FD-METHOD FOR STURM-LIOUVILLE PROBLEMS.
EXPONENTIAL RATE OF CONVERGENCE

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Abstract

In this work there is considered Sturm-Liouvill problem

$$u''(x) + [\lambda - q(x)]u(x) = 0, \qquad x \in (0,1),$$

$$u(0) = u(1) = 0$$

with piecewise smooth coefficient $q(x)$.

The functional-discrete method (FD-method) is used for resolution this problem that was proposed by one of autors.

The aim of this work is to obtain conditions when FD-method has exponential rate of convergence, to obtain explicit estimates of its precision, from which there follow two-sided estimates for exact eigen-values and estimates of remainder terms of classical asumptotic formulas.

1 *Introduction.*

In the works [1, 2, 3] it was shown, that the uniform accuracy of all eigen-values can be obtained by combining standard numerical method for eigen-values with low ordinal numbers with asymptotic formula for high ordinal numbers. However, if method of finite differences or finite elements is used as a numerical method, then a method is obtained, which is non-effective for reaching high accuracy, excluding very low and very high ordinal numbers. Here, when we mention all eigen-values, we mean the number of eigen-values which depends on the numerical method. For example, in the grid method this number is proportional to the inverse step of the grid h. More acceptable methods are received when standard numerical methods are combined with asymptotic correction. Review of results of the above-mentioned approach is contained in [4]. It should be noted that the approaches mentioned above give 1) approximation to the first $\frac{1}{h}$ eigen-values of differential problem only; 2) accuracy, which is rigidly connected with the numerical method being used and with the smoothness of coefficients.

In [5] to calculate eigen-values and eigen-functions of ordinary differential equations with smooth coefficients, the method without saturation of accuracy was proposed. This method is based on the interpolation procedure and gives approximations to the first n eigen-values and eigen-functions, where $(n-1)$ is a power of corresponding polynomial. While eigen-values and eigen-functions with lower ordinal numbers are approximated with accuracy, which is automatically adjusted with Smoothness of eigen-functions, for higher ordinal numbers, which are close to n, convergence does not take place.

In work [61 to solve Sturm-Liouville problem

$$u''(x) + [\lambda - q(x)]u(x) = 0, \quad x \in (0,1),$$
$$u(0) = u(1) = 0 \tag{1}$$

with piecewise smooth coefficient $q(x)$, functional-discrete method (FD-method) was proposed. This method allows, with discretization parameter N fixed, to obtain approximations to all eigen-functions and eigen-values with precision which depends on any power $(Nn)^{-1}$ (see Theorem 1). Here $N + 1$ is equal to the number of steps of piecewise constant function $\overline{q}(x)$,which substitutes function $q(x)$ when we make transition from (1) to discrete-continuous appreximation problem, n is ordinal number of corresponding eigen-value.

FD-method consists in the following. The problem (1) is embedded in more general problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} + [\lambda - w(x,t)]u(x,t) = 0, \quad x \in (0,1),$$
$$u(0,t) = u(1,t) = 0, \tag{2}$$

where

$$w(x,t) = \overline{q}(x) + t[q(x) - \overline{q}(x)],$$

t is parameter from $[0,1]$ - It is clear that when $t = 1$ the solution of problem (2) coincides with the solution of problem (1), i.e,

$$\lambda_n(w(\bullet,1)) = \lambda_n(q(\bullet)),$$
$$n = 1,2,3,... \tag{3}$$
$$u_n(x, w(\bullet,1)) \equiv u_n(x, q(\bullet)).$$

Notations used in (3) accentuate that eigen-values (put in the ascending order) and eigen-functions of problems (1) and (2) are non-linear functionals and non-linear operators (correspondingly) of coefficient $q(x)$ of differential operator.

Let us expand $\lambda_n(w(\bullet,t))$ and $u_n(x, w(\bullet,t))$, as functions of t, into Taylor series in the neighborhood of the point $t = 0$ with remainder term in integral form and let us set $t = 1$. Then we will have

$$\lambda_n(q(\bullet)) = \overset{m}{\lambda}_n(\overline{q}(\bullet)) + R^n_{m+1}(\lambda) = \sum_{j=0}^{m} \frac{1}{j!} \frac{d^j \lambda_n(w(t,\bullet))}{dt^j}\bigg|_{t=0} + \\ + \frac{1}{m!} \int_0^1 (1-t)^m \frac{d^{m+1}\lambda_n(w(t,\bullet))}{dt^{m+1}dt}, \tag{4}$$

$$u_n(x, q(\bullet)) = \overset{m}{u}_n(x, \overline{q}(\bullet)) + R^n_{m+1}(u, x) = \sum_{j=0}^{m} \frac{1}{j!} \frac{\partial^j u_n(x, w(t,\bullet))}{\partial t^j}\bigg|_{t=0} + \\ \frac{1}{m!} \int_0^1 (1-t)^m \frac{\partial^{m+1} u_n(x, w(t,\bullet))}{\partial t^{m+1}dt}. \tag{5}$$

Note, that $\overset{m}{\lambda}_n(\overline{q}(\bullet))$, $\overset{m}{u}_n(x, \overline{q}(\bullet))$ are none other than segments of Volterra type series for a functional $\lambda_n(q(\bullet))$ and operator $u_n(x, q(\bullet))$ correspondingly (see. [6]). Members of the series (4), (5) are obtained recursively according to the following procedure. Let us introduce the notations

$$u_n^{(j)}(x) = \frac{1}{j!} \frac{\partial^j u_n(x, w(t,\bullet))}{\partial t^j}\bigg|_{t=0}, \\ \lambda_n^{(j)}(x) = \frac{1}{j!} \frac{d^j \lambda_n(w(t,\bullet))}{dt^j}\bigg|_{t=0}. \tag{6}$$

Then we will have

$$\begin{cases} \dfrac{d^2 u_n^{(j+1)}(x)}{dx^2} + \left[\lambda_n^{(0)} - \overline{q}(x)\right] u_n^{(j+1)} = -\displaystyle\sum_{s=0}^{j} \lambda_n^{(j+1-s)} u_n^{(s)}(x) + \\ \quad + \left[q(x) - \overline{q}(x)\right] u_n^(j)(x) = -F_n^{(j+1)}(x), \quad x \in (0,1) \\ u_n^{(j+1)}(0) = u_n^{(j+1)}(1) = 0, \\ \quad j = 1, 2, ... \end{cases} \tag{7}$$

$$\lambda_n^{(j+1)} = -\int_0^1 u_n^{(0)}(x) \left\{ \sum_{(s=1)}^{j} \lambda_n^{(j+1-s)} u_n^{(s)}(x) - \left[q(x) - \overline{q}(x)\right] u_n^{(j)}(x) \right\} dx, \tag{8}$$

$$j = 0, 1, ...$$

The initial condition for the recursion process (7), (8) is defined as a solution of the problem

$$\frac{d^2 u_n^{(0)}(x)}{dx^2} + \left[\lambda_n^{(0)} - \overline{q}(x)\right] u_n^{(0)} = 0, \quad x \in (0,1),$$
$$u_n^{(0)}(0) = u_n^{(0)}(1) = 0, \tag{9}$$
$$\|u_n^{(0)}\|_0 = \left[\int_0^1 u_n^{(0)}(x))^2 dx\right]^{1/2} = 1,$$

with piecewise constant coefficient $\overline{q}(x)$. The problem (9) we shall call the basic problem for FD-method. Approximation of the solution of problem (1) with the solution of problem (9) is known in the literature as Pruess method (see [7]).

Let $Q'[0,1]$ be a class of functions which have finite number of discontinuity points $0 < \eta_1 < \eta_2 < ... < \eta_l < 1$, and which on segments of continuity $[\eta_{i-1}, \eta_i]$, $i = \overline{1, l+1}$, $\eta_0 = 0$, $\eta_{l+1} = 1$ have continuous derivatives up to the r-th order, where r is non-negative number. We introduce non-uniform grid

$$\hat{\omega} = \{x_i : \ 0 < x_1 < x_2 < \ldots < x_N < 1, \ h_i = x_1 - x_{i-1} > 0,$$
$$\left.\sum_{i=1}^{N+1} h_i = 1, \quad x_{N+1} = 1, \quad x_0 = 0\right\},$$

such that $\rho = \left\{\eta_i : i = \overline{1, l}\right\} \in \hat{\omega}$ and we introduce the notations

$$h = \max_{1 \le i \le N+1} h_i. \tag{10}$$

In [6] using the condition of normalization $\int\limits_n^2 (x, w(\bullet))dx = 1$ the following theorem was proved:

Theorem 1. Let $q(x) \in Q^2[0,1]$, $\overline{q}(x) = [q(xi-1) + q(x_i)]/2$, $x \in [x_{i-1}, x_i]$, $i = \overline{1, N+1}$, then the following estimates of accuracy of FD-method hold

$$\left|\lambda_n(q(\bullet)) - \overset{m}{\lambda}_n(\overline{q}(\bullet))\right| \le C_m \min\{h^{m+1} n^{-m}, h^{2m+2} n\} \le C_m h^{m+1} n^{-m}, \tag{11}$$

$$\left| u_n(x, q(\bullet)) - \overset{m}{u}_n(x, \overline{q}(\bullet)) \right| \leq D_m \min\left\{ \left(\frac{h}{n}\right)^{m+1}, h^{2m+2} \right\} \leq D_m \left(\frac{h}{n}\right)^{m+1}, \tag{12}$$

where $C_m,\ D_m$ are constants which do not depend on h and n $(h \sim N_{-1})$.

Remark 1. When $m = 0$, from (4), (2) we have

$$\lambda_n(q(\bullet)) - \lambda_n^{(0)}(\overline{q}(\bullet)) = \int_0^1 \frac{d\lambda_n(w(t,\bullet))}{dt} dt =$$

$$\int_0^1 \int_0^1 [q(x) - \overline{q}(x)]\, u_n^2(x, \omega(\bullet, t)) dx dt,$$

and together with the estimate (11), the following estimate follows

$$\left| \lambda_n(q(\bullet)) - \lambda_n^{(0)}(\overline{q}(\bullet)) \right| \leq \|q - \overline{q}\|_\infty$$

which is valid for $q(x) \in Q^{(0)}[0,1]$.

It is clear, that if two piecewise constant functions $\overline{q}_1(x)$ and $\overline{q}_2(x)$ are constructed, such that

$$\overline{q}_1(x) \leq q(x) \leq \overline{q}_2(x),$$

then the following fork holds

$$\lambda_n^{(0)}(\overline{q}_1(\bullet)) \leq \lambda_n(q(\bullet)) \leq \lambda_n^{(0)}(\overline{q}_2(\bullet)), \quad n = 1, 2, ...$$

This idea was used in [8],[9] for more general Stunn-Liouville problem.

The aim of this work is to obtain conditions when FD-method has exponential rate of convergence, to obtain explicit estimates of its precision, from which there follow two-sided estimates for exact eigen-values and estimates of remainder terms of classical asymptotic formulas. To formulate the main result of this work, we introduce notations

$$M_n = \max\left\{ (\lambda_n^{(0)} - \lambda_{n-1}^{(0)})^{-1}, (\lambda_{n+1}^{(0)} - \lambda_n^{(0)})^{-1} \right\}, \quad n \geq 2, \tag{13}$$

$$r_n = 4\|q - \overline{q}\|_\infty M_n \tag{14}$$

and sequence $\{g_i\}_{j=0}^{\infty}$, defined with recursion formula

$$g_{j+1} = \sum_{s=0}^{j} g_s g_{j-s}, \quad j = 0, 1, ..., \quad g_0 = 1 \tag{15}$$

Here $\|\nu\|_\infty = \max\limits_{x\in[0,1]} |\nu(x)|$.

The following theorem is the basic result of this article.

Theorem 2. Let

$$\int\limits_0^1 u_n^{(0)}(x)u_n^{(j)}(x)dx = \delta_{0j}, \quad j = 0, 1, ... \tag{16}$$

and the following condition is fulfilled

$$r_n < 1$$

Then the solution of the problem (1) is represented in form of series

$$\begin{gathered} u_n(x, q(\bullet)) = \sum_{j=0}^{\infty} u_n^{(j)}(x, \overline{q}(\bullet)), \\ \lambda_n(q(\bullet)) = \sum_{j=0}^{\infty} \lambda_n^{(j)}(\overline{q}(\bullet)), \end{gathered} \tag{17}$$

which converge at the rate of geometric progression with denominator r_n, and the following estimates hold

$$\left|\lambda_n(q(\bullet)) - \overset{m}{\lambda}_n(\overline{q}(\bullet))\right| \le \|q - \overline{q}\|_\infty \frac{r_n^m}{1 - r_n}\alpha_m \tag{18}$$

$$\left\|u_n(x, q(\bullet)) - \overset{m}{u}_n(x, \overline{q}(\bullet))\right\|_0 \le \frac{r_n^{m+1}}{1 - r_n}\alpha_{m+1} \tag{19}$$

where

$$\alpha_j = 2\frac{(2j-1)!!}{(2j+2)!!}.$$

Note that from (18), (19) two-sided estimates of precision are obtained at once. They are of a priori - a posteriors nature, because they demand the knowledge of the constant M_n. In case $N = 0, \overline{q}(x) \equiv const$ for M_n, we have explicit expression: $M_n = \frac{1}{\pi^2(2n-1)}$ and the estimates (18), (19) become explicit a priori, which will be used in Example 1.

Corollary 1. Let us set (7)-(9) $\overline{q}(x) \equiv 0$, then, if

$$r_n^0 = \frac{4\|q\|_\infty}{\pi^2(2n-1)} < 1,$$

then the solution of the problem (1) is represented in form of series

$$u_n(x, q(\bullet)) = \sum_{j=0}^{\infty} u_n^{(j)}(x, 0),$$
$$\lambda_n(q(\bullet)) = \sum_{j=0}^{\infty} \lambda_n^{(j)}(0), \tag{17'}$$

which converge at the rate of geometric progression with denominator r_n^0 and the following estimates hold

$$\left|\lambda_n(q(\bullet)) - \overset{m}{\lambda}_n(0)\right| \le \|q\|_\infty \frac{(r_n^0)^m}{1 - r_n^0} \alpha_m$$

$$\left\|u_n(x, q(\bullet)) - \overset{m}{u}_n(x, 0))\right\|_0 \le \frac{(r_n^0)^{m+1}}{1 - r_n^0} \alpha_{m+1}$$

We note that in this case series (17′), when function $q(x)$ is smooth enough, are classic asymptotic series (see, for instance,[10]). For the case $q(x) \in Q^0[0,1]$ we will have [6]

$$\lambda_n(q(\bullet)) =$$

$$= (n\pi)^2 + \int_0^1 (1 - \cos 2n\pi z_1) q(z_1) dz_1 + \frac{1}{n\pi} \int_0^1 \int_0^1 \sin n\pi z_1 \sin n\pi z_2 q(z_1) q(z_2)^*$$

$$^* \left[(z_1 + z_2 - 1)\sin n\pi(z_1 + z_2) - ((|z_1 - z_2| - 1)\sin n\pi|z_1 - z_2|\right] dz_1 dz_2 + R_3^n(\lambda),$$

$$u_n(x, q(\bullet)) = \sqrt{2}\sin n\pi x + \frac{1}{\sqrt{2}n\pi} \int_0^1 q(z) \sin(n\pi z)^*$$

$$^* \left[(z + x - 1)\sin n\pi(z + x) - ((|z - x| - 1)\sin n\pi|z - x|\right] dz + R_2^n(u; x),$$

where

$$|R_3^n(\lambda)| \le \|q\|_\infty \frac{(r_n^0)^2}{1 - r_n^0} 0{,}125 = O\left(\frac{1}{n^2}\right),$$
$$|R_2^n(u; x)| \le \frac{(r_n^0)^2}{1 - r_n^0} 0{,}125 = O\left(\frac{1}{n^2}\right).$$

Estimates of remainder terms, shown here, are based on (18), (19) and make more precise corresponding estimates from [6].

2 *Proof of the theorem 2.*

The solution of the problem (6), (7), (8) can be represented in the form

$$u_n^{(j+1)}(x) = -\sum_{p=1,p\neq n}^{\infty} \frac{(F_n^{j+1}, u_p^{(0)})}{\lambda_p^{(0)} - \lambda_n^{(0)}} u_p^{(0)}(x),$$

from which the estimate follows

$$\|u_n^{(j+1)}\|_0 \leq M_n \|F_n^{j+1}\|_0 \tag{20}$$

which, using inequality,

$$\|F_n^{j+1}\|_0^2 = \left\| \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} - (q-\overline{q}) u_n^{(j)} \right\|_0^2 - \left[\lambda_n^{(j+1)}\right]^2 \leq$$

$$\leq \left\| \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} - (q-\overline{q}) u_n^{(j)} \right\|_0^2 \leq \|q-\overline{q}\|_\infty^2 \left(\sum_{p=0}^{j} \|u_n^{(j-p)}\|_0 \|u_n^{(p)}\|_0 \right)^2,$$

can be represented in the following form

$$\|u_n^{(j+1)}\|_0 \leq \overline{r}_n \left\{ \sum_{s=0}^{j} \|u_n^{(j-s)}\|_0 \|u_n^{(s)}\|_0 \right\}, \tag{21}$$

where $\overline{r}_n = \|q-\overline{q}\|_\infty M_n$.

Using introduced notation (1 5), from (2 1) we obtain

$$\|u_n^{(j+1)}\|_0 \leq \overline{r}_n^{j+1} g_{j+1}. \tag{22}$$

To estimate g_j, $j = \overline{0,\infty}$, we will need an auxiliary statement.

Lemma 1. For members of sequence $\{g_j\}_{j=0}^{\infty}$, defined with recursion formula (15), the following equality holds

$$g_j = 4^j \alpha_j, \quad j = 0, 1, ... \tag{23}$$

Proof. We introduce for the sequence $\{g_j\}_{j=0}^{\infty}$ the generating function

$$f(x) = \sum_{j=0}^{\infty} g_j x^j. \tag{24}$$

Then, according to (1 5), it will satisfy the following equation

$$\frac{1}{x} f(x) = [f(x)]^2 + \frac{1}{x}. \tag{25}$$

From here we find

$$2xf(x) = 1 - \sqrt{1-4x} \tag{26}$$

From (26) we see. that $f(x)$ will be analytical function in the domain $|x| < 1/4$ and the radius of convergence of the series (24) will be $R = 1/4$. Using expansion of the function $\sqrt{1-\alpha}$, $|\alpha| < 1$ into Taylor series, from (26) we obtain

$$2xf(x) = 2x + \sum_{s=2}^{\infty} \frac{(2s-3)!!}{(2s)!!}(4x)^s$$

and the explicit formula for g_i:

$$g_j = 2\frac{(2j-1)!!}{(2j+2)!!}4^j, \quad j = 1, 2, ..., \quad g_0 = 1, \ (-1)!! = 1. \tag{27}$$

Lemma is proved completely.

Taking (23) into account, the inequality (22) obtains the view

$$\|u_n^{(j+1)}\|_0 \leq (4\overline{r}_n)^{j+1}\alpha_{j+1} = r_n^{j+1}\alpha_{j+1}, \tag{28}$$

which together with (8), (16) leads to the estimate

$$|\lambda_n^{j+1}| \leq \|q - \overline{q}\|_\infty \|u_n^{(j)}\|_0 \leq \|q - \overline{q}\|_\infty r_n^j \alpha_j. \tag{29}$$

Finally, we have

$$\left\| u_n(x, q(\bullet)) - \overset{m}{u}_n(x, \overline{q}(\bullet)) \right\| \leq \frac{r_n^{m+1}}{1-r_n}\alpha_{m+1},$$

$$\left| \lambda_n(q(\bullet)) - \overset{m}{\lambda}_n(0) \right| \leq \|q\|_\infty \frac{r_n^m}{1-r_n}\alpha_m$$

which proves the theorem.

Remark 2. It is not hard to transfer the obtained results to the boundary conditions of the third kind

$$u'(0) = \alpha u(0), \quad u'(1) = -\beta u(1), \quad \alpha, \beta \geq 0,$$

while the estimates (1 8), (1 9) are preserved.

3 *Algorithmic Implementation.*

To solve each of the problems (7), (8), let us use techniques of exact three-point difference schemes (e.t.d.s.) [12] (the similar approach to the solution of equations of the (7) kind was proposed in [13]). We introduce the following notations

$$t_i = \lambda_n(\overline{q}(\bullet)) - q_i, \quad \nu_i = |t_i|^{1/2}, \quad i = \overline{1, N+1},$$

$$\mu_1(\nu_i x) = \begin{cases} \nu_i^{-1}\sinh(\nu_i x), & if \quad t_i < 0 \\ x, & if \quad t_i = 0 \\ \nu_i^{-1}\sin(\nu_i x), & if \quad t_i > 0 \end{cases}$$

$$\mu_2(\nu_i x) = \begin{cases} \nu_i^{-1}\cosh(\nu_i x), & if \quad t_i < 0 \\ x, & if \quad t_i = 0 \\ \nu_i^{-1}\cos(\nu_i x), & if \quad t_i > 0 \end{cases}$$

where q_i is one of the possible approximations of $q(x)$ on the interval $[x_{i-1}, x_i]$, for example $q_i = q(x_{i-1/2})$. Then e.t.d.s. for (7), (8) will be written in the following form

$$(ay_{\overline{x}}^{(j+1)})_{\hat{x}} - d(x)y^(j+1) = -\varphi^{(j+1)}(x), \ x \in \hat{\omega}, \ y^{(j+1)}(0) = y^{(j+1)}(1), \quad (30)$$

or in the extended form

$$\frac{1}{\overline{h}}\left[a(x_{i+1}\frac{y^{(j+1)}(x_{i+1}) - y^{(j+1)}(x_i)}{h_{i+1}} - a(x_i)\frac{y^{(j+1)}(x_i) - y^{(j+1)}(x_{i-1})}{h_{i+1}}\right] -$$

$$-d(x_i)y^{(j+1)}(x_i) = -\varphi^{(j+1)}(x_i), \quad x_i \in \hat{\omega},$$

$$y^{(j+1)}(0) = y^{(j+1)}(1) = 0,$$

where

$$a = a(x_i) = \left[\frac{1}{h_i}\mu_1(\nu_i h_i)\right]^{-1}$$

$$d = d(x_i) = \frac{1}{\overline{h}}\sum_{\alpha=1}^{2}\frac{\mu_2(\nu_{i+\alpha-1}h_{i+\alpha-1}) - 1}{\mu_1(\nu_{i+\alpha-1}h_{i+\alpha-1})}, \quad \bar{h} = 0.5(h_{i+1} + h_i),$$

$$\varphi^{(j+1)} = \varphi^{(j+1)}(x_i) = [\bar{h}\mu_1(\nu_i h_i)]^{-1}\int\limits_{x_{i-1}}^{x_i}\mu_1(\nu_i(\xi - x_{i-1}))F_n^{j+1}(\xi)d\xi +$$

$$+[\bar{h}\mu_1(\nu_{i+1}h_{i+1})]^{-1}\int\limits_{x_i}^{x_{i+1}}\mu_1(\nu_{i+1}(x_{i+1} - \xi))F_n^{j+1}(\xi)d\xi$$

The fact that difference scheme (30) is exact means that its solution coinsides with projection of corresponding solution of the problem (7), (8) on the grid $\hat{\omega}$, i.e. $y^{(j+1)} = u_n^{(j+1)}(x), \quad x \in \hat{\omega}$.

Difference scheme (30) is degenerate, but, since it is exact, the condition of its solvability will be automatically satisfied due to the selection of λ_n^{j+1} according to formula (8). We find its solution in the following way. In (30) we throw the last equation away and carry a member, containing $y^{(j+1)}(x_N)$, to the right side in equation next to last. We find the solution of the obtained system with three-diagonal (non-degenerate) matrix $y^{(j+1)}(x_N), \quad i = 1, ..., N-1$, which depends on the unknown parameter $y^{(j+1)}(x_N)$. This parameter is obtained due to the condition of normalization (16). In order to do this, we realize the reconstruction operation of the grid projection of exact solution of the problem (7), (8) at first

$$u_n^{(j+1)}(x) = \frac{\mu_1(\nu_i(x - x_{i-1}))}{\mu_1(\nu_i h_i)} y^{(j+1)}(x_i) + \frac{\mu_1(\nu_i(x_i - x))}{\mu_1(\nu_i h_i)} y^{(j+1)}(x_{i-1}) +$$
$$+ \int\limits_{x_{i-1}}^{x_i} G^i(x,\xi) F_n^{(j+1)}(\xi) d\xi, \quad x \in [x_{i-1}, x_i], \quad i = \overline{1, N+1}, \tag{31}$$

where

$$G^i(x,\xi) = \mu_1^{-1}(\nu_i h_i) \begin{cases} \mu_1(\nu_i(x - x_{i-1}))\mu_1(\nu_i(x_i - \xi)), & x \le \xi, \\ \mu_1(\nu_i(\xi - x_{i-1}))\mu_1(\nu_i(x_i - x)), & \xi \le x, \end{cases}$$

and then normalization condition (16) in terms of $y^{j+1}(x)$ is written down, using expression (31). $y^{j+1}(x_N)$ is obtained from (16). In recursion process (30), (31) j runs from 0 to $m-1$. Initial conditions are defined in the following way. At first, for $y^{(0)}(x)$ the problem (30) is solved with) $j = -1, \quad \underset{(0)}{\varphi}(x) \equiv 0$, which will have a solution provided that determinant $\delta_n(\lambda(\overline{q}(\bullet)))$ of three-diagonal matrix of homogeneous system (30) is equal to zero, i.e.

$$\Delta_n(\lambda \overline{q}(\bullet)) =$$

$$\begin{vmatrix} \left\{\frac{1}{\hbar_1}\left(\frac{a(x_2)}{h_2}+\frac{a(x_1)}{h_1}\right)-d(x_1)\right\} & \frac{a(x_2)}{\hbar_1 h_2} & 0 & 0\cdots 0 \\ \frac{a(x_2)}{\hbar_2 h_2} & \left\{\frac{1}{\hbar_2}\left(\frac{a(x_3)}{h_3}+\frac{a(x_2)}{h_2}\right)-d(x_2)\right\} & \frac{a(x_3)}{\hbar_2 h_3} & 0\cdots 0 \\ \ddots & \ddots & \ddots & \\ 0\cdots 0 \frac{a(x_{N-1})}{\hbar_{N-1}h_{N-1}} & \left\{-\frac{1}{\hbar_{N-1}}\left(\frac{a(x_N)}{h_N}+\frac{a(x_{N-1})}{h_{N-1}}\right)-d(x_{N-1})\right\} & \frac{a(x_N)}{\hbar_{N-1}h_N} & \\ 0\cdots 0 \quad 0 & \frac{a(x_N)}{\hbar_N h_N} & \left\{-\frac{1}{\hbar_N}\left(\frac{a(x_{N+1})}{h_{N+1}}+\frac{a(x_N)}{h_N}\right)-d(x_N)\right\} & \end{vmatrix}$$

$$= 0$$

To find roots of equation (32) $\lambda_i(\overline{q}(\bullet)), \quad i = 1, 2, ...,$ let us obtain two-sided estimates for each of them at first, which is possible to do using Remark 1. We have

$$(\pi n)^2 + q_{\min} = \lambda_n(q_{\min}) \le \lambda_n(\overline{q}(\bullet)) \le \lambda_n(q_{\max}) = (\pi n)^2 + q_{\max}, \qquad (32')$$

where

$$q_{min} = \min_{x\in[0,1]} q(x), \qquad q_{max} = \max_{x\in[0,1]} q(x)$$

Let us find a number N, such that $\forall n \ge N$ segments $(32')$, containing roots of equation (32), will be non-intersecting, i.e. the following inequality will be fulfilled

$$(\pi n)^2 + q_{\max} \le [\pi(n+1)]^2 + q_{min}, \quad n \ge N.$$

Hence

$$N = \left\lceil \frac{q_{\max} - q_{\min} - \pi^2}{2\pi^2} \right\rceil,$$

where $\lceil a \rceil$ means the least whole number greater or equal to a.

Then $\forall n \ge N$ we find roots of equation (32) $\lambda_n(\overline{q}(\bullet))$ with bisection method. We expand determinant (32) for specific $\widetilde{\lambda}$ according to the recursion formula

$$\Delta_k(\widetilde{\lambda}) = \left\{-\frac{1}{\hbar_k}\left[\frac{a(x_{k+1})}{h_{k+1}} + \frac{a(x_k)}{h_k}\right] - d(x_k)\right\}\Delta_{k-1}(\widetilde{\lambda}) - \frac{a^2(x_k)}{\hbar_k h_k^2 h_{k+1}}\Delta_{k-2}(\widetilde{\lambda}),$$

$$k = 1, 2, ..., N, \quad \Delta_{-1}(\widetilde{\lambda}) = 0, \quad \Delta_0(\widetilde{\lambda}) = 1.$$

Roots of equation (32) are found in the simpliest way, if

$$q_{\max} - q_{\min} \le 3\pi^2, \qquad (32'')$$

besause in this case $\forall n \geq 1$ the limits of roots of $(32')$ will be non-intersecting. If condition $(32'')$ is not satisfied, this leads to the technical complications only, which are not fundamental (see detailing in [13]).

Having found roots of the transcendental equation (32) $0 < \lambda_1^{(0)}(\overline{q}(\bullet)) < \lambda_2^{(0)}(\overline{q}(\bullet)) < ... < \lambda_n^{(0)}(\overline{q}(\bullet)) < ...$, we substitute the n-th root into (30) $(j = -1, \ \ \varphi^{(0)} \equiv 0)$ and complement the obtained system of homogeneous equations with condition of normalization (16)

$$1 = \int\limits_0^1 \left[u_n^{(0)}(x, \overline{q}(\bullet))\right]^2 dx =$$

$$= \sum_{i=1}^{N+1} \int\limits_{x_{i-1}}^{x_i} \left[\frac{\mu_1(\nu_i(x - x_{i-1}))}{\mu_1(\nu_i h_i)} y^{(0)}(x_i) + \frac{\mu_1(\nu_i(x_i - x))}{\mu_1(\nu_i h_i)} y^{(0)}(x_{i-1})\right]^2 dx$$

Solution of a new system $y_{(0)}(x)$ is obtained in the same way, as it was described above for $j \geq 0$. When it is found, this solution $y_{(0)}(x)$, together with $\lambda_n^{(0)}(\overline{q}(\bullet))$, forms initial conditions for the recursion process described above.

Definition 1. FD-method for the problem (1) is called exactly realizable, if all operations of integration in the algorithm (9), (10), (8) are performed exactly.

The following statement holds.

Lemma 2. For FD-method for problem (1) to be exactly realizable, it is sufficient that $q(x)$ is piecewise polynomial or piecewise trigonometric.

Proof is obvious.

In case when FD-method for problem (1) is not exactly realizable, to preserve the estimates (18), (19) we change flinction $q(x)$ with function $\widetilde{q}(x)$, which is built in the following way. Each interval $[x_{i-1}, x_i], \ \ i = \overline{1, N+1}$ we divide into $\left[\frac{1}{4M_n}\right]$ equal intervals if n is such that $4M_n < 1$ and make no change otherwise. Between each two knots of the new grid $\hat{\omega}(n)$ we substitute $q(x)$ with interpolation polynomial of m-th degree, which we denote by $\widetilde{q}(x)$. Then we solve problem (1) with FD-method substituting $q(x)$ with $\widetilde{q}(x)$, which will be exactly realizable and the qualitative nature of the estimates (18), (19) will be preserved. Let us illustrate this method with the following examples.

Example 1. Let

$$q(x) = \pi^2(1 - \cos(\pi x)).$$

In this case, the method is exactly realizable. Results of calculations for $N = 1, \quad x_1 = 1/2, \quad q_1 = 1/2, \quad q_2 = 3/2$ are presented in Table 1.

Table 1.

Number of eigen-value	1	2
$\overset{0}{\lambda}_n/\pi_2$	1.93859169064	5.04568495547
$\overset{1}{\lambda}_n/\pi_2$	1.92251551301	5.04199553591
$\overset{2}{\lambda}_n/\pi_2$	1.91809704126	5.03177991276
$\overset{3}{\lambda}_n/\pi_2$	1.91806100532	5.03189728271
$\overset{4}{\lambda}_n/\pi_2$	1.91805813493	5.03192257449

Number of eigen-value	3	4
$\overset{0}{\lambda}_n/\pi_2$	9.99319814005	17.01166208848
$\overset{1}{\lambda}_n/\pi_2$	10.00612909994	17.00875718370
$\overset{2}{\lambda}_n/\pi_2$	10.01441867460	17.00792008125
$\overset{3}{\lambda}_n/\pi_2$	10.01432461311	17.00793764246
$\overset{4}{\lambda}_n/\pi_2$	-	-

One can judge on the quality of approximation of the first four eigen-values from Table 2, taken from [11], p. 250.

Table 2. Estimates for the first four eigen-values, $q(x) = \pi^2(1 - \cos(\pi x))$.

Number of Eigen-value	Lower Estimate	Upper Estimate
1	1.91805812	1.91805816
2	5.031913	5.031922
3	10.011665	10.014381
4	16.538364	17.035639

If we set $\overline{q}(x)$, then, using the corollary 1 with substitution of $q(x)$ with $q(x) - \overline{q}(x) = -\pi^2 \cos \pi x$, we will have

$$\lambda_n^{(0)}(\overline{q}) = \pi^2(n^2 + 1), \qquad u_n^{(0)}(x, \overline{q}) = \sqrt{2} \sin \pi n x,$$

$$\lambda_n^{(1)}(\overline{q}) = 0, \qquad u_n^{(1)}(x,\overline{q}) = \frac{\sin \pi(n+1)x}{\sqrt{2}(2n+1)} - \frac{\sin \pi(n-1)x}{\sqrt{2}(2n-1)},$$

$$\lambda_n^{(2)}(\overline{q}) = \frac{\pi^2}{2(4n^2-1)},$$

$$u_n^{(2)}(x,\overline{q}) = \frac{\sin \pi(n+2)x}{4\sqrt{2}(2n+1)(2n+2)} + \frac{\sin \pi(n-2)x}{4\sqrt{2}(2n-1)(2n-2)},$$

$$\lambda_n^{(3)}(\overline{q}) = 0,$$

$$u_n^{(3)}(x,\overline{q}) = \frac{1}{2\sqrt{2}(2n+1)^2}\left[\frac{1}{4n^2-1} + \frac{1}{8(n+1)}\right] \sin \pi(n+1)x +$$

$$+\frac{1}{2\sqrt{2}(2n-1)^2}\left[\frac{1}{4n^2-1} + \frac{1}{8(n-1)}\right] \sin \pi(n-1)x +$$

$$+\frac{\sin \pi(n+3)x}{16\sqrt{2}(2n+1)(n+1)(6n+9)} - \frac{\sin \pi(n+3)x}{16\sqrt{2}(2n+1)(n+1)(6n+9)},$$

$$\lambda_n^{(4)}(\overline{q}) = \frac{\pi^2(20n^2+7)}{32(4n^2-1)^3(n^2-1)}.$$

Consequently

$$\frac{1}{\pi^2}\lambda(q(\bullet)) = n^2 + 1 + \frac{n}{2(4n^2-1)} + \frac{20n^2+7}{32(4n^2-1)^3(n^2-1)} + \frac{1}{\pi^2}R_5^n(\lambda),$$

where, according to corollary 1,

$$\frac{1}{\pi^2}|R_5^n(\lambda)| \le \frac{14}{(2n-1)^3(2n-5)}.$$

Hence, the estimate follows

$$\left|\frac{1}{\pi^2}\lambda(q(\bullet)) - n^2 - 1 - \frac{n}{2(4n^2-1)} - \frac{20n^2+7}{32(4n^2-1)^3(n^2-1)}\right| \le$$

$$\le \frac{14}{(2n-1)^3(2n-5)},$$

which gives the simplest possibility to continue Table 2. Moreover, the last estimate refines lower and upper bounds for the fourth eigen-value from [11]:

$$16,994334 \le \frac{1}{\pi^2}\lambda_4(q(\bullet)) \le 17,0215446.$$

Example 2.

We consider the problem [5]

$$u''(x) + [\lambda - x^2)]u(x) = 0, \quad x \in (0,1), \quad u(0) = u'(1) = 0.$$

Though in $x = 1$ the boundary condition is not a Dirichlet condition, according to Remark 2, application and substantiation of FD-method remain valid with minor technical changes, and when $\overline{q}(x) \equiv 0, \quad N = 0$ we will have

$$\lambda_n(q(\bullet)) = \left(n - \frac{1}{2}\right)^2 \pi^2 + \int\limits_0^1 [1 - \cos(2n-1)\pi x]x^2 dx + R_2^n(\lambda) =$$

$$= \overset{1}{\lambda}_n(0) + R_2^n(\lambda) = \pi^2 \left(n - \frac{1}{2}\right)^2 + \frac{1}{3} - \frac{2}{\pi^2(2n-1)^2} + R_2^n(\lambda),$$

$$| + R_2^n(\lambda)| \leq \frac{1}{\pi^2(2n-1)} \left[1 - \frac{4}{\pi^2(2n-1)}\right]^{-1} = \frac{1}{\pi^2(2n-1) - 4}$$

Hence

$$\left|\lambda_{100}(q(\bullet)) - \overset{1}{\lambda}_{100}(0)\right| = |\lambda_{100}(q(\bullet)) - 97711{,}884300101| \leq 0{,}000510190734,$$

which is more precise in comparison with the corresponding result from [5], obtained when $n = 180$ (number of interpolation points). Moreover, explicit estimate of accuracy is given here.

Example 3.

Let in the problem (1) $q(x) = \pi^2 e^{\pi x}$. In this case method is exactly realizable. Results of calculations for $N = 1, \; x_1 = 0.8, \; q_1 = 1/2(1+e^{0.8\pi}, q_2 = 1/2(e^{0.8\pi + e^{\pi}}$, are presented in Table 3. Since in this case the codition of Theorem 2 $r_n < 1$ is fulfilled for $n > 1000$ results from [7] are given for comparison in the fourth colunm of the table.

Table 3.

Number of Eigen-value (n)	$\frac{\overset{0}{\lambda}_n}{\pi^2}$	$\frac{\overset{1}{\lambda}_n}{\pi^2}$	$\frac{\lambda_n}{\pi^2}$ (Pryce)
1	7.9235329168	5.74927162957	4.8966693800
2	11.6276625848	9.77545303263	10.045189893
3	17.6043941512	15.1409004557	16.019267250
4	25.4310372553	23.1662032022	23.266270940
5	34.494920344	32.2696978281	32.263707046
6	44.9428991294	43.229889154	43.220019641
7	57.7262032039	56.1268444388	56.181594023
8	72.9480498731	71.0925708125	71.152997537
9	90.1421266411	88.1034910387	88.132119192
10	109.036454774	107.1075478	107.11667614

11	129.815679554	128.094523291	128.10502127
12	152.788412801	151.072130248	151.09604375
13	177.942960741	176.066503867	176.08899681
14	205.039528236	203.071605593	203.08337104
15	233.952647272	232.074213731	232.07881198
16	264.81789293	263.068734187	263.07506796
17	297.818396016	296.059629319	296.07195674
18	332.93272765	331.057744727	331.06934398
19	369.993979431	368.060823881	368.06712902
20	408.923663598	407.062553446	407.06523527
21	449.827052348	448.059596338	448.06360365
22	492.834778464	491.054743814	491.06218803
23	537.92494633	536.053888672	536.06095197
24	584.968873335	583.055958044	583.05986641
25	633.910318641	632.057164377	632.05890789
26	684.835120614	683.055326993	683.05805737
27	737.844948401	736.052330313	736.05729923
28	792.919234235	791.051870641	791.05662058
29	849.953117775	848.053357166	848.05601068
30	908.90308909	907.054239751	907.05546058
31	969.841572225	968.052991363	968.05496270
32	1032.85183612	1031.05096224	1031.0545106
33	1097.91494194	1096.05068678	1096.0540990
34	1164.9423576	1163.05180592	1163.0537230
35	1233.89873671	1232.05247734	1232.0533787
36	1304.84670629	1303.05157536	1303.0530626
37	1377.85679576	1376.0501123	1376.0527718
38	1452.91162242	1451.04993422	1451.0525036
39	1529.93456148	1528.05080691	1528.0522557
40	1608.89591462	1607.05133392	1607.0520262
41	1689.85084448	1688.05065238	1688.0518132
42	1772.86053167	1771.04954833	1771.0516152
43	1857.90898764	1856.04942656	1856.0514309
44	1944.92866169	1943.05012596	1943.0512590
45	2033.89398107	2032.05055021	2032.0510984
46	2124.85423354	2123.05001741	2123.0509481
47	2217.86344431	2216.04915504	2216.0508073
48	2312.9068495	2311.0490681	2311.0506752
49	2409.92404571	2408.04964108	2408.0505511
50	2508.89259866	2507.04998973	2507.0504344
51	2609.85705218	2608.04956193	2608.0503245

100	10008.8881822	10007.0481983	
200	40008.8870789	40007.0477506	
300	90008.8868746	90007.0476677	
400	160008.886803	160007.047639	
500	250008.88677	250007.047625	
600	360008.886752	360007.047618	
700	490008.886741	490007.047614	
800	640008.886734	640007.047611	
900	810008.886729	810007.047609	
1000	1000008.88673	1000007.047607	

If we take $N = 0$, then, when n is sufficiently big, we can take $\dfrac{\overset{1}{\lambda}_n(0)}{\pi^2}$ approximation to $\dfrac{\lambda_n}{\pi^2}$ which has the following view

$$\frac{\overset{1}{\lambda}_n(0)}{\pi^2} = n^2 + \frac{1}{\pi}(e^{\pi} - 1) = n^2 + 7.047601...$$

and characterizes asymptotic behavior of $\dfrac{\lambda_n}{\pi^2}$. From Table 3 one can see asymptotics when n increases.

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