TWO MATHEMATICAL MODELS FOR INVESTIGATING WAVE GENERATION IN RESERVOIRS

F. Criado,[1] T. Gvelesiani,[2] N. Jgamadze,[2] H. Meladze.[2]

Abstract

In this paper, two mathematical models for investigating wave generation in reservoirs are considered: the first one is based on the small amplitude wave theory and the second one is based on the shallow water theory. For numerical solving the system of equations obtained, a completely conservative two-layerd finite difference scheme is constructed. Finally, the results of physical and mathematical simulations are compared.

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Introduction

Our time has regretfully been marked by a great number of technogenic catastrophes. The need for preventing them calls for the investigation of many natural phenomena that, up until now, have been all but ignored by engineers.

Among such phenomena is that of wave generation in reservoirs due to large landslides. These waves (let us call them "landslide-waves") can attain a reasonable height and provoke the overflow of the reservior's protective structures, damaging them, along with constructions occuring lower levels, flooding structures in the vicinity of the shore line, etc. This real threat is emphasised by the large number of past cases (it's enough to recall the waves provoked by landslides in the reservoirs Latuya Bay, Vaiont, etc.[1,2]).

The existence of numerous large reservoirs, together with those under construction, in mountainous and semi-mountainous regions with complex geologic and seismic conditions, makes it important to investigate these phenomena and develop techniques for predicting wave parameters.

These investigations can be carred out using methods based on both physical and mathematical models. The former is time-consuming and expensive, as physical models have to be specially constructed for each case, whether it be a reservoir or its dam site. It is important to bear in mind that, at the present time, the geometric and kinematic parameters of the landslide process usually can not be predicted with sufficient accuracy,

[1]Department of Mathematics. Málaga University. Spain.

[2]Department of Applied Mathematics.Tbilisi State University. Georgia.

so it is necessary to use a wide range of parameters during research, which is rather complicated in case of physical models. For this reason, there is an increasingly urgent need for the construction of mathematical models and for carrying out of a series of numerical experiments.

The process of generating and developing landslide-waves can be mathematically described by means of models based on:
a) the small amplitude wave theory,
b) the shallow water theory.

1 *Mathematical Model Based on the Small Amplitude Wave Theory*

This model is constructed on the assumption that water fraction velocities $u(x,y,t)$ and $u(x,y,t)$, the elevation of free surface $z=\eta(x,y,t)$ and their derivatives are small quantities. We also assume that the liquid is ideal, uniform and incompressible, and its movement is vortexless. These conditions ensure the existence of simple velocity potential $\varphi(x,y,z,t)$ in every singly connected domain Ω, for which the following base equation of small amplitude wave theory - the Laplace equation - is true:

$$\triangle\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0. \tag{1.1}$$

Likewise, we assume that the liquid has a movable or immovable boundary surface S, which separates it from every other medium and with has the property that every fraction on this surface stays on it. Such boundary surfaces have impremable bottoms and free water surfaces. On the solid impremeable boundary the following condition is given:

$$\frac{\partial\varphi}{\partial n}=v_n, \tag{1.2}$$

where n =the initial normal vector of the given section of water area boundary S, v =the projection of velocity of the given side of the boundary in the direction of normal n.

On the surface of water, for $z=\eta(x,y,t)\approx 0$ the following conditions will be fulfilled:

$$\frac{\partial\varphi}{\partial t}=-g\eta,\quad \frac{\partial\eta}{\partial t}=\frac{\partial\varphi}{\partial z}, \tag{1.3}$$

where g is acceleration of gravity.

If we schematise the reservior in the rectangle or parallelepiped rectangle mode (plane and space cases, respectively), according to equations (1.1)-(1.3) we obtain the analytical solution of the boundary value problem. In this case, the landslide process is simulated in the following way: as a result of a horizontal landslide culminating in a wave process on the water surface, the displacement of liquid with intensity v, which determines the volume of liquid discharged at the end face of the wall (in the plane case) or at the corresponding board site (in the space case) during a time span $0 \leq t \leq t_0$, is equal to the given volume of the landslide mass.

The solution to the problem can be obtained using Laplace's integral transformation by various t and Furier's finite cosines - transformation by variables x, y - and in the three-dimentional case the following is given [3]:

$$\eta(x,y,t) = \frac{1}{L}\int_0^L \int_0^{-h} \int_0^t v(x,y,t)dxdzdt+ \tag{1.4}$$
$$+\frac{2}{L}\sum_{n=1}\sum_{m=1}\xi_n^m + \frac{1}{L_1}\sum_{n=1}\xi_n + \frac{1}{L_1}\sum_{m=1}\xi_m,$$

where

$$\xi_n^m = u_n J_n^m \cos a_n \cos a_m y I_n^m(t), \quad a_m = m\pi/L,$$

$$u_n = \int_0^L n(x,z,t)\cos a_n x dx, \quad a_n = \pi/L, \quad a_n^m = (a_n^2 + a_m^2)^{1/2}$$

$$J_n^m = -\frac{1}{\cosh a_n^m h}\int_0^{-h} v(x,z,t)\cosh a_n^m(h+\xi)d\xi,$$

$$I_n^m(t) = \int_0^t v(x,z,t)\cosh \gamma_n^m(t-\tau)d\tau,$$

$$\gamma_n^m = a_n^m g \tanh a_n^m h, \quad \xi_n = \xi_n^0, \quad \xi_m = \xi_m^0,$$

L =lenght, L_1 =width and h =depth of reservoir.

Formula (1.4) allows us to estimate the wave amplitude comparatively quickly, which is extremely important for engineers, above all in the case of the problem's multivariant performance.

In this case, the velocity of the water at the board site and its discharge time can be determined on the basis of any simplified assumptions on the effect of landslides on water.

The problem of the influence of water on its movement arises on account of the former's dynamic interaction with the latter. Therefore, it is necessary to find a combined solution to equations (1.1)-(1.3) and equations describing the landslide movement so to determine the latter's kinematic characteristics more precisely. The numerical solution to this problem in a two-dimentional application was discussed in paper [4]. With the aim of investigating this case, we consider the most important section of the real reservoir and landslide medium. The landslide movement is described by the following system of equations:

$$\frac{dl}{dt} = v_{sl}, \quad \frac{dv_{sl}}{dt} = \frac{F_g + F_0 + F_h + F_p}{M}, \tag{1.5}$$

where v_{sl} =landslide velocity, as a whole, $l(t)$ =the direction of the landslide from the moment it began to move until moment t, M =landslide mass.

$F_g = M_1 \sin\alpha$ gravitational force, M_1 =mass of the slide part lying over the board slope, α =board slope angle.

$F_0 = k(M_1 \cos\alpha + (M - M_1)) + cL_{sl}$ opposing force, k =opposing coefficient, c =bond coefficient, L_{sl} =slide length.

$F_h = -\int\limits_G pn_\tau dG - k\int\limits_G pn_\sigma dG$ hydrodynamic force, $p = \frac{\partial\varphi}{\partial n} + y$ surplus, in comparison with the atmospheric pressure of water, G and dG, respectively, slide bound converted to water and its infinitely small element (integral is taken over the wet side of the slide bound),y = ordinate of site G, n_γ and n_σ = normal projection of initial normal vector, respectively, on the direction of the movement and perpendicularly to it

$F_p = k\int\limits_G y' dG'$ force, conditioned by blowing pressure and activated at the foot of the slide and perpendicular to the slope, G' = foot of the slide, being below the surface of the water.

The force producing the slide can be given as follows:

$$F(a) = F_1(a) + aF_2(a),$$

where a = slide acceleration

$$F_1 = F_0 - \int\limits_G \xi_2 n_\tau dG - k\int\limits_G \xi_2 n_\sigma dG, \quad F_2 = -\int\limits_G \xi_1 n_\tau dG - k\int\limits_G \xi_1 n_\sigma dG,$$

$$F_0 = F_g + F_0 + F_p - \int\limits_G y n_\tau dG - k\int\limits_G y n_\sigma dG,$$

ξ_1 and ξ_2 =functions that satisfy the Laplace equation with the following boundary conditions on the surface of the water

$$\xi_1(x, 0; t) = 0, \quad \xi_2(x, 0; t) = -\eta(x; t),$$

and on moving solid bound of domain engaged by the water

$$\frac{\partial\xi_1}{\partial n} = -\eta, \quad \frac{\partial\xi_2}{\partial n} = -v_x \frac{\partial}{\partial x}\frac{\partial\varphi}{\partial n} - v_y \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial n}, \quad \gamma = (\overline{m}, \overline{n}),$$

$\overline{m}$ =initial vector, parallel to v, $\overline{n}$ =initial vector, perpendicular to an infinitely small section of the solid bound of domaim.

Acceleration a is determined by equation

$$a = \frac{F(a)}{M} = \frac{F_1(a)}{M} + a\frac{F_2(a)}{M}.$$

By solving the system of differential equations (1.1)-(1.3), combined with (1.5), using net method and some standard numerical method, we can obtain the unknown functions (the initial data) for carrying out calculations being determined by formula (1.4), which describes the wave generation process in a reservoir schematised in the parallelepiped rectangle mode.

2 *Mathematical Model Based on the Shallow Water Theory*

This model is based on a rough theory which springs from the assumption that the distribution of pressure in liquid obeys hydrostatic laws.

If we consider the ideal liquid and the potential stream, then the water plane movement can be described by three-dimentional Eulerian equations. If we average these equations by vertical axis z with the condition of impermeability on the bottom

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w = 0, \quad z = -h,$$

where h is the water surface mark and on free surface of water

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} - w = 0 \ \ and \ \ p = 0 \ \ for \ \ z = \eta,$$

this gives us the main equation - Saint-Venant equation - of the shallow water theory.

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -g\frac{\partial \eta}{\partial x}, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -g\frac{\partial \eta}{\partial y}, \\ \frac{\partial}{\partial x}(u(\eta + h)) = \frac{\partial}{\partial y}(v(\eta + h)) &= -\frac{\partial \eta}{\partial t}. \end{aligned} \tag{2.6}$$

After introducing the mass produced on the init. of area $\rho = \overline{\rho}(\eta + h)$ and the force produced on the init. of surface $p = g/2\overline{\rho}\rho^2$, this equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0,$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y} = -\frac{\partial p}{\partial x}, \tag{2.7}$$
$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho u v}{\partial x} + \frac{\partial \rho v^2}{\partial y} = -\frac{\partial p}{\partial y},$$

The sound velocity, determined as , is the local velocity of small perturbations relatively spreading through the water.

The initial conditions are

$$u(x, y, 0) = 0, \quad v(x, y, 0) = 0, \quad \rho(x, y, 0) = \rho_0(x, y).$$

So as to determine fall-wave parameters, the slide is simulated by moving the bound site at a given velocity. As a boundary condition, the velocity of the reservoir board site during time span $0 \le t \le t_0$ is therefore given in the following way:

$$u(x, y, t) = u_0, \quad v(x, y, y) = v_0, \quad (x, y) \in S,$$

where S is the slide section.

For solving the system of equation (2.2), the following completely conservative two-layer difference scheme [5,6]

$$\rho_t + j^{(0.5)}_{1\overset{0}{x}} + j^{(0.5)}_{2\overset{0}{y}} = 0,$$
$$j_{1t} + \left(j_1^{(0.5)} u^{(\sigma)}\right)_{\overset{0}{x}} + \left(j_2^{(0.5)} u^{(\sigma)}\right)_{\overset{0}{y}} + p^{(0.5)}_{\overset{0}{x}} = \omega_{1x} + f_1 + gF_1', \tag{2.8}$$
$$j_{2t} + \left(j_1^{(0.5)} v^{(\sigma)}\right)_{\overset{0}{x}} + \left(j_2^{(0.5)} v^{(\sigma)}\right)_{\overset{0}{y}} + p^{(0.5)}_{\overset{0}{y}} = \omega_{2y} + f_2 + gF_2',$$
$$j_1 = \rho u, \quad j_2 = \rho v, \quad \rho = \rho_0(\eta + h), \quad p = \frac{g}{2\overline{\rho}}\rho^2,$$
$$\omega_1 = \mu\rho h^2 u_{\overline{x}} |u_{\overline{x}}|, \quad \omega_2 = \mu\rho h^2 v_{\overline{y}} |v_{\overline{y}}|,$$
$$f_1 = -\frac{u c_f}{\rho}(u^2 + v^2)^{1/2}, \quad f_2 = -\frac{v c_f}{\rho}(u^2 + v^2)^{1/2},$$

is constructed. Here, ρ_0 =water density, μ =viscosity coefficient, f_1 and f_2 =friction forces in directions x and y, respectively, ω_1 and ω_2 =viscosity forces in directions x and y, respectively, $F_1 = F_1(x)$ and $F_2 = F_2(x)$ =functions describing the bottom relief of the reservoir.

The initial conditions are

$$u(ih, jh, 0) = 0, \quad v(ih, jh, 0) = 0, \quad \rho(ih, jh, 0) = \rho_0(ih, jh, 0).$$

For boundary conditions on the fictitious knots grid, where velocity and "density" functions are determined by the following formulae

$$\hat{u}_{l-1} = 2\hat{u}_l - u_{l+1}, \quad \hat{v}_{l-1} = 2\hat{v}_l - v_{l+1}, \quad \hat{\rho}_{l-1} = \rho_{l+1}.$$

Here, l =boundary knot, $l-1$ =the nearest functions knot, $l+1$ =the nearest preboundary knot.

An interval process may be used for solving these non-linear differential equations.

The choice of the definite friction computation model is determined by utility. It is usefull to take into consideration the computations can be easily made for the wide set of probable values of landslide parameters by means of computational dependence (1.4), which allows us to reread their less favorable combination. If, together with this, the maximum water increase near the dam is less then the excess of its crest above the back water level, the problem may be considered as solved. Otherwise, it might also be necessary to make more time-consuming calculations along the lines of the shallow water theory, which will allow us estimate the influence of the reservoir's real shape. It is also necessary to bear in mind that in some cases the results of the computations condition the need for physical experiments.

3 *Comparison of the Results of Physical and Mathematical Simulations.*

In order to control analytical and calculated dependencies, as well as revealing the applicability limits of mathematical models, the results obtained from the said models were compared with each other and with those of the labour experiments.

Because the analytical dependencies were obtained with the assumption that the domain occupied by the water is rectangle parallelepiped, in which the slide is simulated by the vertical fall of the solid mass and has the same rectangle parallelepiped form, the comparison was first carried out for the domain with the corresponding form.

As the slide solid is pulled down at the butt-end of the rectangle parallelepiped chute, it generates a wave that spreads along the chute, which, with a sufficient amount of accuracy, can be considered to be the plane wave corresponding to the solution obtained from the two-dimensional boundary problem. The comparison of the results obtained from the Baronin and Noda experiments [3],[7] are given below.

In the Baronin experiment, the chute had a rectangle cross-section of width $L_1 = 0.3m$ and length $L = 3m$. The depth of the water was $H = 0.202m$. A slide solid mass of thickness $D = 0.101m, 0.20Pm, 0.404m$ was pulled along the whole width of the chute during a time span of $t_0 = 0.375sec, 0.25sec$ and $0.6sec$. Simulating the dam, the vertical wall was located opposite the "slide" butt-end The wave amplitude was measured at different cross-sections of the chute by means of wave-meters situated over the butt-end.

The results of the above experiment were compared with those of calculations obtained by simulating the fall of a slide solid mass at a given velocity at the butt- end. Both mathematical models adequately described the wave generation process and their results coincided well with experimental results when $D/H \leq 1.0$. This is clearly shown in fig. 1.

In the case of a slide at the butt-end [7], Noda obtained a theoretical solution to the wave generation two-dimensional problem in a half-infinity domain. To verify this solution, an experiment using a chute of length $L = 32m$ and a rectangle parallelepiped box with a thickness $D = 7.5sm$ (for different depths) was carried out, in which the slide solid mass fell vertically at the butt-end along the whole width $L_1 = 0.3m$ during a time span t_0. The wave amplitude was measured at a definite distance from the butt-end. The depth was $H = 0.61m, 0.305m, 0.229m$. The wave amplitude was observed at distance $x = 20H, 20H$ and $26.7H$.

The comparison of the results of physical and mathematical simulation shows that the model based on the small amplitude wave theory describes the wave generation process just as well as the model based on the shallow water theory. The results of the computations coincide with the experimental results, especially in the case of the former, i.e. the main wave This comparison is shown in fig. 2.

The experiments for investigating waves generated by the lateral fall of slide solids mass, simulating a landslide on the shore of the reservoir,were conducted by authors [8]. A slide solid, length $b = 1.6m$ and thickness $D = 1.5sm$, fell vertically at the middle of lateral wall of a chute rectangle of length $L = 7,53m$ and width $L_1 = 1.3m$. Experiments were conducted for different depths of water and falling times. The waves generated were measured by wave-meters, located in the corners of the vertical walls at the butt-end of the chute (dam-site). The wave generation process for a water depth of $H = 8sm$ and a time span of $t_0 = 0.2sec$ is shown in fig. 3: (a) wave formation at $t = 1sec$, (b) wave spreading along the chute at $t = 4sec$. On the basis of comparison experimental and calculation data the limits of application the theoretical methods were determined [9].

Methods developed for investigating the wave generation process were used for carrying out the numerical investigation of the for the Getic reservoir in Armenia (fig. 4). An unstable mass of volcanogenic rock, with a total volume of $5 - 6mln.m^3$, is located on the right shore of the reservoir. Geological investigations have shown that as much as $1mln.m^3$ of rock could fall into the water at a fall-line with an extension of $2b = 100m$, approximately $x_0 = 960m$ away from the dam. After determination the probable velocity and duration of the fall, calculations were made in a two-dimensional application using the aforementioned methods [4].

It is not surprising that the calculations based on the shallow water theory, taking into consideration the lateral bay of the reservoir, local widening near the dam, bottom relief and the back of the dam (fig. 4), gave a smaller maximal amplitude wave (rise of the water level) near the dam $\eta = 6.3m$ than the calculations made using the small amplitude wave theory $\eta = 8.3m$, which were used in the case of the schemed rectangle reservoir.

Fig.1

Fig.2

Fig.3

Fig.4

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