientation of $T \times \mathbb{R}$ being given by the form $dt \wedge d\tau$.

The trace property $\operatorname{tr} p \hat{\circ} a = \operatorname{tr} a \hat{\circ} p$ is obviously satisfied if either a or p belongs to $\hat{\mathcal{I}}$.

When using this definition of trace, we have the following formula, of which the right-hand side will be referred to as the *topological index*.

Theorem 3.6.1 *For any $J \geq 1$, we have*

$$\operatorname{ind} \mathrm{A} = \operatorname{tr} (1 - \mathrm{p_0} \hat{\circ} \mathrm{a})^{\hat{\circ}(\mathrm{J}+1)} - \operatorname{tr} (1 - \mathrm{a} \hat{\circ} \mathrm{p_0})^{\hat{\circ}(\mathrm{J}+1)}, \tag{3.6.2}$$

where p_0 is the leading term of the parametrix of a.

Proof. We begin with the algebraic index formula (3.5.3) taking

$$\begin{aligned} p &= \sum_{j=0}^{J} (1 - p_0 \circ a)^j \circ p_0 \\ &= p_0 \circ \sum_{j=0}^{J} (1 - a \circ p_0)^j, \end{aligned}$$

with J large enough. Then

$$\begin{aligned} 1 - p \circ a &= (1 - p_0 \circ a)^{\circ(J+1)}, \\ 1 - a \circ p &= (1 - a \circ p_0)^{\circ(J+1)}, \end{aligned}$$

whence

$$\operatorname{ind} \mathrm{A} = \operatorname{tr} (1 - \mathrm{p_0} \circ \mathrm{a})^{\circ(\mathrm{J}+1)} - \operatorname{tr} (1 - \mathrm{a} \circ \mathrm{p_0})^{\circ(\mathrm{J}+1)}. \tag{3.6.3}$$

According to Lemma 3.4.5, we need to extract the constant term in (3.6.3). It follows that we may calculate both $(1 - p_0 \circ a)^{\circ(J+1)}$ and $(1 - a \circ p_0)^{\circ(J+1)}$ keeping merely the terms of degree ≤ 1 in h.

We have

$$1 - p_0 \circ a = 1 - p_0 a - h\, (\partial/\partial\tau) p_0\, \boldsymbol{D} a - \ldots,$$

"dots" meaning the terms of higher degree in h. By induction, one easily arrives at the equality

$$\begin{aligned} (1 - p_0 \circ a)^{\circ(J+1)} &= (1 - p_0 a)^{J+1} \\ &- h \left(\sum_{j=0}^{J} (1 - p_0 a)^j\, (\partial/\partial\tau) p\, \boldsymbol{D} a\, (1 - p_0 a)^{J-j} \right. \\ &\left. - \sum_{i+j+k=J-1} (1 - p_0 a)^j\, (\partial/\partial\tau)(1 - p_0 a)\, (1 - p_0 a)^i\, \boldsymbol{D}(1 - p_0 a)\, (1 - p_0 a)^k \right) \\ &+ \ldots . \end{aligned} \tag{3.6.4}$$

The sum $\sum_{i+j+k=J-1}(\cdot)$ on the right of (3.6.4) may be written as

$$\sum_{j=0}^{J} (\partial/\partial\tau)(1-p_0a)^j\, \boldsymbol{D}(1-p_0a)^{J-j} \tag{3.6.5}$$

or

$$\sum_{j=0}^{J} (1-p_0a)^j\, (\partial/\partial\tau)(1-p_0a)\, \boldsymbol{D}(1-p_0a)^{J-j}.$$

Using "integration by parts", we can transform the latter expression to the form

$$\begin{aligned} \boldsymbol{D} \sum_{j=0}^{J} & (1-p_0a)^j\, (\partial/\partial\tau)(1-p_0a)\, (1-p_0a)^{J-j} \\ & - \sum_{j=0}^{J} \boldsymbol{D}(1-p_0a)^j\, (\partial/\partial\tau)(1-p_0a)\, (1-p_0a)^{J-j} \\ & - \sum_{j=0}^{J} (1-p_0a)^j\, \boldsymbol{D}(\partial/\partial\tau)(1-p_0a)\, (1-p_0a)^{J-j}. \end{aligned} \tag{3.6.6}$$

If $J \geq 1$, all the written terms belong to the trace ideal $\mathcal{I}$ for they contain a factor $1-p_0a \in \mathcal{I}$ or its derivatives.

Let us now write down the constant term of the trace of $(1-p_0\circ a)^{\circ(J+1)}$. We represent the second sum in the right-hand side of (3.6.4) as the half-sum of expressions (3.6.5) and (3.6.6). We may drop the first sum in (3.6.6) for the complete derivative in t vanish under integration. Then, permutting cyclically the factors under the trace sign, we obtain

$$\begin{aligned} -\frac{1}{2\pi i} \iint_{T\times\mathbb{R}} \operatorname{tr} \Bigg(& (J+1)\,(1-p_0a)^J \left(\frac{\partial p_0}{\partial\tau}\frac{\partial a}{\partial t} - \frac{\partial^2}{\partial t\partial\tau}(p_0a) \right) \\ - \frac{1}{2}\sum_{j=0}^{J} & \left(\frac{\partial}{\partial\tau}(1-p_0a)^j \frac{\partial}{\partial t}(1-p_0a) - \frac{\partial}{\partial t}(1-p_0a)^j \frac{\partial}{\partial\tau}(1-p_0a) \right) (1-p_0a)^{J-j} \Bigg) dt d\tau. \end{aligned}$$

Since

$$\frac{\partial p_0}{\partial\tau}\frac{\partial a}{\partial t} - \frac{\partial^2}{\partial t\partial\tau}(p_0a) = \frac{1}{2}\frac{\partial p_0}{\partial\tau}\frac{\partial a}{\partial t} - \frac{1}{2}\frac{\partial p_0}{\partial t}\frac{\partial a}{\partial\tau} - \frac{1}{2}\frac{\partial^2 p_0}{\partial t\partial\tau}a - \frac{1}{2}p_0\frac{\partial^2 a}{\partial t\partial\tau}$$

and

$$\left(\frac{\partial p_0}{\partial\tau}\frac{\partial a}{\partial t} - \frac{\partial p_0}{\partial t}\frac{\partial a}{\partial\tau} \right) dt\wedge d\tau$$

$$= -dp_0 \wedge da,$$

$$\left(\frac{\partial}{\partial\tau}(1-p_0a)^j\frac{\partial}{\partial t}(1-p_0a) - \frac{\partial}{\partial t}(1-p_0a)^j\frac{\partial}{\partial\tau}(1-p_0a)\right) dt \wedge d\tau$$

$$= -d(1-p_0a)^j \wedge d(1-p_0a),$$

we get, for the constant term of $\operatorname{tr}(1 - p_0 \circ a)^{\circ(J+1)}$, the expression

$$\begin{aligned}&\frac{1}{2\pi i}\iint_{T\times\mathbb{R}}\frac{1}{2}\\ &\quad\times \operatorname{tr}\left((J+1)\,(1-p_0a)^J dp_0 \wedge da - \sum_{j=0}^{J} d(1-p_0a)^j \wedge d(1-p_0a)(1-p_0a)^{J-j}\right)\\ &\quad+ \frac{1}{2\pi i}\iint_{T\times\mathbb{R}}\frac{1}{2}\operatorname{tr}\left((J+1)\,(1-p_0a)^J\left(\frac{\partial^2 p_0}{\partial t\partial\tau}a + p_0\frac{\partial^2 a}{\partial t\partial\tau}\right)\right) dtd\tau.\end{aligned} \tag{3.6.7}$$

A similar expression can be written for the constant term of the trace of $(1 - a \circ p_0)^{\circ(J+1)}$ by interchanging a and p_0. Note that

$$\operatorname{tr}\left((1-p_0a)^J\,\frac{\partial^2 p_0}{\partial t\partial\tau}\,a\right) = \operatorname{tr}\left(a\,(1-p_0a)^J\,\frac{\partial^2 p_0}{\partial t\partial\tau}\right) = \operatorname{tr}\left((1-ap_0)^J\,a\,\frac{\partial^2 p_0}{\partial t\partial\tau}\right),$$

$$\operatorname{tr}\left((1-p_0a)^J\,p_0\,\frac{\partial^2 a}{\partial t\partial\tau}\right) = \operatorname{tr}\left(p_0\,(1-ap_0)^J\,\frac{\partial^2 a}{\partial t\partial\tau}\right) = \operatorname{tr}\left((1-ap_0)^J\,\frac{\partial^2 a}{\partial t\partial\tau}\,p_0\right).$$

Hence it follows that the last integral in (3.6.7) does not change under permutation of a and p_0. Thus, taking the difference of (3.6.7) and the corresponding expression obtained by interchanging a and p_0, we find

$$\begin{aligned}\operatorname{ind}\mathrm{A} &= \frac{1}{2\pi\mathrm{i}}\iint_{\mathrm{T}\times\mathbb{R}}\frac{1}{2}\\ &\times\Bigg(\operatorname{tr}\left((J+1)(1-p_0a)^J dp_0 \wedge da - \sum_{j=0}^{J} d(1-p_0a)^j \wedge d(1-p_0a)\,(1-p_0a)^{J-j}\right)\\ &\quad-\operatorname{tr}\left((J+1)(1-ap_0)^J da \wedge dp_0 - \sum_{j=0}^{J} d(1-ap_0)^j \wedge d(1-ap_0)\,(1-ap_0)^{J-j}\right)\Bigg).\end{aligned} \tag{3.6.8}$$

Taking into account the rule for passing from the $\circ$-product to the $\hat{\circ}$-product, one easily recognizes formula (3.6.2) in (3.6.8). In particular, we obtain that the right-hand side of (3.6.8) or (3.6.2) is independent of J, provided J is large enough.

Using the $\hat{\circ}$-product, we prove now that (3.6.2) is valid for $J \geq 1$. Indeed,

$$\begin{aligned} &\operatorname{tr}(1-p_0\hat{\circ}a)^{\hat{\circ}(J+1)} - \operatorname{tr}(1-a\hat{\circ}p_0)^{\hat{\circ}(J+1)} \\ &\quad = \Big(\operatorname{tr}(1-p_0\hat{\circ}a)^{\hat{\circ}J} - \operatorname{tr}(1-a\hat{\circ}p_0)^{\hat{\circ}J}\Big) \\ &\qquad - \Big(\operatorname{tr}(1-p_0\hat{\circ}a)^{\hat{\circ}J}\hat{\circ}(p_0\hat{\circ}a) - \operatorname{tr}(1-a\hat{\circ}p_0)^{\hat{\circ}J}\hat{\circ}(a\hat{\circ}p_0)\Big). \end{aligned}$$

However, by the associativity of the $\hat{\circ}$-product, we have

$$\begin{aligned} \operatorname{tr}(1-p_0\hat{\circ}a)^{\hat{\circ}J}\hat{\circ}(p_0\hat{\circ}a) &= \operatorname{tr} p_0\hat{\circ}(1-a\hat{\circ}p_0)^{\hat{\circ}J}\hat{\circ}a \\ &= \operatorname{tr}(1-a\hat{\circ}p_0)^{\hat{\circ}J}\hat{\circ}(a\hat{\circ}p_0) \end{aligned}$$

for $J \geq 1$. The last equality is due to the fact that $(1-a\hat{\circ}p_0)^{\hat{\circ}J} \in \hat{\mathcal{I}}$ for $J \geq 1$, and hence a cyclic permutation of factors under the trace sign is possible. The proof is complete.

□

For $J = 1$, formula (3.6.8) becomes

$$\mathrm{ind\,A} = \frac{1}{2\pi \mathrm{i}} \iint_{\mathrm{T}\times\mathbb{R}} \operatorname{tr}((1-\mathrm{p_0 a})\mathrm{dp_0} \wedge \mathrm{da}) - \operatorname{tr}((1-\mathrm{ap_0})\mathrm{da} \wedge \mathrm{dp_0})$$

for

$$\begin{aligned} \operatorname{tr} d(1-p_0a) \wedge d(1-p_0a) &= \operatorname{tr} d(1-ap_0) \wedge d(1-ap_0) \\ &= 0. \end{aligned}$$

Integrating by parts in the first summand yields

$$\begin{aligned} \iint_{T\times\mathbb{R}} \operatorname{tr}((1-p_0a)dp_0 \wedge da) &= -\iint_{T\times\mathbb{R}} \operatorname{tr}(d(1-p_0a) \wedge p_0da) \\ &= \iint_{T\times\mathbb{R}} \operatorname{tr}(dp_0 \wedge ap_0da + p_0da \wedge p_0da). \end{aligned}$$

On the other hand, the second summand may be transformed as follows:

$$-\operatorname{tr}((1-ap_0)da \wedge dp_0) = \operatorname{tr}(dp_0 \wedge (1-ap_0)da).$$

So, (3.6.8) for $J = 1$ gives formula (3.1.3).

Acknowledgement

The authors are greatly indebted to Professor Fedosov for numerous discussions on his techniques.

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