INVESTIGATION OF THE NONLOCAL INITIAL-BOUNDARY VALUE PROBLEMS FOR THE STRING OSCILLATION AND TELEGRAPH EQUATIONS

D. G. Gordeziani, G. A. Avalishvili

Abstract

The present article is devoted to the initial-boundary value problems with the so called nonlocal conditions, where, in contrast to classical initial-boundary value problems, there is given a certain relation between the boundary meanings of unknown function and its inherent ones. On the example of string oscillation and telegraph equations there are studied different types of nonlocal problems, proved existence and uniqueness theorems and given algorithms for direct construction of solutions. The mentioned problems could be interpreted as problems with boundary control, when maintenance of a certain link between boundary and inherent meanings of the unknown function is requested.

In order to solve the above problems we use the method of reflected wave in the case of string oscillation equation, and the case of telegraph equation is reduced to Volterra type integral equations. The proof of uniqueness theorems are basicly based on the theory of characteristics.

1 *Introduction*

Nonlocal problems arise while mathematical modelling of different processes in physics, ecology, chemistry, biology and other fields [14]-[19]. The above mentioned problems are very important from the point of their practical application in solving mathematical problems of the mechanic of solid body. They allow to control stress-strain state of body and therefore, from a certain point, are similar to the control problems, and particularly, to those of exact controlability [13].

It must be pointed out that the theoretical study of nonlocal problems is connected with the great difficulties. Too many things are expected to be done in this direction, though a lot of interesting works are already devoted to these issues [1]-[12]. Naturally, arises a question: what prevents us from solving nonlocal problems even in the case of linear three-dimensional equations of the theory of elasticity? First of all, here usually it is impossible to apply traditional approach for proving uniqueness theorem, since nonlocal conditions contain inherent points of the region, and obtaining integral of energy, none of the members become zero in homogeneous nonlocal conditions on the boundary. Even in the simplest case it is obvious that direct application of the method of singular integral equations does not give a result. Here does not even work the method of functional analysis which considers the proof of coerciveness on the basis of Korn's inequality with the following application of Lax-Milgram's theorem. This is the reason for existing only separate results.

First, a certain class, particularly the class of three dimensional nonlocal problems, was formulated and studied in the article [1]. Further, in the works [4, 5] the problem stated in [1] was called Bitsadze-Samarskii problem and resolution methods were suggested for the such type problems in the case of rather general elliptic equations. In the work [6] there were considered boundary conditions, similar to those of Bitsadze-Samarskii ones, for the equations of shell and elasticity theory. Under rather strict conditions there is proved uniqueness of the solution of the nonlocal problem for the three dimensional models of the elasticity theory. The stated nonlocal problems was effectively solved in the case of round plates for the Kirchhoff model. Later, in the works [17, 19] were suggested interesting generalizations of Bitsadze-Samarskii conditions.

As was already mentioned, investigation of nonlocal problems in the mechanics of solid body for general mathematical models is rather difficult. Therefore, we limit ourselves by considering simple models, especially as the issues of existence and uniqueness of the solution are not studied previously. But it should be mentioned that the methods applied herein can be used even for more general nonlocal problems.

Secs. 2 and 3 of the present work are devoted to the investigation of one-dimensional problem of the mechanic of solid medium with different nonlocal boundary conditions. More precisely, in Sec. 2 there is considered string oscillation equation with the classical initial and discrete nonlocal conditions, which represent the generalization of Bitsadze-Samarskii conditions. There we prove the theorem of existence and uniqueness of the solution, which can be constructed directly using algorithms given ibidem. In Sec. 3 there is considered the telegraph equation. As in the case of string oscillation equation there is studied nonlocal problem with discrete nonlocal conditions. Finally, in Sec. 4 we state the general theorem of uniqueness for multidimensional nonlocal problem with discrete nonlocal conditions.

2 *The nonlocal problem for string oscillation equation*

As we mentioned in the introduction, we study rather general nonlocal problem, state the theorem of existence and uniqueness and give an algorithm for direct construction of the solution. Main tools for finding the solution are the method of a reflected wave and properties of the solution on characteristics.

Consider nonlocal problem for the equation of string oscillation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad 0 < t < T, \tag{2.1}$$

with the classical initial conditions

$$\begin{array}{l} u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x), \end{array} \quad 0 \le x \le l, \tag{2.2}$$

and nonlocal boundary conditions

$$\begin{array}{l} \alpha(t)u(0,t) + \beta(t)\dfrac{\partial u}{\partial x}(0,t) = \displaystyle\sum_{i=1}^{m} a^i(t)u(\xi_i(t),t) + f(t), \\ \gamma(t)u(l,t) + \theta(t)\dfrac{\partial u}{\partial x}(l,t) = \displaystyle\sum_{j=1}^{n} b^j(t)u(\eta_j(t),t) + g(t), \end{array} \quad 0 \le t \le T, \tag{2.3}$$

where $\alpha, \beta, \gamma, \theta, f, g, a^i(t), b^j(t)$ $(i = 1, ..., m;\ j = 1, ..., n)$ are prescribed functions, $\xi_i(t),\ \eta_j(t)$ $(i = \overline{1,m};\ j = \overline{1,n})$ are sliding points of the string $(0,l)$ which define nonlocal condition, i.e. if all a^i, b^j are zero, then we get classical initial-boundary value problem. The following theorem is true.

Theorem 2.1. Assume that the following conditions are in force:

(i) $f, g, \alpha, \beta, \gamma, \theta, a^i, b^j \in C^2[0,T] (i = \overline{1,m};\ j = \overline{1,n}),\ \varphi \in C^2[0,l]$, $\psi \in C^1[0,l],\ \alpha(t)\beta(t) \ne 0,\ \gamma(t)\theta(t) \ne 0, \quad 0 \le t \le T$;

(ii) $\xi_i, \eta_j \in C^2[0,T],\ 0 < \xi_i(t), \eta_j(t) < l$, when $t \in [0,T],\ i = 1, ..., m$, $j = 1, ..., n$;

(iii) each of the functions $\beta(t), \theta(t)$ either does not equal to zero for any $t \in [0,T]$, or is equal to zero for all t. Then nonlocal problem (2.1)-(2.3) has unique solution $u(x,t)$, which is twice continuously differentiable on the set $\bar{D} = \{0 \le x \le l,\ 0 \le t \le T\}$, satisfies the equation (2.1) and the conditions (2.2), (2.3).

Proof. Note, that if the solution of the problem (2.1)-(2.3) is found, then we get some functions on the ends of the string

$$\begin{array}{l} u(0,t) = \mu_1(t), \\ u(l,t) = \mu_2(t), \end{array} \quad 0 \le t \le T, \tag{2.4}$$

and then $u(x,t)$ is the solution of classical Cauchy-Dirichlet problem for the equation (2.1) with the initial and boundary conditions (2.2), (2.4), which has unique solution

$$
\begin{aligned}
u(x,t) = {} & \frac{\Phi(x+t)+\Phi(x-t)}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} \Psi(\alpha)d\alpha \\
& -\frac{\varphi''(0)}{\lambda^2 \sin\lambda l}\sin(\lambda(l-x))\cos\lambda t - \frac{\varphi''(l)}{\lambda^2\sin\lambda l}\sin\lambda x\cos\lambda t \\
& +\sum_{n=0}^{\infty}\bar{\mu}_1(t-2nl-x) - \sum_{n=1}^{\infty}\bar{\mu}_1(t-2nl+x) \\
& +\sum_{n=0}^{\infty}\bar{\mu}_2(t-(2n+1)l+x) - \sum_{n=0}^{\infty}\bar{\mu}_2(t-(2n+1)l-x),
\end{aligned}
\tag{2.5}
$$

where $\lambda = \dfrac{\pi}{l+1}$, $\Phi(x)$, $\Psi(x)$ represent respectively the functions $\varphi(x) + \dfrac{\varphi''(0)}{\lambda^2\sin\lambda l}\sin(\lambda(l-x)) + \dfrac{\varphi''(l)}{\lambda^2\sin\lambda l}\sin\lambda x, \psi(x)$ expanded on the whole axle retaining smoothness in such a way, that

$$
\begin{aligned}
&\Phi(x)+\Phi(-x) = 2\Phi(0), && \Psi(x)+\Psi(-x) = 2\Psi(0), \\
&\Phi(l-x)+\Phi(l+x) = 2\Phi(l), && \Psi(l-x)+\Psi(l+x) = 2\Psi(l),
\end{aligned}
$$

and the function

$$
\bar{\mu}_1(t) = \begin{cases} \mu_1(t) - (\varphi(0) + \dfrac{\varphi''(0)}{\lambda^2}) - \psi(0)t + \dfrac{\varphi''(0)}{\lambda^2}\cos\lambda t, & t \geq 0 \\ 0, & t < 0 \end{cases}
\tag{2.6}
$$

and analogiously for $\bar{\mu}_2(t)$, where 0 is changed by l. Thus, any solution of the problem (2.1)-(2.3) can be expressed by the form (2.5). If we find twice continuously differentiable functions $\mu_1(t)$, $\mu_2(t)$, then the problem is solved. Consequently, due to this fact under the solution of the problem (2.1)-(2.3) sometimes we mean the couple $\{\mu_1, \mu_2\}$. To make further calculations easier we bring in the following notation:

$$
\begin{aligned}
F(x,t) = {} & \frac{\Phi(x+t)+\Phi(x-t)}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t}\Psi(\alpha)d\alpha \\
& -\frac{\varphi''(0)}{\lambda^2\sin\lambda l}\sin(\lambda(l-x))\cos\lambda t - \frac{\varphi''(l)}{\lambda^2\sin\lambda l}\sin\lambda x\cos\lambda t.
\end{aligned}
$$

Taking into account nonlocal conditions (2.3) we get, that the problem (2.1)-(2.3) is solved, if we find the couple $\{\mu_1, \mu_2\}$, which satisfies the equations

$$\begin{aligned}
&\alpha(t)\mu_1(t) + \beta(t)(F_x(0,t) - \bar{\mu}_1'(t) - 2\sum_{n=1}^{\infty}\bar{\mu}_1'(t-2nl) \\
&\qquad +2\sum_{n=0}^{\infty}\bar{\mu}_2'(t-(2n+1)l)) = \sum_{i=1}^{m} a^i(t)u(\xi_i(t),t) + f(t), \\
&\gamma(t)\mu_2(t) + \theta(t)(F_x(l,t) + \bar{\mu}_2'(t) + 2\sum_{n=1}^{\infty}\bar{\mu}_2'(t-2nl) \\
&\qquad -2\sum_{n=0}^{\infty}\bar{\mu}_1'(t-(2n+1)l)) = \sum_{j=1}^{n} b^j(t)u(\eta_j(t),t) + g(t),
\end{aligned} \tag{2.7}$$

when $0 \le t \le T$.

It should be mentioned, that from the above discussions it follows that resolution of the problem is completely reduced to finding of the pair $\{\mu_1, \mu_2\}$, i.e. existence and uniqueness of the solution $u(x,t)$ and the pair $\{\mu_1, \mu_2\}$ are equivalent.

Since the functions $\xi_i(t)$ and $\eta_j(t)$ $(i = 1, ..., m;\ j = 1, ..., n)$ are continuous on $[0,T]$ and for all $t \in [0,T]$ they belong to the interval $(0,l)$, then there exist

$$\varepsilon_1 = \min_{\substack{0\le t\le T\\ 1\le i\le m}} \xi_i(t), \qquad \tilde{\varepsilon}_1 = \max_{\substack{0\le t\le T\\ 1\le i\le m}} \xi_i(t),$$

$$\varepsilon_2 = \min_{\substack{0\le t\le T\\ 1\le j\le n}} \eta_j(t), \qquad \tilde{\varepsilon}_2 = \max_{\substack{0\le t\le T\\ 1\le j\le n}} \eta_j(t),$$

where each of the numbers $\varepsilon_1, \varepsilon_2, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2$ belongs to $(0,l)$. Let's denote by $t^* = \min\{\varepsilon_1, \varepsilon_2, l-\tilde{\varepsilon}_1, l-\tilde{\varepsilon}_2\}$ and then all the curves ξ_i, η_j are located in the stripe $[t^*, l-t^*] \times [0,T]$.

Taking (2.7) into account and the definition of $\bar{\mu}_1(t), \bar{\mu}_2(t)$, we get, that if the pair $\{\mu_1, \mu_2\}$ is the solution of the problem, then it has to satisfy the following equalities

$$\begin{aligned} \alpha(t)\mu_1(t) - \beta(t)\mu_1'(t) &= \sum_{i=1}^{m} a^i(t)F(\xi_i(t),t) + \tilde{f}(t), \\ \gamma(t)\mu_2(t) + \theta(t)\mu_2'(t) &= \sum_{j=1}^{n} b^j(t)F(\eta_j(t),t) + \tilde{g}(t), \end{aligned} \qquad 0 \le t \le t^*,$$

where $\tilde{f}(t)$, $\tilde{g}(t)$ are prescribed continuously differentiable functions. Consequently, for defining $\mu_1(t)$ and $\mu_2(t)$ we get an ordinary differential equations of the first order. Assume, that one of the conditions of the point (iii) in the theorem 2.1 is true, i.e. $\beta(t) \neq 0$, when $t \in [0,T]$. Then taking into account compatibility condition $\mu_1(0) = \varphi(0)$, for $0 \le t \le t^*$ we obtain

$$\mu_1(t) = e^{\int_0^t \frac{\alpha(\tau)}{\beta(\tau)} d\tau} \left(\varphi(0) - \int_0^t e^{-\int_0^\tau \frac{\alpha(s)}{\beta(s)} ds} \left(\sum_{i=1}^{m} \frac{a^i(\tau)F(\xi_i(\tau),\tau)}{\beta(\tau)} + \frac{\tilde{f}(\tau)}{\beta(\tau)} \right) d\tau \right).$$

In the second case, $\mu_1(t)$ can be directly expressed by the functions standing in the right part of the equation. Here, correspondent functions $\tilde{f}(t)$ or $\tilde{g}(t)$ will be twice continuously differentiable. In both cases, as we see, $\mu_1(t)$ is equal to twice continuously differentiable function, when $0 \le t \le t^*$. Therefore, in the time period $[0,t^*]$ we can define the unknown pair of functions $\{\mu_1, \mu_2\}$ and, using the formula (2.5), get the solution of the problem (2.1)-(2.3) in this time interval.

Now, take for the initial moment t^*, i.e. we bring in a new time variable $\tau = t - t^*$. Then the nonlocal problem for the function $v(x,\tau) = u(x,\tau+t^*)$ considered in $[0,l] \times [0,t^*]$ gets the following form

$$v_{\tau\tau} = v_{xx}, \quad 0 < x < l, \quad 0 < \tau < t^*, \tag{2.8}$$

$$\begin{aligned} &\alpha(\tau+t^*)v(0,\tau) + \beta(\tau+t^*)\frac{\partial v}{\partial x}(0,\tau) \\ &\qquad = \sum_{i=1}^{m} a^i(\tau+t^*)v(\xi_i(\tau+t^*),\tau) + f(\tau+t^*), \\ &\gamma(\tau+t^*)v(l,\tau) + \theta(\tau+t^*)\frac{\partial v}{\partial x}(l,\tau) \\ &\qquad = \sum_{j=1}^{n} b^j(\tau+t^*)v(\eta_j(\tau+t^*),\tau) + g(\tau+t^*), \end{aligned} \qquad 0 \le \tau \le t^*, \tag{2.9}$$

where the initial conditions are

$$\begin{aligned} v(x,0) &= u(x,t^*), \\ v_\tau(x,0) &= u_t(x,t^*), \end{aligned} \qquad 0 \le x \le l. \tag{2.10}$$

As in the previous case we can find the solution of the nonlocal problem (2.8)-(2.10) on $[0, t^*]$ and it will be an expansion of $u(x,t)$ on the time interval $[t^*, 2t^*]$. Let's show now that obtained $u(x,t)$ is the solution of the problem (2.1)-(2.3), when $0 \leq t \leq 2t^*$. Obviously, it's sufficient to check twice continuously differentiability of $u(x,t)$ in the moment $t = t^*$. Since $u(x,t)$ is the solution of (2.1)-(2.3) on $[0, t^*]$, then it is twice continuously differentiable by x, when $t = t^*$, and

$$\lim_{t-t^* \to 0^-} u(x,t) = u(x,t^*) = v(x,0) = \lim_{t-t^* \to 0^+} u(x,t),$$

and consequently $u(x,t)$ is continuous in the point t^*. Analogically,

$$\lim_{\Delta \to 0^-} \frac{u(x,t^*+\Delta) - u(x,t^*)}{\Delta} = u_t(x,t^*)$$

$$= v_\tau(x,0) = \lim_{\Delta \to 0^+} \frac{u(x,t^*+\Delta) - u(x,t^*)}{\Delta},$$

$$\lim_{t-t^* \to 0^-} u_t(x,t) = u_t(x,t^*) = v_\tau(x,0) = \lim_{t-t^* \to 0^+} u_t(x,t).$$

Therefore, $u_t(x,t)$ exists and is continuous for $t = t^*$.

In the same way we can check that $u_{tt}(x,t)$ is continuous for $t = t^*$. Consequently $u(x,t)$ is the solution of the nonlocal problem (2.1)-(2.3), when $0 \leq t \leq 2t^*$.

Applying the same method we find $u(x,t)$ on the intervals $[0, nt^*]$ $(n = 2, 3, ...)$ up to the moment T. Therefore we can find $u(x,t)$ for the whole time interval $[0, T]$, i.e. the solution of the problem (2.1)-(2.3) exists, is unique and expressed through the given functions and their integrals. □

The stated problem can be interpreted as the problem of exact controlability by the boundary conditions, where the boundary meanings of unknown function are required to differ from the linear combination of its meanings in certain points by a given beforehand number. This type of problems arise in building constructions, generators, etc.

3 *Nonlocal problem for telegraph equation*

As in the case of string oscillation equation we consider nonlocal problem for telegraph equation. However, in contrast to the case of string oscillation, here the main method of constructing solution is the application of a special type potential, which allows to reduce stated nonlocal problem to integral equations. Here we also use corresponding notations of the Sec. 2.

Consider the nonlocal problem for telegraph equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + c^2 u, \quad 0 < x < l, \quad 0 < t < T, \tag{3.1}$$

with homogeneous initial conditions

$$u(x,0) = u_t(x,0) = 0, \quad 0 \le x \le l, \tag{3.2}$$

and nonlocal boundary conditions

$$\begin{aligned} &\alpha(t)u(0,t) + \beta(t)\frac{\partial u}{\partial x}(0,t) = \sum_{i=1}^{m} a^i(t)u(\xi_i(t),t) + f(t), \\ &\gamma(t)u(l,t) + \theta(t)\frac{\partial u}{\partial x}(l,t) = \sum_{j=1}^{n} b^j(t)u(\eta_j(t),t) + g(t), \end{aligned} \quad 0 \le t \le T, \tag{3.3}$$

where $0 \ne c = const$ is real or imaginary number, and $u(x,t)$ is unknown function, twice continuously differentiable on $[0,l] \times [0,T]$, satisfying the equation (3.1) and conditions (3.2), (3.3). The following theorem is true.

Theorem 3.1. If all the conditions of the Theorem 2.1 are in force, then nonlocal problem (3.1)-(3.3) has unique solution.

Proof. Note that if we find the solution $u(x,t)$ of the problem (3.1)-(3.3), then it gets certain meanings on the boundary and, consequently, is the solution of the telegraph equation with classical Dirichlet conditions on the boundary. In this case we simply can show that

$$\begin{aligned} u(x,t) = \frac{\partial}{\partial x}\Bigg[&\int_0^{t-x} \varphi(\tau) I(c\sqrt{(t-\tau)^2 - x^2})d\tau \\ &+ \int_0^{t-l+x} \psi(\tau) I(c\sqrt{(t-\tau)^2 - (l-x)^2})d\tau \Bigg], \end{aligned} \tag{3.4}$$

where $I(z) = \sum_{s=0}^{\infty} \frac{1}{(s!)^2}\left(\frac{z}{2}\right)^{2s}$, $\varphi, \psi \in C^2[0,T]$, $\varphi(\tau) = \psi(\tau) = 0$, for $\tau \le 0$ and if the expression under the square root is negative, then we consider imaginary meaning of the root. Therefore the solution of the problem (3.1)-(3.3) is uniquely defined by the functions φ, ψ and due to this fact resolution of the stated problem is reduced to finding of functions $\varphi(t)$ and $\psi(t)$. In order to make our reasoning more clear we consider the case when $\beta(t) \ne 0$, $\theta(t) \ne 0$ for $0 \le t \le T$ in (3.3). Though in the other cases they are slightly different.

Substituting formula (3.4) into first condition (3.3) we get an equation relatively to $\varphi(t)$ and $\psi(t)$

$$\alpha(t)u(0,t)+\beta(t)\frac{\partial u}{\partial x}(0,t)=\sum_{i=1}^{m}a^i(t)\left[-\varphi(t-\xi_i(t))\right.$$

$$-\int_0^{t-\xi_i(t)}\varphi(\tau)\frac{c\xi_i(t)I'(c\sqrt{(t-\tau)^2-\xi_i^2(t)})}{\sqrt{(t-\tau)^2-\xi_i^2(t)}}d\tau+\psi(t-l+\xi_i(t))$$

$$\left.+\int_0^{t-l+\xi_i(t)}\psi(\tau)\frac{c(l-\xi_i(t))I'(c\sqrt{(t-\tau)^2-(l-\xi_i(t))^2}}{\sqrt{(t-\tau)^2-(l-\xi_i(t))^2}}d\tau\right]+f(t).$$

Analogous equation follows from the second boundary condition (3.3). As in the proof of the Theorem 2.1 we consider the same time interval $[0,t^*]$ for which we have

$$\begin{gathered}-\alpha(t)\varphi(t)+\beta(t)\varphi'(t)-\beta(t)\int_0^t\varphi(\tau)\frac{cI'(c(t-\tau))}{t-\tau}d\tau=f(t),\\ \gamma(t)\psi(t)+\theta(t)\psi'(t)-\theta(t)\int_0^t\psi(\tau)\frac{cI'(c(t-\tau))}{t-\tau}d\tau=g(t).\end{gathered}\tag{3.5}$$

From (3.5) denoting $\dfrac{I'(c(t-\tau))}{t-\tau}=I_1(c(t-\tau))$, we get, that

$$\left.\begin{aligned}\varphi(t)&=\int_0^t\int_0^s\varphi(\tau)cI_1(c(s-\tau))d\tau ds+\int_0^t\frac{\alpha(\tau)}{\beta(\tau)}\varphi(\tau)d\tau\\&\qquad+\int_0^t\frac{f(\tau)}{\beta(\tau)}d\tau,\\ \psi(t)&=\int_0^t\int_0^s\psi(\tau)cI_1(c(s-\tau))d\tau ds-\int_0^t\frac{\gamma(\tau)}{\theta(\tau)}\psi(\tau)d\tau\\&\qquad+\int_0^t\frac{g(\tau)}{\theta(\tau)}d\tau,\end{aligned}\right\}\quad 0\le t\le t^*.\tag{3.6}$$

The equations (3.6) are almost the same Volterra type integral equations. Due to this fact we consider them only for $\varphi(t)$. Let's bring in the operator

$$(K\varphi)(t) = \int\limits_0^t \int\limits_0^s \varphi(\tau) c I_1(c(s-\tau)) d\tau ds + \int\limits_0^t \frac{\alpha(\tau)}{\beta(\tau)} \varphi(\tau) d\tau.$$

Then the first equation (3.6) takes the following form

$$\varphi = K\varphi + f^*, \tag{3.7}$$

where $f^*(t) = \int\limits_0^t \frac{f(\tau)}{\beta(\tau)} d\tau$ and obviously $f^* \in C^2[0, t^*]$. Let's prove now that K is the compact operator from $C[0, t^*]$ to $C[0, t^*]$. Consider bounded set $X \subset C[0, t^*]$ and prove that KX is supercompact in $C[0, t^*]$. In order to do so, according to Arcella's theorem, we have to check uniform boundness and uniform continuity of KX. Indeed, if we denote a norm in $C[0, t^*]$ by $\|\cdot\|$, we get

$$\|K\varphi\| \leq T^2 C_1 \|\varphi\| + T C_2 \|\varphi\|,$$

where $C_1 = \max\limits_{[0,t^*]} |cI_1(t)|$, $C_2 = \max\limits_{[0,t^*]} \left| \frac{\alpha(t)}{\beta(t)} \right|$, and since $\|\varphi\|$ is bounded, KX is uniformly bounded. Also

$$|(K\varphi)(t_1) - (K\varphi)(t_2)| = \left| \int\limits_{t_1}^{t_2} \int\limits_0^s \varphi(\tau) c I_1(c(s-\tau)) d\tau ds + \int\limits_{t_1}^{t_2} \frac{\alpha(\tau)}{\beta(\tau)} \varphi(\tau) d\tau \right|$$

$$\leq |t_2 - t_1| (T C_1 \|\varphi\| + C_2 \|\varphi\|)$$

and, consequently, KX is uniformly continuous. This fact proves that K is compact.

Taking the latter into account for the equation (3.7) the theorems of Fredholm are true. If we prove, that the homogeneous equation has trivial solution, then the equation (3.7) has unique solution. Let's consider the homogeneous equation

$$\varphi = K\varphi$$

or

$$\varphi(t) = \int\limits_0^t \int\limits_0^s \varphi(\tau) c I_1(c(s-\tau)) d\tau ds + \int\limits_0^t \frac{\alpha(\tau)}{\beta(\tau)} \varphi(\tau) d\tau.$$

Let's show that the following estimation is true:

$$|\varphi(t)| \leq C_1^n \frac{t^{2n}}{(2n)!} \|\varphi\| + C_2^n \frac{t^n}{n!} \|\varphi\|. \tag{3.8}$$

Indeed, (3.8) is obvious for $n = 1$. Assume that the above estimation is true for n and let it show for $n + 1$.

$$|\varphi(t)| \leq C_1 \int\limits_0^t \int\limits_0^s |\varphi(\tau)| d\tau ds + C_2 \int\limits_0^t |\varphi(\tau)| d\tau$$

$$\leq C_1^{n+1} \|\varphi\| \int\limits_0^t \int\limits_0^s \frac{\tau^{2n}}{(2n)!} d\tau ds + C_2^{n+1} \|\varphi\| \int\limits_0^t \frac{\tau^n}{n!} d\tau$$

$$= C_1^{n+1} \frac{t^{2n+2}}{(2n+2)!} \|\varphi\| + C_2^{n+1} \frac{t^{n+1}}{(n+1)!} \|\varphi\|.$$

Therefore the inequality (3.8) is true for any n. For $n \to \infty$ we get $\varphi(t) \equiv 0,\ t \in [0, t^*]$.

So, the first equation (3.6) has unique continuous solution. However the form of the equation provides twice continuously differentiability of φ on $[0, t^*]$. Analogically we can check existence and uniqueness of the function $\psi(t)$, which allows to conclude that the pair $\{\varphi, \psi\}$ is defined single-valued and, consequently, the solution of the problem (3.1)-(3.3) is also single-valued defined on $[0, t^*]$.

Now, consider the time interval $[t^*, 2t^*]$. Here we also get equations similar to (3.6), where f and g are changed by combinations of functions $\varphi(t)$ and $\psi(t)$, already defined on $[0, t^*]$, since $t^* \leq \xi_i(t), \eta_j(t) \leq l - t^*$ and consequently for $t^* \leq t \leq 2t^*$

$$2t^* - l \leq t - \xi_i(t) \leq t^*, \quad 2t^* - l \leq t - l + \eta_j(t) \leq t^*,$$

$$2t^* - l \leq t - l + \xi_i(t) \leq t^*, \quad 2t^* - l \leq t - \eta_j(t) \leq t^*.$$

Repeating above reasoning for these equations we will be able to determine $\varphi(t)$ and $\psi(t)$ for $t \in [t^*, 2t^*]$. It is not difficult to check that $\varphi(t)$ and $\psi(t)$ functions found in that way, will be twice continuously differentiable on $[0, 2t^*]$. Therefore $u(x, t)$ solution of the stated problem will be uniquely found on $[0, 2t^*]$. Analogically we can define $u(x, t)$ on $[0, nt^*], n \in \mathbb{N}$ up to the moment T. Consequently, for any $t \in [0, T]$ the solution (3.1)-(3.3) is uniquely defined. $\square$

4 *Nonlocal problem for multidimensional medium oscillation equation.*

As was pointed out in the introduction we encounter significant difficulties in resolving nonlocal problems when the number of space variables increases. Though, it should be mentioned that nevertheless the theorem of uniqueness is correct under rather general assumptions.

Consider bounded region $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, where Γ is the boundary of Ω. Let $\Omega_i(t)$ $(i = 1, ..., m)$ be the subsets of Ω, where each $\bar{\Omega}_i(t)$ is strictly included into $\bar{\Omega}$. Assume that $\Gamma_i(t)$ boundaries of $\Omega_i(t)$ are diffeomorphic images of Γ, i.e. $x^{(i)}(t) = I_i(x, t)$, where $x^{(i)} \in \Gamma_i(t), x \in \Gamma, I_i(\cdot, t)$ is diffeomorphism, Γ and Γ_i are Lyapunov surfaces for all $i = \overline{1, m}$.

Let L be uniformly elliptic operator

$$L \equiv \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where a_{ik}, b_i, c $(c \leq 0)$ are rather smooth prescribed functions.

Consider nonlocal problem for hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = f(x, t), \quad (x, t) \in Q_T = \Omega \times (0, T), \tag{4.1}$$

with classical initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \\ u_t(x, 0) &= u_1(x), \end{aligned} \qquad x \in \Omega, \tag{4.2}$$

and nonlocal boundary conditions

$$u(x, t) = \sum_{i=1}^{m} p_i(x, t) u(x^{(i)}, t) + g(x, t), \ (x, t) \in S_T = \Gamma \times [0, T], \tag{4.3}$$

where all the prescribed functions and the regions, where the equation is considered, are such that the theory of characteristics is applicable, and $u(x, t)$ is unknown function which is the classical solution of the equation (4.1) satisfying, at the same time, conditions (4.2) and (4.3). Under such assumptions the following theorem is correct.

Theorem 4.1. The nonlocal problem (4.1)-(4.3) has no more than one regular solution.

Proof. Assume that there exist two $u(x, t)$ and $v(x, t)$ solutions of the problem. Then obviously their difference $w(x, t) = u(x, t) - v(x, t)$ is the solution of the homogeneous equation (4.1) under homogeneous initial and nonlocal conditions. Note, that

$$\rho_i(t) = dist(\Gamma_i(t), \Gamma) = \inf_{x,y\in\Gamma} \rho(I_i(x,t), y)$$

continuously depends on t

$$|\rho_i(t_0 + \Delta t) - \rho_i(t_0)| = |dist(\Gamma_i(t_0 + \Delta t), \Gamma) - dist(\Gamma_i(t_0), \Gamma)|$$

$$\leq dist(\Gamma_i(t_0 + \Delta t), \Gamma_i(t_0)) \leq \inf_{x\in\Gamma} \rho(I_i(x, t_0), I_i(x, t_0 + \Delta t)) \to 0,\ \Delta t \to 0,$$

since $I_i(x,t)$ is continuous function.

Taking into account that each $\bar{\Omega}_i(t)$ is the proper subset of $\bar{\Omega}$, we get that for all $t \in [0,T]$, $\rho_i(t) > 0$, and consequently, there exists such a $\delta > 0$, that $\rho_i(t) > \delta$, $t \in [0,T]$ $(i = \overline{1,m})$. Therefore, for any point (x,t) belonging to some curvilinear cylinder $\Omega_i(t)$, the sphere with center x of a radius δ will entirely be in a "horizontal" cut $\Omega \times \{t\}$.

Uniform ellipticness of the operator L allows to inscribe as well as to overdraw cones respectively inside and outside of a characteristic conoid, defined by the operator L. Tangents of angles between the axle and rulings of the cones we denote by α and β $(\alpha \leq \beta)$ and call them spreads of the cones.

Note that since $w(x,0) = w_t(x,0) = 0$, $x \in \Omega$, then $w(x,t)$ equals to zero in any point (x,t) for which the base of the characteristic conoid, passing through this point, lies in Ω. Consider now an interval $0 \leq t \leq t^*$, where $t^* = \delta/\beta$. Then for any point $(\bar{x}, \bar{t})$ belonging to the curvilinear cylinder $\Omega_i(t)$ $(i = 1, ..., m)$, base of the cone, with a top in $(\bar{x}, \bar{t})$, and axle parallel to the axle t and with spread β, lies in Ω, as $\bar{t}\beta \leq t^*\beta = \delta/\beta \cdot \beta = \delta$. Therefore, $w(\bar{x}, \bar{t}) = 0$, i.e. in any point of the curvilinear cylinders $\Omega_i(t)(i = \overline{1,m})$, $w(x,t) = 0$, for $0 \leq t \leq t^*$. Taking into account that $w(x,t)$ satisfies the homogeneous nonlocal boundary conditions, we obtain

$$w(x,t) = 0, \qquad (x,t) \in S_{t^*},$$

and therefore $w(x,t)$ is the solution of the homogeneous equation (4.1) under homogeneous initial and boundary conditions. Since classical initial-boundary value problem has unique solution, then

$$w(x,t) \equiv 0, \qquad 0 \leq t \leq t^*.$$

Now take for an initial moment of time t^*, i.e. change the variable $\tau = t - t^*$. Then the function $w^*(x,\tau) = w(x, \tau + t^*)$ satisfies the following problem

$$w^*_{\tau\tau} = Lw^*, \quad (x,\tau) \in Q_{T-t^*}, \tag{4.4}$$

$$w^*(x,0) = w^*_\tau(x,0) = 0, \ \ x \in \Omega, \tag{4.5}$$

$$w^*(x,\tau) = \sum_{i=1}^{m} p_i(x,\tau+t^*) w^*(I_i(x,\tau+t^*),\tau), \ (x,\tau) \in S_{T-t^*}. \tag{4.6}$$

Repeating the proceeding reasoning we get, that $w^*(x,\tau) \equiv 0, \ 0 \le \tau \le t^*$, and consequently, $w(x,t) \equiv 0, \ 0 \le t \le 2t^*$. Analogically, $w(x,t) \equiv 0$ for $t \in [0, nt^*], \ n \in \mathbb{N}$ up to the moment T. So, $w(x,t) \equiv 0, \ (x,t) \in Q_T$, which means, that $u(x,t) \equiv v(x,t)$ and the solution is unique. $\square$

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